# LIST EDGE COLORING OF PLANAR GRAPHS WITHOUT 6-CYCLES WITH TWO CHORDS 

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#### Abstract

A graph $G$ is edge- $L$-colorable if for a given edge assignment $L=\{L(e)$ : $e \in E(G)\}$, there exists a proper edge-coloring $\varphi$ of $G$ such that $\varphi(e) \in L(e)$ for all $e \in E(G)$. If $G$ is edge- $L$-colorable for every edge assignment $L$ such that $|L(e)| \geq k$ for all $e \in E(G)$, then $G$ is said to be edge- $k$-choosable. In this paper, we prove that if $G$ is a planar graph without 6 -cycles with two chords, then $G$ is edge- $k$-choosable, where $k=\max \{7, \Delta(G)+1\}$, and is edge- $t$-choosable, where $t=\max \{9, \Delta(G)\}$.


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## 1. InTRODUCTION

Graphs considered in this paper are finite, simple and undirected. The terminologies and notations used but undefined in this paper can be found in [2]. Let $G=(V, E)$ be a graph. We use $V(G), E(G), \Delta(G)$ and $\delta(G)$ (or simply $V, E$, $\Delta$ and $\delta$ ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of $G$, respectively. A cycle $C$ of length $k$ is called a $k$-cycle in the graph $G$. If $x y \in E(G) \backslash E(C)$ and $x, y \in V(C), x y$ is called to be a chord of $C$ in the graph $G$.

[^0]An edge coloring of a graph $G$ is a mapping $\varphi$ from $E(G)$ to the set of colors $\{1,2, \ldots, k\}$ for some positive integer $k$. An edge coloring is called proper if every two adjacent edges receive different colors. The edge chromatic number $\chi^{\prime}(G)$ is the smallest integer $k$ such that $G$ has a proper edge-coloring into the set $\{1,2, \ldots, k\}$.

We say that $L$ is an edge assignment for the graph $G$ if it assigns a list $L(e)$ of possible colors to each edge $e$ of $G$. If $G$ has a proper edge-coloring $\varphi$ such that $\varphi(e) \in L(e)$ for each edge $e$ of $G$, then we say that $G$ is edge-L-colorable or $\varphi$ is an edge-L-coloring of $G$. The graph $G$ is edge- $k$-choosable if it is edge-$L$-colorable for every edge assignment $L$ satisfying $|L(e)| \geq k$ for all $e \in E(G)$. The list edge chromatic number $\chi_{\text {list }}^{\prime}(G)$ of $G$ is the smallest $k$ such that $G$ is edge- $k$-choosable.

On the list edge coloring of a graph, there is a celebrated conjecture known as the list edge coloring conjecture, which was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris (see [8, 13]).

Conjecture 1 [9]. If $G$ is a multigraph, then $\chi_{\text {list }}^{\prime}(G)=\chi^{\prime}(G)$.
The conjecture has been proved for a few classes of graphs, such as graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [4], outerplanar graphs [19], bipartite multigraphs [4, 7], complete graphs of odd order [9]. Vizing [15] proposed a weaker conjecture than Conjecture 1.

Conjecture 2 [9]. Every graph $G$ is edge- $(\Delta(G)+1)$-choosable.
Harris [10] showed that $\chi_{\text {list }}^{\prime}(G) \leq 2 \Delta(G)-2$ if $G$ is a graph with $\Delta(G) \geq 3$. This implies Conjecture 2 for the case $\Delta(G)=3$. Juvan et al. [14] settled the case for $\Delta(G)=4$ in 1999. And there are some other special cases of Conjecture 2 which have been confirmed, such as complete graphs [8], graphs with girth at least $8 \Delta(G)(\ln \Delta(G)(G)+1.1)$ [15], planar graphs with $\Delta(G) \geq 8$ [1], and planar graphs with $\Delta(G) \neq 5$ and without intersecting 3 -cycles [20]. Suppose that $G$ is a planar graph without $k$-cycles for some fixed integer $3 \leq k \leq 6$. Then it was proved that Conjecture 2 holds if $G$ satisfies one of the four following conditions:
(i) either $k=3$ or $k=4$ and $\Delta(G) \neq 5$ [22],
(ii) $k=4$ [17],
(iii) $k=5$ [20],
(iv) $k=6$ and $\Delta(G) \neq 5[18]$.

Other related known results on this topic can be found in [5, 11, 12, 16].
Cai [6] proved that if $G$ is a planar graph without chordal 6 -cycles, then $G$ is edge- $k$-choosable, where $k=\max \{8, \Delta(G)+1\}$. In this paper, we will strengthen this result and obtain that if $G$ is a planar graph and each 6 -cycle of $G$ contains at most one chord, then $\chi_{\text {list }}^{\prime}(G) \leq \max \{7, \Delta(G)+1\}$ and $\chi_{\text {list }}^{\prime}(G) \leq \max \{9, \Delta(G)\}$.

## 2. Main Results and Their Proofs

In the section, we always assume that all graphs are planar graphs that have been embedded in the plane and $G$ is a planar graph without 6 -cycles with two chords. We use $d_{G}(x)$, or simply $d(x)$, to denote the degree of a vertex $x$ in $G$. For $f \in F(G)$, if $u_{1}, u_{2}, \ldots, u_{n}$ are the vertices on the boundary walk, then we write $f=u_{1} u_{2} \cdots u_{n} u_{1}$. The degree of a $f$, denoted by $d(f)$, is the number of edges incident with $f$, where each cut-edge is counted twice. We denote by $\delta(f)$ the minimum degree of vertices incident with the face $f$. A vertex (face) $x$ is called to be a $k$-vertex ( $k$-face), $k^{+}$-vertex ( $k^{+}$-face) and $k^{-}$-vertex ( $k^{-}$-face), if $d(x)=k, d(x) \geq k$ and $d(x) \leq k$, respectively. $f_{i}(v)$ is the number of $i$-faces incident with $v$ for each $v \in V(G)$.

First, we give some properties on $G$.
Lemma 3. If $v$ is a $5^{+}$-vertex of $G$, then $f_{3}(v) \leq\left\lfloor\frac{3}{4} d(v)\right\rfloor$.
Proof. Since $G$ contains no 6 -cycles with two chords, $v$ is not incident with four consecutive 3 -faces. So $f_{3}(v) \leq\left\lfloor\frac{3}{4} d(v)\right\rfloor$.

Lemma 4. Let $u$ be a 4-vertex of $G$.
(1) If $f_{3}(u)=3$, then $f_{4}(u)=0$, that is, $u$ is incident with a $5^{+}$-face.
(2) If $f_{3}(u)=2$, then $f_{4}(u) \leq 1$.

Proof. Let neighbors of $u$ be $u_{1}, u_{2}, u_{3}, u_{4}$ and faces incident with $u$ be $f_{1}, f_{2}, f_{3}$, $f_{4}$ in the clockwise order, where $f_{1}$ is incident with $u_{1}, u_{2}$.
(1) Without loss of generality, we assume that $f_{1}, f_{2}, f_{3}$ are 3 -faces. If $f_{4}$ is a 4 -face $u u_{1} v u_{4} u$, then the 6 -cycle $u u_{2} u_{3} u_{4} v u_{1} u$ contains two chords $u u_{3}$ and $u u_{4}$, a contradiction. So $d\left(f_{4}\right) \geq 5$, that is, $f_{5^{+}}(u)=1$.
(2) Suppose that two 3 -faces incident with $u$ are not adjacent, without loss of generality, we assume that $f_{1}, f_{3}$ are 3 -faces. If $f_{2}$ is a 4 -face $u u_{2} v u_{3} u$, then the 6 -cycle $u u_{1} u_{2} v u_{3} u_{4} u$ contains two chords $u u_{2}$ and $u u_{3}$, a contradiction. So $d\left(f_{2}\right) \geq 5$. By the same argument, we have $d\left(f_{4}\right) \geq 5$.

Suppose that two 3 -faces incident with $u$ are adjacent, without loss of generality, we assume that $f_{1}, f_{2}$ are 3 -faces. If $f_{3}$ is a 4 -face $u u_{3} v u_{4} u$, then we must have $v=u_{1}$. Since $d(u)=4, d\left(u_{4}\right) \geq 5$. Thus if $f_{4}$ is a 4 -face $u u_{1} w u_{4} u$, then we also have $w=u_{3}$, it is impossible. So $d\left(f_{4}\right) \geq 5$. By the same argument, if $d\left(f_{4}\right)=4$, then $d\left(f_{3}\right) \geq 5$. Hence $f_{4}(u) \leq 1$.

Lemma 5. $G$ satisfies at least one of the following conditions.
(1) $G$ has an edge uv with $d(u)+d(v) \leq \max \{8, \Delta(G)+2\}$.
(2) $G$ has an even cycle $C=v_{1} v_{2} \cdots v_{2 n} v_{1}$ with $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 n-1}\right)$ $=3$.
(3) $G$ has a 6-vertex $u$ with five neighbors $v, w, x, y, z$ such that $d(v)=d(y)=3$ and $v w, x y, y z \in E(G)$ (see Figure 1).


Figure 1. The subgraph for Lemma 5(3).

Proof. Let $G$ be a minimal counterexample to the lemma. It is easy to check that $G$ is connected. By the choice of $G$, we have the following observations.
(P1) For any edge $u v, d(u)+d(v) \geq \max \{9, \Delta(G)+3\}$ by (1). Then $\delta(G) \geq 3$ and all neighbors of a $i$-vertex must be $(9-i)^{+}$-vertices, where $i=3,4$ or 5 .
(P2) Let $G_{3}$ be the subgraph induced by the edges incident with 3 -vertices of $G$. Then $G_{3}$ is a forest.

By ( P 1 ), every two 3 -vertices are not adjacent, and it follows that $G_{3}$ is a bipartite subgraph. By (2), $G_{3}$ contains no even cycles. So $G_{3}$ is a forest and (P2) holds. Let $V_{1}$ be the set of 3 -vertices of $G$. Thus for any component of $G_{3}$, we select a vertex $u \notin V_{1}$ as a root of the tree. Then every 3 -vertex has exactly two children. If $u v \in E\left(G_{3}\right), u \in V_{1}$ and $v$ is a child of $u$, then $v$ is called a 3 -master of $u$. Note that each 3 -vertex has exactly two 3 -masters and each vertex of degree at least 6 can be the 3 -master of at most one 3 -vertex.

According to the Euler's formula $|V(G)|-|E(G)|+|F(G)|=2$ of a planar graph $G$, we have
$\sum_{v \in V(G)}(3 d(v)-10)+\sum_{f \in F(G)}(2 d(f)-10)=-10(|V(G)|-|E(G)|+|F(G)|)=-20<0$.
Now we define the initial weight function on $V(G) \cup F(G)$ by letting $w(x)=$ $3 d(x)-10$ for any $x \in V(G)$ and $w(x)=2 d(x)-10$ for any $x \in F(G)$. Thus the total sum of weights is the negative number -20 . We use the following rules to redistribute the initial charge that leads to a new charge $w^{\prime}(x)$.
R1. Every 3-vertex $v$ receives $\frac{1}{2}$ from each of its 3-masters.
R2. Let $f=u u^{\prime} v v^{\prime} u$ be a 4-face in $G$ with $d(u) \leq \min \left\{d\left(u^{\prime}\right), d(v), d\left(v^{\prime}\right)\right\}$. If $d(u) \geq 4$, then $f$ receives $\frac{1}{2}$ from each of its incident vertices. Otherwise, $f$ receives nothing from $u$, receives $\frac{1}{2}$ from $v, \frac{3}{4}$ from $u^{\prime}$ and $\frac{3}{4}$ from $v^{\prime}$.

R3. Let $f$ be a 3 -face incident with a $4^{+}$-vertex $v$. Then $f$ receives $a$ from $v$.
R3.1. If $d(v)=4$, then
$a= \begin{cases}\frac{1}{2} & \text { if } f_{4}-(v)=4 \text { or if } f_{3}(v)=3 \text { and } f \text { is located in the middle } \\ \text { of three consecutive } 3 \text {-faces incident with } v, \\ \frac{3}{4} & \text { if } f_{3}(v)=2 \text { and } f_{4}(v)=1, \text { or if } f_{3}(v)=3 \text { and } f \text { is located } \\ \text { in one side of three consecutive } 3 \text {-faces incident with } v, \\ 1 & \text { otherwise. }\end{cases}$
R3.2. If $d(v)=5$, then

R3.3. If $d(v) \geq 6$, then
$a= \begin{cases}\frac{3}{2} & \text { if } f \text { is adjacent to two non-adjacent }\left(3,6,6^{+}\right) \text {-faces at } v \\ \text { and } d(v)=6, \\ \text { if } f \text { is incident with a 3-vertex, } \\ \frac{7}{4} & \text { otherwise. }\end{cases}$
In the following, we will check that $w^{\prime}(x) \geq 0$ for all elements $x \in V(G) \cup F(G)$ to obtain the following obvious contradiction.

$$
0 \leq \sum_{v \in V \cup F} w^{\prime}(x)=\sum_{v \in V \cup F} w(x)=-20 .
$$

First, we consider the final charge of any face $f$. If $d(f) \geq 5$, then it retains its initial charge and it follows that $w^{\prime}(f)=w(f)=2 d(f)-10 \geq 0$. Suppose that $d(f)=4$. Then $w(f)=8-10=-2$. If $\delta(f)=3$, then $w^{\prime}(f)=w(f)+\frac{3}{4}+\frac{1}{2}+\frac{3}{4}=0$ by R2. Otherwise $w^{\prime}(f)=w(f)+4 \times \frac{1}{2}=0$. So $w^{\prime}(f) \geq 0$ if $d(f)=4$.

Suppose that $d(f)=3$. Then $w(f)=6-10=-4$. If $\delta(f)=3$, then $f$ is incident with two $6^{+}$-vertices by $(P 1)$ and it follows that $w^{\prime}(f)=w(f)+2+2=0$ by R3.3. If $\delta(f) \geq 5$, then $f$ receives at least $\frac{3}{2}$ from each of its incident vertices by R3.2 and R3.3, so $w^{\prime}(f) \geq w(f)+3 \times \frac{3}{2}>0$. In the following, we assume that $\delta(f)=4$. Let $f$ be a 3 -face uvwu such that $d(u)=4$. Then $d(v) \geq 5$ and $d(w) \geq 5$ by ( $P 1$ ). According to R3.1, we consider the following three cases.

Case 1. $f$ receives $\frac{1}{2}$ from $u$, that is, $f_{4^{-}}(u)=4$ or $f_{3}(u)=3$ and $f$ is located in the middle of three consecutive 3 -faces incident with $u$.

It suffices to check that $f$ receives at least $\frac{7}{4}$ from each of $v$ and $w$. Thus $w^{\prime}(f) \geq w(f)+\frac{1}{2}+\frac{7}{4}+\frac{7}{4}=0$, a contradiction.

Subcase 1.1. $f_{4^{-}}(u)=4$, that is, $u$ is incident with four faces of degree at most 4. Then $f_{3}(u)=4$ or $f_{3}(u)=1$ by Lemma 4. If $f_{3}(u)=1$, then all faces adjacent to $f$ are $4^{+}$-faces, and it follows from R3.2 and R3.3 that $f$ receives at least $\frac{7}{4}$ from $v, w$ respectively. If $f_{3}(u)=4$, then any 3 -face incident with $u$ must be adjacent to a $5^{+}$-face and it follows from R 3.2 and R 3.3 that $f$ receives at least $\frac{7}{4}$ from $v, w$ respectively.

Subcase 1.2. $f_{3}(u)=3$ and $f$ is located in the middle of three consecutive 3 -faces incident with $u$. If $d(v) \geq 6$, then two faces adjacent to $f$ at $v$ are not $\left(3,6,6^{+}\right)$-faces (since $d(u)=4$ and $u v$ is incident with two $\left(4,5^{+}, 6^{+}\right)$-faces) and it follows from R3.3 that $f$ receives at least $\frac{7}{4}$ from $v$. Suppose that $d(v)=5$. Let five faces incident with $v$ be $f, f_{1}, \ldots, f_{4}$ in clockwise order, where $u v$ is incident with $f$ and $f_{1}$ (see Figure 2). Then $d\left(f_{4}\right) \geq 5$ since $G$ contains no 6 -cycles with two chords. If $f_{3}(v)=3$, then $f_{4}(v)=0$, and $f$ is not located in the middle of three consecutive 3 -faces incident with $v$ (since $d\left(f_{4}\right) \geq 5$ ), and only one face adjacent to $f$ at $v$ is a $5^{+}$-face (since $d\left(f_{1}\right)=3$ ). So $f$ receives at least $\frac{7}{4}$ from $v$ by R3.2. By symmetry, $f$ receives at least $\frac{7}{4}$ from $w$.


Figure 2. $d(u)=4, f_{3}(u)=3$ and $f$ is located in the middle of three consecutive 3 -faces incident with $u$.

Case 2. $f$ receives $\frac{3}{4}$ from $u$. Then $f_{3}(u)=2$ and $f_{4}(u)=1$, or $f_{3}(v)=3$ and $f$ is located in the one side of these 3 -faces by R3.1. Suppose that $f_{3}(u)=2$ and $f_{4}(u)=1$. Then the induced subgraph of $u$ and its neighbors must be isomorphic to a configuration as Figure 3, where $w=x$ or $w=y$. If $v x$ is incident with two 3 -faces $u v x u$ and $v x x^{\prime} v$, then the 6 -cycle $x x^{\prime} v y z u x$ contains two chords $u v$ and $u y$, a contradiction. If $v x$ is incident with a 4 -face $v x x^{\prime} x^{\prime \prime} v$, then the 6 cycle $x x^{\prime} x^{\prime \prime} v y u x$ contains two chords $u v$ and $x v$, a contradiction, too. So $v x$ is incident with a $5^{+}$-face. By the same argument, $v y$ is incident with a $5^{+}$-face, too. By R3.2 and R3.3, $f$ receives at least $\frac{7}{4}$ from $v$, at least $\frac{3}{2}$ from $w$. So $w^{\prime}(f) \geq w(f)+\frac{3}{4}+\frac{3}{2}+\frac{7}{4}=0$ by R 3 .


Figure 3. $w=x$ or $w=y$.
Suppose that $u$ is incident with three 3 -faces and $f$ is located in the one side of these 3 -faces. Then $u$ is incident with a $5^{+}$-face by Lemma 4 . Without loss of generality, we assume that $u v$ is incident with two 3 -faces. By the similar arguments with Subcase $1.2, v$ sends at least $\frac{7}{4}$ to $f$. So $w^{\prime}(f) \geq w(f)+\frac{3}{4}+\frac{3}{2}+$ $\frac{7}{4}=0$.

Case 3. $f$ receives 1 from $u$. Since $d(v) \geq 5, v$ sends at least $\frac{3}{2}$ to $f$ by R3.2 and R3.3. Similarly, $w$ sends at least $\frac{3}{2}$ to $f$. So $w^{\prime}(f) \geq w(f)+1+\frac{3}{2}+\frac{3}{2}=0$.

Till now, we have checked that $w^{\prime}(f) \geq 0$ for any face $f \in F(G)$. Next, we begin to check the new charge of all vertices of $G$. Let $v$ be a vertex of $G$. If $d(v)=3$, then $w^{\prime}(v) \geq w(v)+2 \times \frac{1}{2}=0$ by R1 since $v$ has exactly two 3 masters. Suppose that $d(v)=4$. If $f_{4^{-}}(v) \leq 2$, then $w^{\prime}(v)=w(v)-2 \times 1=0$ by R3.1. If $f_{4^{-}}(v)=4$, then $w^{\prime}(v)=w(v)-4 \times \frac{1}{2}=0$ by R3.1. If $f_{4^{-}}(v)=3$, then $f_{3}(v)=3$ and $f_{4}(v)=0$, or $f_{4}(v)=1$ and $f_{3}(v)=2$ by Lemma 4. So $w^{\prime}(v) \geq w(v)-\frac{1}{2}-2 \times \frac{3}{4}=0$.

Suppose that $d(v)=5$. Then $w(v)=15-10=5$ and $f_{3}(v) \leq 3$ by Lemma 3. If $f_{3}(v) \leq 2$, then $w^{\prime}(v) \geq w(v)-2 \times \frac{7}{4}-3 \times \frac{1}{2}=0$ by R2 and R3.2. Suppose that $f_{3}(v)=3$. If $f_{4}(v)=1$, then $f_{5^{+}}(v) \geq 1$ and it follows that $w^{\prime}(v) \geq w(v)-3 \times \frac{3}{2}-\frac{1}{2}=0$ by R2 and R3.2. Otherwise $f_{5^{+}}(v)=2$ and it follows that $w^{\prime}(v) \geq w(v)-2 \times \frac{7}{4}-\frac{3}{2}=0$ by R3.2.

Suppose that $d(v)=6$. Then $w(v)=18-10=8$ and $f_{3}(v) \leq\left\lfloor\frac{3}{4} \times 6\right\rfloor=4$ by Lemma 3. It follows from ( $P 2$ ) that it may be the 3 -master of some 3 -vertex $u$, that is, $v$ needs to send at most $\frac{1}{2}$ to its neighbors by R1. If $f_{3}(v) \leq 2$, then $w^{\prime}(v) \geq w(v)-2 \times 2-4 \times \frac{3}{4}-\frac{1}{2}>0$ by R1-R3. If $f_{3}(v)=3$, then $f_{5^{+}}(v) \geq 1$ and it follows that $w^{\prime}(v) \geq w(v)-3 \times 2-2 \times \frac{3}{4}-\frac{1}{2}=0$. Suppose that $f_{3}(v)=4$. Then $f_{4}(v)=0$. If $v$ is incident with at most two $\left(3,6,6^{+}\right)$faces, then $w^{\prime}(v) \geq w(v)-2 \times 2-2 \times \frac{7}{4}-\frac{1}{2}=0$. Otherwise, $v$ is incident with three $\left(3,6,6^{+}\right)$-faces by $(P 2)$ and $(3)$ of the lemma, and $v$ is incident with three consecutive 3 -faces in which the middle 3 -face is incident with two non-adjacent $\left(3,6,6^{+}\right)$-faces. So $w^{\prime}(v) \geq w(v)-3 \times 2-\frac{3}{2}-\frac{1}{2}=0$ by R1 and R3.3.

Suppose that $d(v)=7$. Then $f_{3}(v) \leq 5$ by Lemma 3 . If $f_{3}(v)=5$, then
$f_{4}(v)=0$ and $w^{\prime}(v) \geq w(v)-5 \times 2-\frac{1}{2}>0$. If $f_{3}(v)=4$, then $f_{4}(v) \leq 1$ and $w^{\prime}(v) \geq w(v)-4 \times 2-\frac{3}{4}-\frac{1}{2}>0$. If $f_{3}(v) \leq 3$, then $w^{\prime}(v) \geq w(v)-3 \times 2-4 \times \frac{3}{4}-$ $\frac{1}{2}>0$.

If $d(v) \geq 8$, then $f_{3}(v) \leq\left\lfloor\frac{3 d(v)}{4}\right\rfloor$ by Lemma 3, and it follows that $w^{\prime}(v) \geq$ $w(v)-2 \times\left\lfloor\frac{3 d(v)}{4}\right\rfloor-\frac{3}{4}\left(d(v)-\left\lfloor\frac{3 d(v)}{4}\right\rfloor\right)-\frac{1}{2}=\frac{21(d(v)-8)}{16} \geq 0$.

Hence, we complete the proof of Lemma 5.

Theorem 6. $G$ is edge- $k$-choosable, where $k=\max \{7, \Delta(G)+1\}$.
Proof. Let $G$ be a minimal counterexample to the theorem. Then there is an edge assignment $L$ with $|L(e)| \geq k$ for all $e \in E(G)$, where $k=\max \{7, \Delta(G)+1\}$, such that $G$ is not edge- $L$-colorable. By Lemma 5 , we consider three cases as follows.

Case 1. $G$ contains an edge $u v$ with $d(u)+d(v) \leq \max \{8, \Delta(G)+2\}$. Let $G^{\prime}=G-u v$. Then $G^{\prime}$ has an edge- $L$-coloring $\psi$. Since there exist at most $\max \{6, \Delta(G)\}$ edges adjacent to $u v$ and $|L(u v)| \geq \max \{7, \Delta(G)+1\}$, we can color $u v$ with some color from $L(u v)$ that was not used by $\psi$ on the edges adjacent to $u v$. It is easy to show that any edge- $L$-coloring of $G^{\prime}$ can be extended to an edge- $L$-coloring of $G$. This contradicts the choice of the graph $G$.

Case 2. $G$ contains an even cycle $C=v_{1} v_{2} \cdots v_{2 n} v_{1}$ with $d\left(v_{1}\right)=d\left(v_{3}\right)=$ $\cdots=d\left(v_{2 n-1}\right)=3$. Let $G^{\prime}$ be the subgraph of $G$ obtained by deleting the edges of $C$. Then $G^{\prime}$ has an edge- $L$-coloring $\psi$. We define an edge assignment $L^{\prime}$ of $C$ such that $L^{\prime}(e)=L(e) \backslash\left\{\psi\left(e^{\prime}\right) \mid e^{\prime} \in E\left(G^{\prime}\right)\right.$ is adjacent to $e$ in $\left.G\right\}$ for each $e \in E(C)$. It is easy to see that $L^{\prime}(e) \geq 2$ for each $e \in E(C)$. It is showed in [3] that any even cycle is edge-2-choosable. So $C$ is edge- $L^{\prime}$-colorable and it follows that $G$ is edge- $L$-colorable, a contradiction.

Case 3. $G$ has a 6 -vertex $u$ with five neighbors $v, w, x, y, z$ such that $d(v)=$ $d(y)=3$ and $v w, x y, y z \in E(G)$. Let $v^{\prime} \in N(v) \backslash\{u, w\}$. According to Case 1, we assume that $d\left(v_{1}\right)+d\left(v_{2}\right) \geq \max \{9, \Delta(G)+3\}$ for any edge $v_{1} v_{2} \in E(G)$. Since $d(u)+d(v)=6+3, \Delta(G)=6$ and $d(w)=d(x)=d(z)=d\left(v^{\prime}\right)=6$. Without loss of generality, we consider the worst case that $|L(e)|=7$ for all $e \in E(G)$. By minimality of $G, G^{\prime}=G-\{y, v\}$ has an edge- $L$-coloring $\psi$. For each $e \in E(G)$, let $L^{\prime}(e)=L(e) \backslash\left\{\psi\left(e^{\prime}\right) \mid e^{\prime} \in E\left(G^{\prime}\right)\right.$ is adjacent to $e$ in $\left.G\right\}$.

If $\left|L^{\prime}(x y)\right| \geq 3$, then we can color $v v^{\prime}, v w, v u, y u, y z$ and $x y$ successively to obtain an edge- $L$-coloring of $G$, a contradiction. So $\left|L^{\prime}(x y)\right|=2$. By the same argument, we have $\left|L^{\prime}(y z)\right|=\left|L^{\prime}(v w)\right|=\left|L^{\prime}\left(v v^{\prime}\right)\right|=2$. If $\left|L^{\prime}(u y)\right| \geq 4$, then we can color $v v^{\prime}, v w, v u, x y, y z$ and $u y$ successively, a contradiction. So $\left|L^{\prime}(u y)\right|=3$. By the same argument, we have $\left|L^{\prime}(u v)\right|=3$. Hence $\left|L^{\prime}(x y)\right|=$ $\left|L^{\prime}(y z)\right|=\left|L^{\prime}(v w)\right|=\left|L^{\prime}\left(v v^{\prime}\right)\right|=2$ and $\left|L^{\prime}(u y)\right|=\left|L^{\prime}(u v)\right|=3$.

If $L^{\prime}(x y) \neq L^{\prime}(y z)$, without loss of generality, we assume that there is a color $a \in L^{\prime}(x y) \backslash L^{\prime}(y z)$, then we color $x y$ with $a$ firstly, and then color $v v^{\prime}, v w, v u, y u$ and $y z$ successively, a contradiction. So $L^{\prime}(x y)=L^{\prime}(y z)$. By the same argument, we have $L^{\prime}(v w)=L^{\prime}\left(v v^{\prime}\right)$.

Without loss of generality, we assume that $\psi(u x)=1, \psi(u z)=2, \psi(u w)=3$, $L^{\prime}(x y)=L^{\prime}(y z)=\{\alpha, \beta\}$. Then $1 \in L(x y)$ and $2 \in L(y z)$ for otherwise $\left|L^{\prime}(x y)\right| \geq$ 3 or $\left|L^{\prime}(y z)\right| \geq 3$. Thus the colors $1,2, \alpha, \beta$ are all distinct. At the same time, we have that $L^{\prime}(u x) \subseteq\{1,2,3\}$ for otherwise we can recolor $u x$ with a color in $L^{\prime}(u x) \backslash\{1,2,3\}$, color $x y$ with 1 , and color $v v^{\prime}, v w, v u, y u$ and $y z$ successively to obtain an edge- $L$-coloring of $G$, a contradiction. By the same argument, we have $L^{\prime}(u z) \subseteq\{1,2,3\}$ and $L^{\prime}(u w) \subseteq\{1,2,3\}$. So $L^{\prime}(u x) \cup L^{\prime}(u z) \cup L^{\prime}(u w)=\{1,2,3\}$.

Now if $1 \in L^{\prime}(u z)$ and $2 \in L^{\prime}(u x)$, that is, $\{1,2\} \subseteq L^{\prime}(u z) \cap L^{\prime}(u x)$, then we recolor $u x$ with 2 , and $u z$ with 1 to obtain a contradiction. So $\{1,2\} \nsubseteq L^{\prime}(u z) \cap$ $L^{\prime}(u x)$. Similarly, we have $\{1,3\} \nsubseteq L^{\prime}(u x) \cap L^{\prime}(u w)$ and $\{2,3\} \nsubseteq L^{\prime}(u z) \cap L^{\prime}(u w)$. These three results imply that $\left|L^{\prime}(u x)\right|=\left|L^{\prime}(u z)\right|=\left|L^{\prime}(u w)\right|=2$. Let $a \in$ $L^{\prime}(u x) \backslash\{1\}, b \in L^{\prime}(u z) \backslash\{2\}$ and $c \in L^{\prime}(u w) \backslash\{3\}$. Then $\{a, b, c\}=\{1,2,3\}$. Thus we recolor $u x$ with $a, u z$ with $b$ and $u w$ with $c$ to obtain a final contradiction.

This completes the proof of Theorem 6.
According to the theorem, it is easy to obtain the following corollary.
Corollary 7. If $\Delta(G) \geq 6$, then $\chi_{l i s t}^{\prime}(G) \leq \Delta(G)+1$.
The following result is about edge- $\Delta$-choosable of embedded planar graphs without 6 -cycles with two chords.

Theorem 8. $G$ is edge- $k$-choosable if $k=\max \{9, \Delta(G)\}$.
This theorem implies that if $G$ is a planar graph $G$ with $\Delta(G) \geq 9$ and every 6 -cycle of $G$ contains at most one chord, then $G$ is edge- $\Delta$-choosable.

Proof. Suppose that there is an edge assignment $L$ with $|L(e)| \geq k$ for all $e \in$ $E(G)$ such that $G$ is not edge- $L$-colorable, but all subgraphs of $G$ are edge- $L$ colorable.

Lemma 9 [4]. The graph $G$ has the following properties.
(1) $G$ is connected and $\delta(G) \geq 2$.
(2) $G$ contains no edges uv with $d(u)+d(v) \leq 10$.
(3) $G$ contains no 2-alternating cycles, that is, $G$ does not contain an even cycle $C=v_{1} v_{2} \cdots v_{2 n} v_{1}$ with $d\left(v_{1}\right)=d\left(v_{3}\right)=\cdots=d\left(v_{2 n-1}\right)=2$.
Suppose $G_{2}$ be the subgraph induced by the edges incident with the 2-vertices of $G$. By Lemma $9(2)$, any two 2 -vertices are not adjacent in $G$, so $G_{2}$ does not contain any odd cycle. By Lemma $9(3), G_{2}$ contains no even cycle. So $G_{2}$ is a
forest. It follows that $G_{2}$ contains a matching $M$ such that all 2 -vertices in $G_{2}$ are saturated. If $u v \in M$ and $d(u)=2$, then $v$ is called the 2-master of $u$. It is easy to see that each 2 -vertex has one exactly 2 -master and each $9^{+}$-vertex can be the 2 -master of at most one 2 -vertex.

Lemma 10 [21]. Let $X=\left\{x \in V(G) \mid d_{G}(x) \leq 3\right\}$ and $Y=\bigcup_{x \in X} N(x)$. If $X \neq \emptyset$, then there exists a bipartite subgraph $M^{\prime}$ of $G$ with partite sets $X$ and $Y$ such that $d_{M^{\prime}}(x)=1$ for any $x \in X$ and $d_{M^{\prime}}(y) \leq 2$ for any $y \in Y$. Here, we call $w$ the 3 -master of $u$ if $u w \in M^{\prime}$ and $u \in X$.

Now we use the method of redistribution of charge in order to obtain a contradiction. We assign an "initial charge" $c(x)$ to each element $x \in V(G) \cup$ $F(G)$, where $c(x)=3 d(x)-10$ if $x \in V(G)$ and $c(x)=2 d(x)-10$ if $x \in F(G)$. Then

$$
\begin{equation*}
\sum_{x \in V(G) \cup F(G)} c(x)=\sum_{v \in V(G)}(3 d(v)-10)+\sum_{f \in F(G)}(2 d(f)-10)<0 . \tag{1}
\end{equation*}
$$

Our discharging rules are defined as follows.
R1. Let $v$ be a 2 -vertex. If $v$ is incident with a 3 -face and a $6^{+}$-face $f$, then $v$ receives 2 from $f$ and 2 from its 2 -master. Otherwise, $v$ receives 2 from its 2 -master and 2 from its 3 -master.

R2. Every 3 -vertex $v$ receives 1 from its 3 -master.
R3. Let $f$ be a 3 -face and $v$ be a $4^{+}$-vertex incident with $f$. Then $f$ receives $a$ from $v$, where

$$
a= \begin{cases}\frac{1}{2} & \text { if } d(v)=4, \\ \frac{3}{2} & \text { if } 5 \leq d(v) \leq 6, \\ \frac{7}{4} & \text { if } d(v)=7, \\ 2 & \text { if } d(v) \geq 8\end{cases}
$$

R4. Let $f$ be a 4 -face incident with a $4^{+}$-vertex $v$. Then $f$ receives $a$ from $v$, where

$$
a= \begin{cases}\frac{1}{2} & \text { if } 4 \leq d(v) \leq 5 \\ \frac{3}{4} & \text { if } 6 \leq d(v) \leq 7 \\ 1 & \text { if } 8 \leq d(v)\end{cases}
$$

Let $c^{\prime}(x)$ be the final charge on $x \in V(G) \cup F(G)$. Then $\sum_{x \in V(G) \cup F(G)} c^{\prime}(x)=$ $\sum_{x \in V(G) \cup F(G)} c(x)<0$. In the following, we will check that $c^{\prime}(x) \geq 0$ for all $x \in V(G) \cup F(G)$ to get a contradiction.

Let $f$ be a face of $G$. If $d(f) \geq 6$, then $f$ is incident with at most $(d(f)-5)$ 2 -vertices each of which is incident with a 3 -face, and it follows that $c^{\prime}(f) \geq$ $c(f)-2(d(f)-5)=0$. If $d(f)=5$, then $f$ retains its initial charge and we have
$c^{\prime}(f)=c(f)=2 d(f)-10 \geq 0$. Suppose that $d(f)=4$. If $\delta(f) \leq 3$, then $f$ is incident with at least two $8^{+}$-vertices by Lemma $9(2)$ and it follows from R4 that $c^{\prime}(f) \geq c(f)+2 \times 1=0$. Otherwise $c^{\prime}(f) \geq c(f)+2 \times \frac{1}{2}+2 \times \frac{3}{4}>0$. Suppose that $d(f)=3$. If $\delta(f) \leq 3$, then $f$ is incident with two $8^{+}$-vertices by Lemma $9(2)$ and it follows from R 3 that $c^{\prime}(f)=c(f)+2+2=0$. If $\delta(f)=4$, then $f$ is incident with two $7^{+}$-vertices by Lemma $9(2)$. Note that any 4 -vertex sends at least $\frac{1}{2}$ to each of its incident 3 -face. So $c^{\prime}(f) \geq c(f)+\frac{1}{2}+2 \times \frac{7}{4}=0$. If $\delta(f) \geq 5$, then $c^{\prime}(f) \geq c(f)+3 \times \frac{3}{2}>0$. So $c^{\prime}(f) \geq 0$ if $d(f)=3$.

Let $v$ be a vertex of $G$. If $d(v)=2$, then $c^{\prime}(v)=c(v)+2+2=0$ by R1. If $d(v)=3$, then $c^{\prime}(v)=c(v)+1=0$ by R2. If $d(v)=4$, then $c^{\prime}(v) \geq c(v)-\frac{1}{2} \times 4=0$ by R3 and R4. Suppose that $d(v)=5$. Then $c(v)=15-10=5$ and $f_{3}(v) \leq 3$ by Lemma 3. If $f_{3}(v)=3$, then $f_{4}(v) \leq 1$ and it follows from R3 and R4 that $c^{\prime}(v) \geq c(v)-3 \times \frac{3}{2}-1 \times \frac{1}{2}=0$. If $f_{3}(v) \leq 2$, then $c^{\prime}(v) \geq c(v)-2 \times \frac{3}{2}-$ $3 \times \frac{1}{2} \geq 0$ by R 3 and R4. If $d(v)=6$, then $f_{3}(v) \leq 4$ by Lemma 3 and we have $c^{\prime}(v) \geq c(v)-4 \times \frac{3}{2}-2 \times \frac{3}{4}>0$. If $d(v)=7$, then $f_{3}(v) \leq 5$ and we have $c^{\prime}(v) \geq c(v)-5 \times \frac{7}{4}-2 \times \frac{3}{4}>0$. Suppose that $d(v)=8$. Then $f_{3}(v) \leq 6$ by Lemma 3 , and it may be the 3 -master of two 3 -vertices by Lemma 10 . If $f_{3}(v)=6$, then $f_{4}(v)=0$ and it follows that $c^{\prime}(v) \geq c(v)-6 \times 2-2=0$. If $f_{3}(v)=5$, then $f_{4}(v) \leq 1$ and it follows that $c^{\prime}(v) \geq c(v)-5 \times 2-1-2>0$. If $f_{3}(v) \leq 4$, then $c^{\prime}(f) \geq c(v)-4 \times 2-4 \times 1-2=0$ by R3 and R4. So $c^{\prime}(v) \geq 0$ if $d(v)=8$.

Now we assume that $d(v) \geq 9$. By Lemmas 9 and $10, v$ may be the 3 master of two $3^{-}$-vertices and the 2 -master of a 2 -vertex, that is, $v$ sends at most 5 to its incident $3^{-}$-vertices. Suppose that $d(v)=9$. Then $f_{3}(v) \leq 6$. If $f_{3}(v) \leq 3$, then $c^{\prime}(v) \geq c(v)-3 \times 2-6 \times 1-5=0$. If $f_{3}(v)=4$, then $f_{4}(v) \leq 4$ and $c^{\prime}(v) \geq c(v)-4 \times 2-4 \times 1-5=0$. If $f_{3}(v)=5$, then $f_{4}(v) \leq 2$ and $c^{\prime}(v) \geq c(v)-5 \times 2-2 \times 1-5=0$. For $f_{3}(v)=6$, we have $f_{4}(v) \leq 1$. If $f_{4}(v)=0$, then $c^{\prime}(v) \geq c(v)-6 \times 2-5=0$. Otherwise, $v$ and its neighbors must induce a configuration isomorphic to Figure 4. Thus, if $d(w)=2$ or $d(x)=2$, then $f_{1}$ is a $6^{+}$-face. If $d(y)=2$ or $d(z)=2$, then $f_{2}$ is a $6^{+}$-face. By R1, $v$ sends at most 2 to its adjacent 2 -vertices. By R2, $v$ sends at most $2 \times 1$ to its adjacent 3 -vertices. So $c^{\prime}(v) \geq c(v)-6 \times 2-1-4=0$.

Suppose that $d(v)=10$. Then $f_{3}(v) \leq 7$. If $f_{3}(v)=7$, then $f_{4}(v) \leq 1$ and it follows that $c^{\prime}(v) \geq c(v)-7 \times 2-1-5=0$. If $f_{3}(v)=6$, then $f_{4}(v) \leq 2$ and it follows that $c^{\prime}(v) \geq c(v)-6 \times 2-2 \times 1-5>0$. If $f_{3}(v) \leq 5$, then $c^{\prime}(v) \geq$ $c(v)-5 \times 2-5 \times 1-5=0$. Suppose that $d(v)=11$. Then $c(v)=3 \times 11-10=22$ and $f_{3}(v) \leq 8$. If $7 \leq f_{3}(v) \leq 8$, then $f_{4}(v) \leq 1$ and it follows that $c^{\prime}(v) \geq$ $22-8 \times 2-1-5=0$. If $f_{3}(v) \leq 6$, then $c^{\prime}(v) \geq 22-6 \times 2-5 \times 1-5=0$. If $d(v) \geq 12$, then $c^{\prime}(v) \geq c(v)-\left\lfloor\frac{3 d(v)}{4}\right\rfloor \times 2-\left(d(v)-\left\lfloor\frac{3 d(v)}{4}\right\rfloor\right) \times 1-5=2 d(v)-\left\lfloor\frac{3 d(v)}{4}\right\rfloor-15 \geq 0$.

Till now, we have checked that $c^{\prime}(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This contradiction completes the proof of Theorem 8.


Figure 4. $d(v)=9, f_{3}(v)=6$ and $f_{4}(v)=1$.

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