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ARC-DISJOINT HAMILTONIAN PATHS IN STRONG ROUND DECOMPOSABLE LOCAL TOURNAMENTS

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Abstract

Thomassen, [Edge-disjoint Hamiltonian paths and cycles in tournaments, J. Combin. Theory Ser. B 28 (1980) 142–163] proved that every strong tournament has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if it is not an almost transitive tournament of odd order. As a subclass of local tournaments, Li *et al.* [*Arc-disjoint Hamiltonian cycles in round decomposable local tournaments*, Discuss. Math. Graph Theory 38 (2018) 477–490] confirmed the existence of such two paths in 2-strong round decomposable local tournaments. In this paper, we show that every strong, but not 2-strong, round decomposable local tournament contains a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices except for three classes of digraphs. Thus Thomassen's result is partly extended to round decomposable local tournaments. In addition, we also characterize strong round digraphs which contain a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

Keywords: local tournament, round-decomposable, arc-disjoint Hamiltonian paths.

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1. Terminology and Introduction

In this article all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. If xy is an arc of a digraph D, then we say that x *dominates* y and write $x \rightarrow y$. For a subset X of $V(D)$, the subdigraph induced by X in D is denoted by $D\langle X\rangle$

and $D - X$ is the subdigraph obtained by deleting X. A subdigraph H of D with $V(H) = V(D)$ is called a *spanning* subdigraph of D. Let $H_1, H_2, \ldots, H_\ell$ be subdigraphs of D, then the new subdigraph induced by $V(H_1) \cup V(H_2) \cup \cdots \cup V(H_\ell)$ in D is denoted by $D\langle H_1, H_2, \ldots, H_\ell\rangle$.

The *out-set* $N^+(x)$ of a vertex x is the set of vertices dominated by x in D, and the *in-set* $N^-(x)$ is the set of vertices dominating x in D. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively.

By a cycle (respectively, path) we mean a directed cycle (respectively, directed path). A path in a digraph D is *Hamiltonian* if it includes all the vertices of D. A path from u to v is called a (u, v)*-path.* A *chord* of a cycle C in a digraph D is an arc in $A(D) \setminus A(C)$, whose two ends lie on C.

The *underlying graph* of D is the graph obtained by ignoring the orientation of arcs in D and deleting parallel edges. We say that D is *connected* if its underlying graph is connected.

A digraph D is *strong*, if for any two vertices $x, y \in V(D)$, the digraph D contains a path from x to y and a path from y to x. A digraph D is k*-strong* if $|V(D)| \geq k + 1$ and for any set X of at most $k - 1$ vertices, the subdigraph $D - X$ is strong. If D is k-strong, but not $(k+1)$-strong, then we call k the *strong connectivity number* of D, denoted by $\kappa(D) = k$. If D is strong and x is a vertex of D such that $D - \{x\}$ is not strong, then we say that x is a *cut-vertex* of D.

A digraph D is *semicomplete* if for any two different vertices x and y, there is at least one arc between them. A semicomplete digraph without a 2-cycle is a *tournament.* An acyclic tournament is called *transitive.* It is easy to see that, for a transitive tournament T, there is a unique vertex ordering $v_1, v_2, \ldots, v_n$ of T, such that $v_i \to v_j$ for all $1 \leq i < j \leq n$. A tournament is *almost transitive* if it is obtained from the transitive tournament T by reversing the arc v_1v_n. In this paper, if we say that T is an almost transitive tournament with the vertex set $\{v_1, v_2, \ldots, v_n\}$, it will be always assumed that $v_i \to v_j$ for all $1 \leq i < j \leq n - 1$, $v_k \to v_n$ for $k = 2, 3, \ldots, n - 1$ and $v_n \to v_1$.

We call a digraph D *locally semicomplete*, if $D\langle N^+(x)\rangle$ and $D\langle N^-(x)\rangle$ are both semicomplete for every vertex x of D. A locally semicomplete digraph containing no cycle of length 2 is called a *local tournament.* It is clear that every tournament is a local tournament.

A digraph on n vertices is called a *round digraph* if we can label its vertices $x_1, \ldots, x_n$ such that for each i, $N^+(x_i) = \{x_{i+1}, \ldots, x_{i+d^+(x_i)}\}$ and $N^-(x_i) = \{x_{i-d^-(x_i)}, \ldots, x_{i-1}\}$, where the subscripts are taken modulo n, and the sequence $x_1, \ldots, x_n$ is called a *round sequence* of D.

The *second power of a cycle* C_n, denoted by C_n^2, is the digraph obtained from C_n by adding the arcs $\{x_ix_{i+2} \mid i = 1, 2, \ldots, n\}$, where $C_n = x_1x_2 \cdots x_nx_1$ and the subscripts are taken modulo n. Clearly, C_n^2 is a round digraph.

Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_r\}$ and let $H_1, H_2, \ldots, H_r$ be a collection of digraphs. Then $D[H_1, H_2, \ldots, H_r]$ is the new digraph obtained from D by replacing each vertex v_i of D with H_i and by adding the arcs from every vertex of H_i to every vertex of H_j if $v_i v_j$ is an arc of D for all i and j satisfying $1 \leq i \neq j \leq r$.

A locally semicomplete digraph D is *round decomposable*, if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[D_1, D_2, \ldots, D_r]$, where each D_i is a strong semicomplete digraph for $i = 1, 2, \ldots, r$. We call $R[D_1, D_2, \ldots, D_r]$ a *round decomposition* of D. Especially, when D is a round decomposable local tournament, each component D_i is a strong tournament. In this paper, if we say that D is a round decomposable local tournament, it will be always assumed that $R[D_1, D_2, \ldots, D_r]$ is a round decomposition of D, where $V(R) = \left\{u_1^1, u_1^2, \ldots, u_1^r\right\}$ with $u_1^i \in V(D_i)$ for $i = 1, 2, \ldots, r$, and $u_1^1, u_1^2, \ldots, u_1^r$ is a round sequence of R.

In the following, we shall use the abbreviations RD's to denote round digraphs, RDLT's to denote round decomposable local tournaments and ATTOO's to denote almost transitive tournaments with odd order at least three.

In 1980, Thomassen characterized the tournaments with two arc-disjoint Hamiltonian paths.

Theorem 1 [11]**.** *Every strong tournament T has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if T is not an almost transitive tournament of odd order.*

It is an interesting problem whether this result can be extended to local tournaments. Since Bang-Jensen [1] introduced the class of locally semicomplete digraphs in 1990, it has been intensively studied and the most interesting results can be found in [3–5, 8]. In 1997, Bang-Jensen, Guo, Gutin and Volkmann presented a full classification of locally semicomplete digraphs.

Theorem 2 [2]**.** *Let D be a connected locally semicomplete digraph. Then exactly one of the following possibilities holds:*

(a) *D is round decomposable with a unique decomposition $R[D_1, D_2, \ldots, D_r]$, where R is a round local tournament on $r \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, \ldots, r$;*

(b) *D is not round decomposable and not semicomplete;*

(c) *D is a not round decomposable, semicomplete digraph.*

Based on the above, many nice properties of semicomplete digraphs (tournaments) were extended to locally semicomplete digraphs (local tournaments), such as universal arcs, out-arc pancyclicity, kings and so on, see [9, 10, 12]. Recently, Li *et al.* proved the following result.

Theorem 3 [7]**.** *Every 2-strong round decomposable local tournament has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.*

In this paper we further characterize the strong, but not 2-strong, round decomposable local tournaments containing such a pair of paths. In addition, we also present a characterization of strong round digraphs which contain a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices which is a correction of a result in [6].

2. Arc-Disjoint Hamiltonian Paths in RD's

In [6] the authors presented a characterization of the round digraphs which have a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. According to this a strong round digraph D has a pair of such paths if and only if $C_n^2 - e$ is a spanning subdigraph of D when n is odd, or $C_n^2 - \{e_1, e_2\}$ is a spanning subdigraph of D when n is even, where e is a chord of C_n^2 when n is odd, or e_1, e_2 are two chords with no common end-vertex in C_n^2 when n is even. But this characterization is not correct (see the following example) and a new characterization is given in Theorem 9.

Example 4. Let $D = C_8^2 - \{e_1, e_2\}$, where $C_8 = u_1u_2 \cdots u_8u_1$, $e_1 = u_8u_2$ and $e_2 = u_4u_6$. Then D is a strong round digraph of even order and e_1, e_2 are two chords with no common end-vertex in C_8^2. Note that D has exactly two cut-vertices u_1 and u_5. Then by the proof of Claim 2 in the proof of Theorem 9, where $r = 8$ and $\ell = 2$, there do not exist two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

To present a revised version of the above characterization, we need the following lemmas, where all subscripts are taken modulo r.

Lemma 5. *Let D be a strong round digraph with a round sequence $u_1, u_2, \ldots, u_r$. If u_i and u_j $(1 \leq i < j \leq r)$ are two cut-vertices and D has two arc-disjoint Hamiltonian paths, then such two paths must start at u_{i+1} and u_{j+1} and end at u_{i-1} and u_{j-1}.*

Proof. Since $N^-(u_{i+1}) = \{u_i\}$, one of such two paths must start at u_{i+1}, as otherwise u_iu_{i+1} is a common arc of such two paths, a contradiction. Similarly, the other path must start at u_{j+1} since $N^-(u_{j+1}) = \{u_j\}$. Moreover, such two paths must end at u_{i-1} and u_{j-1} due to $N^+(u_{i-1}) = \{u_i\}$ and $N^+(u_{j-1}) = \{u_j\}$. ■

Lemma 6. *Let D be a strong round digraph of odd order. If D has two consecutive cut-vertices with respect to the round sequence, then there do not exist two arc-disjoint Hamiltonian paths in D.*

Proof. Let $u_1, u_2, \ldots, u_r$ be a round sequence of D and assume without loss of generality that u_1, u_2 are two cut-vertices. If D has two arc-disjoint Hamiltonian paths, then $r \geq 5$ and by Lemma 5, such two paths must start at u_2 and u_3 and end at u_r and u_1. Since $N^-(u_2) = \{u_1\}$, no (u_3, u_1)-path can contain the vertex u_2. So there is no Hamiltonian (u_3, u_1)-path, and thus, such two paths must be a (u_3, u_r)-path P_1 and a (u_2, u_1)-path P_2. It is easy to see that $P_2 = u_2u_3 \cdots u_ru_1$. Hence, there is at least one common arc in P_1 and P_2, since r is odd. This yields a contradiction. So there do not exist two arc-disjoint Hamiltonian paths in D. ∎

Lemma 7. *Let D be a strong round digraph with a round sequence $u_1, u_2, \ldots, u_r$. If u_i and u_j are two non-consecutive cut-vertices of D, then D has no Hamiltonian (u_{i+1}, u_{i-1})-path and no Hamiltonian (u_{j+1}, u_{j-1})-path.*

Proof. If there is a Hamiltonian (u_{i+1}, u_{i-1})-path P, then it must pass through the vertex u_i. Since u_j is a cut vertex, both of the subpaths $P[u_{i+1}, u_i]$ and $P[u_i, u_{i-1}]$ must contain the vertex u_j, which is impossible. So there is no Hamiltonian (u_{i+1}, u_{i-1})-path. Similarly, there is no Hamiltonian (u_{j+1}, u_{j-1})-path. ∎

Lemma 8. *Let D be a strong round digraph with a round sequence $u_1, u_2, \ldots, u_r$. If u_i and u_{i+2} are two cut-vertices of D for some $i \in \{1, 2, \ldots, r\}$, then there do not exist two arc-disjoint Hamiltonian paths in D.*

Proof. Assume for contradiction that D has two arc-disjoint Hamiltonian paths P_1 and P_2. Since $N^+(u_{i+1}) = \{u_{i+2}\}$ and $N^-(u_{i+1}) = \{u_i\}$, one of such two paths must start at u_{i+1}, say P_1, and the other path P_2 must end at u_{i+1}. By Lemma 5, P_1 is a (u_{i+1}, u_{i-1})-path and P_2 is a (u_{i+3}, u_{i+1})-path. This contradicts Lemma 7. ∎

Theorem 9. *Let D be a strong round digraph with a round sequence $u_1, u_2, \ldots, u_r$. Then D has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if either D has at most one cut-vertex, or D has exactly two cut-vertices whose subscripts are of different parity and r is even.*

Proof. (*Sufficiency*) First we consider the case that D has at most one cut-vertex. Then $C_r^2 - \{e\}$ is a spanning subdigraph of D, where $C_r = u_1u_2 \cdots u_ru_1$ and e is a chord of C_r in C_r^2. Assume without loss of generality that $e = u_1u_3$. When r is odd, $u_3u_5 \cdots u_ru_2u_4 \cdots u_{r-1}u_1$ and $u_1u_2 \cdots u_r$ are the desired two arc-disjoint Hamiltonian paths. When r is even, $u_3u_5 \cdots u_{r-1}u_1u_2u_4 \cdots u_r$ and $u_2u_3 \cdots u_ru_1$ are the desired two arc-disjoint Hamiltonian paths.

Now we consider the case that r is even and D has exactly two cut-vertices whose subscripts are of different parity. Assume without loss of generality that they are u_1 and $u_{2\ell}$, where $\ell \in \{1, 2, \ldots, r/2\}$. If $\ell = 1$, then $u_3u_5 \cdots u_{r-1}u_1u_2u_4$

$\cdots u_r$ and $u_2u_3\cdots u_ru_1$ are the desired two arc-disjoint Hamiltonian paths. If $\ell \in \{2,3,\ldots,r/2-1\}$, then $u_2u_4\cdots u_{2\ell}u_{2\ell+1}\cdots u_ru_1u_3\cdots u_{2\ell-1}$ and $u_{2\ell+1}u_{2\ell+3}\cdots u_1u_2\cdots u_{2\ell}u_{2\ell+2}\cdots u_r$ are the desired two paths. If $\ell = r/2$, then it is covered by renaming $u_{2\ell}$ as u_1.

(*Necessity*) Suppose D has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

If D has at least three cut-vertices and two of them are consecutive, then we may assume without loss of generality that they are u_1, u_2 and u_i. By Lemma 8 we know that $5 \le i \le r-2$. Since u_1 and u_i are two non-consecutive cut-vertices, then by Lemmas 5 and 7 one of such two paths is a (u_2, u_{i-1})-path and the other is a (u_{i+1}, u_r)-path. But u_1 cannot lie on the (u_2, u_{i-1})-path since $N^+(u_1) = \{u_2\}$. This yields a contradiction.

If D has at least three cut-vertices and none of them are consecutive, then we may assume that they are u_1, u_i and u_j $(4 \le i+1 < j \le r-1)$. By Lemma 8, we know that $7 \le i+3 \le j \le r-2$ and $r \ge 9$. Since u_1 and u_i are cut-vertices, by Lemmas 5 and 7 such two paths must be a (u_{i+1}, u_r)-path and a (u_2, u_{i-1})-path. But they have a common arc $u_{j-1}u_j$ since $N^+(u_{j-1}) = \{u_j\}$. It is a contradiction.

From the discussion above we know that D has at most two cut-vertices. If D contains at most one cut-vertex, then we are done. Assume in the following that D has exactly two cut-vertices, say u_1 and u_i, where $2 \le i \le r$. It follows from Lemmas 5 and 7 that one of such two paths is a (u_2, u_{i-1})-path P_1 and the other is a (u_{i+1}, u_r)-path P_2.

Since u_1 and u_i are cut-vertices, the path P_1 contains $P_1' = u_iu_{i+1}\cdots u_ru_1$ as a subpath and P_2 contains $P_2' = u_1u_2\cdots u_i$ as a subpath. If the round sequence from u_2 to u_{i-1} contains an odd number of vertices, then P_1 contains at least one common arc on P_2'. Similarly, if the round sequence from u_{i+1} to u_r contains an odd number of vertices, then P_2 contains at least one common arc on P_1'. So i and r are both even. That is to say the subscripts of such two cut-vertices are of different parity.

Altogether, we have shown that D has at most one cut-vertex or D has exactly two cut-vertices whose subscripts are of different parity and r is even. ■

3. Structure of RDLT's

In this section we only consider RDLT's with strong connectivity number 1 in view of Theorem 3. First we give the following definition which will be used to construct two arc-disjoint Hamiltonian paths in our main result (Theorem 15).

Definition 10. Let D be a round decomposable local tournament with a round decomposition $D = R[D_1, D_2, \ldots, D_r]$ and let $i \in \{1, 2, \ldots, r\}$.

(1) If D_i is a single vertex, say x, then define $P_1^i = P_2^i = P^i = P^{i'} = x$.

(2) If D_i is an ATTOO with the vertex set $\{x_1, x_2, \ldots, x_t\}$, then define $P_1^i = x_t x_1 x_3 \cdots x_{t-2}$, $P_2^i = x_2 x_4 \cdots x_{t-1}$ and $P^{i'} = x_1 x_2 \cdots x_t$ (Figure 1 gives an example, where D_i is an ATTOO with order five, $P_1^i = x_5 x_1 x_3$, $P_2^i = x_2 x_4$ and $P^{i'} = x_1 x_2 x_3 x_4 x_5$).

(3) If D_i is not an ATTOO and $|V(D_i)| \geq 3$, then $|V(D_i)| \geq 4$ and define P^i, $P^{i'}$ to be the two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices in D_i (Theorem 1 guarantees the existence of such two paths).

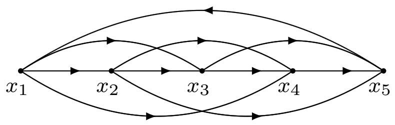

Figure 1. D_i

Proposition 11. *Let $P^i = x_1 x_2 \cdots x_t$ and $P^{i'} = y_1 y_2 \cdots y_t$ be the two paths in Definition* 10(3), *where $t = |V(D_i)|$, $x_1 \neq y_1$ and $x_t \neq y_t$. Then P^i can be partitioned into an (x_1, x_k)-subpath P_1^i and an (x_{k+1}, x_t)-subpath P_2^i such that $1 \leq k \leq t-1$, $x_{k+1} \neq y_1$ and $x_k \neq y_t$.*

Proof. Recall that $t \geq 4$. If $x_1 \neq y_t$ and $x_2 \neq y_1$, then $P_1^i = x_1$ and $P_2^i = x_2 x_3 \cdots x_t$ are the desired two subpaths of P^i. Assume in the following that $x_1 = y_t$ or $x_2 = y_1$.

If $x_1 = y_t$ and $x_3 \neq y_1$, then $x_2 \neq y_t$ and $P_1^i = x_1 x_2$, $P_2^i = x_3 \cdots x_t$ are the desired two subpaths of P^i.

If $x_1 = y_t$ and $x_3 = y_1$, then $x_3 \neq y_t$ and $x_4 \neq y_1$. So $P_1^i = x_1 x_2 x_3$ and $P_2^i = x_4 \cdots x_t$ are the desired two subpaths of P^i.

If $x_2 = y_1$, then $x_3 \neq y_1$ and $x_2 \neq y_t$. So $P_1^i = x_1 x_2$ and $P_2^i = x_3 x_4 \cdots x_t$ are the desired two subpaths of P^i. ■

Note that if x is a cut-vertex of an RDLT, then the component that x belongs to contains a single vertex. In order to present the counterexamples of our main result, we define the following substructures of D.

Definition 12. Let D be an RDLT with a round decomposition $D = R[D_1, D_2, \ldots, D_r]$ and let $u_1^{k_1}, u_1^{k_2}, \ldots, u_1^{k_p}$ be all cut-vertices of D, where $p \geq 1$, $1 \leq k_1 < k_2 < \cdots < k_p \leq r$ and $V(D_{k_i}) = \{u_1^{k_i}\}$ for $i = 1, 2, \ldots, p$. Then the subdigraphs $D\langle D_{k_1}, D_{k_1+1}, \ldots, D_{k_2}\rangle$, $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_{k_3}\rangle, \ldots, D\langle D_{k_p}, D_{k_p+1}, \ldots, D_{k_{p+1}}\rangle$ of D are called p *segments* of D, where $k_{p+1} \triangleq k_1 + r$ and all subscripts are taken modulo r. Note that when $p = 1$, the unique segment is D itself.

(1) A segment $D\langle D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}\rangle$ with at least one $|V(D_t)| \geq 3$ for some $t \in \{k_i+1, k_i+2, \ldots, k_{i+1}-1\}$ is called a *good-type segment* of D if none of the components $D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}$ is an ATTOO and no consecutive components are both a single vertex.

(2) A segment $D\langle D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}\rangle$ with at least one $|V(D_t)| \geq 3$ for some $t \in \{k_i+1, k_i+2, \ldots, k_{i+1}-1\}$ is called a *bad-type*-I *segment* of D if at least one component D_α is an ATTOO for some $\alpha \in \{k_i+1, \ldots, k_{i+1}-1\}$ or at least two consecutive components are a single vertex. The number of bad-type-I segments in D is denoted by $b_1(D)$.

(3) A segment $D\langle D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}\rangle$ is called a *bad-type*-II *segment* of D if $k_{i+1} - k_i$ is an even number and each component D_t contains a single vertex for $t = k_i, k_i+1, \ldots, k_{i+1}$. The number of bad-type-II segments in D is denoted by $b_2(D)$.

(4) A segment $D\langle D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}\rangle$ is called a *bad-type*-III *segment* of D if $k_{i+1} - k_i$ is an odd number and each component D_t contains a single vertex for $t = k_i, k_i+1, \ldots, k_{i+1}$. The number of bad-type-III segments in D is denoted by $b_3(D)$.

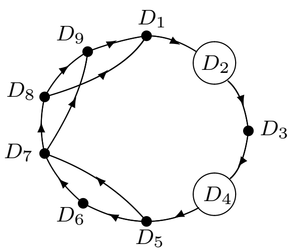

Figure 2

To illustrate Definition 12, we give an RDLT D in Figure 2, where D_i is a single vertex for $i = 1, 3, 5, 6, 7, 8, 9$, D_2 is a strong tournament with order four and D_4 is a strong tournament with order three. Then D_4 is an ATTOO and the unique vertex in $V(D_j)$ is a cut-vertex of D for $j = 1, 3, 5, 7$. Thus there are four segments $D\langle D_1, D_2, D_3\rangle$, $D\langle D_3, D_4, D_5\rangle$, $D\langle D_5, D_6, D_7\rangle$, $D\langle D_7, D_8, D_9, D_1\rangle$ in D, and they are of good-type, bad-type-I, bad-type-II and bad-type-III, respectively.

Remark 13. Any segment of D is either good-type or bad-type-I or bad-type-II or bad-type-III. The bad-type-I, bad-type-II and bad-type-III segments are collectively called *bad-type segments*. Every *good*-type segment contains two arc-disjoint Hamiltonian paths, but any bad-type segment does not have this property. That is the reason we call it a good-type or bad-type segment.

Now we define three special classes of RDLT's as follows which are the exceptions of our main result.

$\mathcal{D}_1 = \{D \mid D$ is an RDLT with $\kappa(D) = 1$ and $b_2(D) \geq 2\}$;

$\mathcal{D}_2 = \{D \mid D$ is an RDLT with $\kappa(D) = 1, b_2(D) = 1$ and $b_3(D) + b_1(D) \geq 1\}$;

$\mathcal{D}_3 = \{D \mid D$ is an RDLT with $\kappa(D) = 1, b_2(D) = 0$ and $b_3(D) + b_1(D) \geq 3\}$.

It is clear that every round local tournament is an RDLT, where each component consists of a single vertex. Moreover, if D is a strong round local tournament of order r, then D has exactly two cut-vertices whose subscripts are of different parity and r is even if and only if $b_2(D) = 0$, $b_3(D) = 2$ and $b_1(D) = 0$. Combining Theorem 9 with the definitions of $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ we can obtain the following result.

Corollary 14. *A strong round local tournament D has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if D is not in $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$.*

4. Arc-Disjoint Hamiltonian Paths in RDLT's

Theorem 15 (Main result)**.** *Let D be a round decomposable local tournament with $\kappa(D) = 1$. Then D has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices if and only if D is not in $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$.*

Proof. Let $R[D_1, D_2, \ldots, D_r]$ be a round decomposition of D. If each component D_i is a single vertex, then D is a round local tournament and we are done by Corollary 14. So assume in the following that at least one component of D is not a single vertex.

Let $u_1^{k_1}, u_1^{k_2}, \ldots, u_1^{k_p}$ be all cut-vertices of D, where $p \geq 1$, $1 \leq k_1 < k_2 < \cdots < k_p \leq r$ and $V(D_{k_i}) = \{u_1^{k_i}\}$ for $i = 1, 2, \ldots, p$. Assume without loss of generality that $k_1 = 1$. Then divide D into p segments $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$, $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_{k_3}\rangle$,..., $D\langle D_{k_p}, D_{k_p+1}, \ldots, D_{k_{p+1}}\rangle$, where $k_{p+1} \triangleq r + 1$ and $D_{r+1} \triangleq D_1$. Denote $\ell_i = |V(D_i)|$ and $u_1^i \in V(D_i)$ for $i = 1, 2, \ldots, r$. The symbols $P^i, P^{i'}, P_1^i, P_2^i$ refer to Definition 10 and Proposition 11.

(*Sufficiency*) Suppose D is not in $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Then $b_2(D) \leq 1$, and when $b_2(D) = 1$, we have $b_3(D) = b_1(D) = 0$; when $b_2(D) = 0$, we have $b_3(D)+b_1(D) \leq 2$. Consider the following seven cases.

Case 1. $b_2(D) = 1$, $b_1(D) = b_3(D) = 0$. In this case $p \geq 2$, as otherwise, the unique segment is bad-type-II, which implies that each component D_i consists of a single vertex, a contradiction. Assume without loss of generality that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ is a bad-type-II segment. Then other segments are all good-type, and hence, there are two arc-disjoint Hamiltonian $(u_1^{k_2}, u_1^1)$-paths P^* and

P^{**} in $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_1\rangle$ by Remark 13. Now $u_1^2u_1^4\cdots u_1^{k_2-1}P^*u_1^3u_1^5\cdots u_1^{k_2-2}$ and $P^{**}u_1^2u_1^3\cdots u_1^{k_2-1}$ are the desired two paths.

Case 2. $b_2(D) = b_1(D) = 0$, $b_3(D) = 2$. In this case $p \geq 3$. Assume that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ and $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_{k_3}\rangle$ are bad-type-III segments. If this is not the case, then we will have two sets of paths instead of P^* and P^{**}, but the structure is the same, so we only give the proof in the first case and other cases are similar (just with 2 extra paths). It is clear that k_2 is an even number and k_3 is an odd number. Moreover, other segments are all good-type, and hence, there are two arc-disjoint Hamiltonian $\left(u_1^{k_3}, u_1^1\right)$-paths P^* and P^{**} in $D\langle D_{k_3}, D_{k_3+1}, \ldots, D_1\rangle$. Now $u_1^2u_1^4\cdots u^{k_2}u^{k_2+1}\cdots u^{k_3-1}P^*u_1^3u_1^5\cdots u^{k_2-1}$ and $u^{k_2+1}u^{k_2+3}\cdots P^{**}u_1^2u_1^3\cdots u_1^{k_2}u_1^{k_2+2}\cdots u_1^{k_3-1}$ are the desired two paths.

Case 3. $b_2(D) = b_3(D) = 0$, $b_1(D) = 1$. Suppose first that $p = 1$ and u_1^1 is the unique cut-vertex. If r is odd, then let $P = P_1^2(P_1^3)P_1^4\cdots P_1^1(P_2^2)P_2^3(P_2^4)\cdots P_2^r$ and $P' = P^{3'}P^{4'}\cdots P^{r'}P^{1'}P^{2'}$, where the symbol (P_1^3) denotes that when $\ell_3 \geq 3$, the path P passes through P_1^3; when $\ell_3 = 1$, the path P skips the path P_1^3. Other similar symbols express the same meaning. It is not difficult to check that P and P' are the desired two paths.

If r is even, then it is easy to see that $P = P_1^2(P_1^3)P_1^4\cdots P_1^rP_1^1(P_2^2)P_2^3(P_2^4)\cdots P_2^{r-1}(P_2^r)$ and $P' = P^{1'}P^{2'}P^{3'}\cdots P^{r'}$ are the desired two paths.

Suppose now that $p \geq 2$ and assume without loss of generality that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ is a bad-I-type segment. Then $k_2 \geq 3$ and other segments are all good-type. Thus, there are two arc-disjoint Hamiltonian $(u_1^{k_2}, u_1^1)$-paths P^* and P^{**} in $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_1\rangle$.

If $k_2 = 3$, then D_2 is an ATTOO and $P = P_1^2P^*P_2^2$, $P' = P^{2'}P^{**}$ are the desired two paths.

If $k_2 \geq 5$ is odd, then $P = P_1^2(P_1^3)P_1^4\cdots P_1^{k_2-1}P^*(P_2^2)P_2^3\cdots P_2^{k_2-2}\left(P_2^{k_2-1}\right)$ and $P' = P^{**}P^{2'}P^{3'}\cdots P^{k_2-1'}$ are the desired two paths.

If $k_2 \geq 4$ is even, then $P = P_1^2(P_1^3)P_1^4\cdots P^*(P_2^2)P_2^3(P_2^4)\cdots P_2^{k_2-1}$ and $P' = P^{2'}P^{3'}\cdots P^{k_2-1'}P^{**}$ are the desired two paths.

Case 4. $b_2(D) = b_3(D) = 0$, $b_1(D) = 2$. Similarly to Case 2, we may assume that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ and $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_{k_3}\rangle$ are bad-type-I segments. Note that when $p \geq 3$, other segments are all good-type. Let $s_1 = \min\{2 \leq j \leq k_2 - 1 \mid \ell_j \geq 3\}$ and $s_2 = \min\{k_2 + 1 \leq j \leq k_3 - 1 \mid \ell_j \geq 3\}$. Define the following paths:

$$\begin{aligned}
P_1 &= P_1^2\left(P_1^3\right)P_1^4\cdots\left(P_1^{k_2-1}\right)P^{k_2'}P^{k_2+1'}\cdots P^{r'}P^1\left(P_2^2\right)P_2^3\cdots P_2^{k_2-1};\\
P_2 &= P_1^{k_2+1}\left(P_1^{k_2+2}\right)P_1^{k_2+3}\cdots P^{k_3}\left(P^{k_3+1}\cdots P^rP^1\right)P^{2'}\\
&\quad\cdots P^{k_2'}\left(P_2^{k_2+1}\right)P_2^{k_2+2}\left(P_2^{k_2+3}\right)\cdots P_2^{k_3-1};
\end{aligned}$$

$$\begin{aligned}
P_3 &= P^{k_2+1}P^{k_2+3}\cdots P_1^{s_2}P_1^{s_2+1}\left(P_1^{s_2+2}\right)P_1^{s_2+3}\cdots\left(P_1^{k_3-1}\right)P^{k_3}\left(P^{k_3+1}\cdots P^1\right)P^{2'}\\
&\quad\cdots P^{k_2'}P^{k_2+2}\cdots P^{s_2-1}P_2^{s_2}\left(P_2^{s_2+1}\right)P_2^{s_2+2}\cdots P_2^{k_3-1};\\
P_4 &= P^{k_2+1}P^{k_2+3}\cdots P^{s_2-1}P_1^{s_2}\left(P_1^{s_2+1}\right)P_1^{s_2+2}\cdots P^{k_3}\left(P^{k_3+1}\cdots P^rP^{1'}\right)\\
&\quad\cdots P^{k_2'}P^{k_2+2}\cdots P_2^{s_2}P_2^{s_2+1}\left(P_2^{s_2+2}\right)\cdots P_2^{k_3-1};\\
P_5 &= P^2P^4\cdots P^{s_1-1}P_1^{s_1}\left(P_1^{s_1+1}\right)P_1^{s_1+2}\cdots P^{k_2'}P^{k_2+1'}\cdots P^{r'}P^1P^3\\
&\quad\cdots P_2^{s_1}P_2^{s_1+1}\left(P_2^{s_1+2}\right)\cdots P_2^{k_2-1};\\
P_6 &= P^2P^4\cdots P_1^{s_1}P_1^{s_1+1}\left(P_1^{s_1+2}\right)P_1^{s_1+3}\cdots P^{k_2'}P^{k_2+1'}\cdots P^{r'}P^1P^3\\
&\quad\cdots P^{s_1-1}P_2^{s_1}\left(P_2^{s_1+1}\right)P_2^{s_1+2}\cdots P_2^{k_2-1}.
\end{aligned}$$

If k_2 is even and k_3 is odd, then P_1 and P_2 are the desired two paths. An example is shown in Figure 3, where $k_2 = 4$, $k_3 = 7$, $V(D_i) = \{u_1^i\}$ for $i = 1, 3, 4, 5$, D_j is a 3-cycle $u_1^ju_2^ju_3^ju_1^j$ for $j = 2, 6$ and u_1^1, u_1^4 are two cut-vertices. Then $P_1 = u_3^2u_1^2u_1^4u_1^5u_1^6u_2^6u_3^6u_1^1u_2^2u_1^3$ and $P_2 = u_1^5u_3^6u_1^6u_1^1u_1^2u_2^2u_3^2u_1^3u_1^4u_2^6$ are two arc-disjoint Hamiltonian paths.

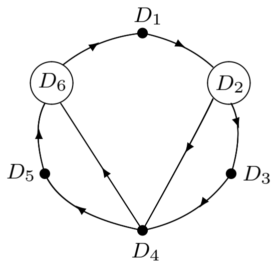

Figure 3

If k_2 and k_3 are both even, then in the case when s_2 is odd, P_1 and P_3 are the desired paths; in the case when s_2 is even, P_1 and P_4 are the desired two paths.

If k_2 is odd and k_3 is even, then in the case when s_1 is odd, P_5 and P_2 are the desired paths; in the case when s_1 is even, P_6 and P_2 are the desired paths.

If k_2 and k_3 are both odd, then in the case when s_1 and s_2 are even, P_6 and P_3 are the desired paths; in the case when s_1 is even and s_2 is odd, P_6 and P_4 are the desired paths; in the case when s_1 and s_2 are odd, P_5 and P_4 are the desired paths; in the case when s_1 is odd and s_2 is even, P_5 and P_3 are the desired paths.

Case 5. $b_2(D) = 0$, $b_3(D) = b_1(D) = 1$. Assume without loss of generality that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ is a bad-type-III segment and $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_{k_3}\rangle$ is a bad-type-I segment. Note that when $p \geq 3$, other segments are all good-type. Let $P = P^2P^4\cdots P^{k_2}P^{k_2+1'}\cdots P^{r'}P^1P^3\cdots P^{k_2-1}$. Then P is a Hamiltonian path in D. Now we look for another Hamiltonian path P' in D such that P and P' are arc-disjoint.

Subcase 5.1. k_3 is an odd number. In this case $P' = P_1^{k_2+1}\big(P_1^{k_2+2}\big)P_1^{k_2+3}\cdots P_1^{k_3}\big(P^{k_3+1}\cdots P^1\big)P^2P^3\cdots P^{k_2}\big(P_2^{k_2+1}\big)P_2^{k_2+2}\big(P_2^{k_2+3}\big)\cdots P_2^{k_3-1}$ is another desired path.

Subcase 5.2. k_3 is an even number. Define $s = \min\{k_2+1 \leq j \leq k_3-1 \mid \ell_j \geq 3\}$. In the case when s is an odd number, the path $P' = P^{k_2+1}P^{k_2+3}\cdots P_1^sP_1^{s+1}\left(P_1^{s+2}\right)P_1^{s+3}\cdots P_1^{k_3}\left(P^{k_3+1}\cdots P^1\right)P^2P^3\cdots P^{k_2}P^{k_2+2}\cdots P^{s-1}P_2^s\left(P_2^{s+1}\right)P_2^{s+2}\big(P_2^{s+3}\big)\cdots P_2^{k_3-1}$ is just we desired. In the other case, when s is an even number, $P' = P^{k_2+1}P^{k_2+3}\cdots P^{s-1}P_1^s\big(P_1^{s+1}\big)P_1^{s+2}\cdots P_1^{k_3}\big(P^{k_3+1}\cdots P^1\big)P^2P^3\cdots P^{k_2}P^{k_2+2}\cdots P_2^sP_2^{s+1}\big(P_2^{s+2}\big)\cdots P_2^{k_3-1}$ is the desired path.

Case 6. $b_2(D) = b_1(D) = 0$, $b_3(D) = 1$. In this case $p \geq 2$. Assume without loss of generality that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ is a bad-type-III segment. Then other segments are all good-type. So there are two arc-disjoint Hamiltonian $\big(u_1^{k_2}, u_1^1\big)$-paths P^* and P^{**} in $D\langle D_{k_2}, D_{k_2+1}, \ldots, D_1\rangle$. Now $P = P^2P^4\cdots P^{k_2-2}P^*P^3P^5\cdots P^{k_2-1}$ and $P' = P^{**}P^2P^3\cdots P^{k_2-1}$ are the desired two paths.

Case 7. $b_1(D) = b_2(D) = b_3(D) = 0$. In this case all segments of D are good-type, and then, every segment $D\langle D_{k_i}, D_{k_i+1}, \ldots, D_{k_{i+1}}\rangle$ has two arc-disjoint Hamiltonian paths P_i and P_i' for $i = 1, 2, \ldots, p$. Hence, $P_1P_2\cdots P_p$ and ${P_2}'{P_3}'\cdots {P_p}'{P_1}'$ are two arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices.

(*Necessity*) Suppose $D \in \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Then $p \geq 2$. In the following we show that D does not contain two arc-disjoint Hamiltonian paths.

Claim 1. *If there is a bad-type-*II *segment and besides it there is another bad-type segment in D, then D does not contain two arc-disjoint Hamiltonian paths.*

Proof. Assume without loss of generality that $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$ is a bad-type-II segment. Then k_2 is an odd number. Note that D contains at least two bad-type segments and any of them does not contain two arc-disjoint Hamiltonian paths. Since $u_1^{k_1}, u_1^{k_2}, \ldots, u_1^{k_p}$ are cut-vertices of D, we deduce that any Hamiltonian path of D starting at some one segment must pass through the Hamiltonian path of any other segment. So if D contains a pair of arc-disjoint Hamiltonian paths P and P', then they must start at different segments, and then, at least one path, say P, does not start at the segment $D\langle D_1, D_2, \ldots, D_{k_2}\rangle$. So P must contain $u_1^1u_1^2\cdots u_1^{k_2}$ as a subpath. Since k_2 is odd, the path P' contains at least one arc on this subpath, a contradiction. Therefore, D does not contain two arc-disjoint Hamiltonian paths. $\square$

If $D \in \mathcal{D}_1 \cup \mathcal{D}_2$, then we are done by Claim 1. So we only need to consider the case that $D \in \mathcal{D}_3$. This implies that there are at least three bad-type segments in

D. Note that any Hamiltonian path of D starting at some one segment must pass through the Hamiltonian path of any other segment and each bad-type segment does not have two arc-disjoint Hamiltonian paths. So D does not contain two arc-disjoint Hamiltonian paths. ■

5. DISCUSSION

Combining Theorem 15 with Theorem 3, we partly extend Theorem 1 from tournaments to round decomposable local tournaments. According to the classification of local tournaments, it would be interesting whether Theorem 1 can be further extended to non-round decomposable local tournaments. In [6] it was proved that every 2-strong non-round decomposable local tournament has a pair of arc-disjoint Hamiltonian paths with distinct initial vertices and distinct terminal vertices. So it remains to consider the existence of such two paths in strong, but not 2-strong, non-round decomposable local tournaments. We leave this as an open problem.

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