Discussiones Mathematicae
Graph Theory 41 (2021) 199–211
doi:10.7151/dmgt.2183

LIST EDGE COLORING OF PLANAR GRAPHS WITHOUT 6-CYCLES WITH TWO CHORDS

Linna Hu [a], Lei Sun [b]

and

Jian-Liang Wu [a,1]

[a] *School of Mathematics, Shandong University, Jinan, 250100, China*
[b] *School of Mathematics, Shandong Normal University, Jinan, 250358, China*

e-mail: jlwu@sdu.edu.cn.

Abstract

A graph G is edge-L-colorable if for a given edge assignment $L = \{L(e) : e \in E(G)\}$, there exists a proper edge-coloring φ of G such that $\varphi(e) \in L(e)$ for all $e \in E(G)$. If G is edge-L-colorable for every edge assignment L such that $|L(e)| \geq k$ for all $e \in E(G)$, then G is said to be edge-k-choosable. In this paper, we prove that if G is a planar graph without 6-cycles with two chords, then G is edge-k-choosable, where $k = \max\{7, \Delta(G) + 1\}$, and is edge-$t$-choosable, where $t = \max\{9, \Delta(G)\}$.

Keywords: planar graph, edge choosable, list edge chromatic number, chord.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

Graphs considered in this paper are finite, simple and undirected. The terminologies and notations used but undefined in this paper can be found in [2]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. A cycle C of length k is called a *k-cycle* in the graph G. If $xy \in E(G) \backslash E(C)$ and $x, y \in V(C)$, xy is called to be a *chord* of C in the graph G.

[1] Corresponding author.

An *edge coloring* of a graph G is a mapping φ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ for some positive integer k. An edge coloring is called *proper* if every two adjacent edges receive different colors. The *edge chromatic number* $\chi'(G)$ is the smallest integer k such that G has a proper edge-coloring into the set $\{1, 2, \ldots, k\}$.

We say that L is an *edge assignment* for the graph G if it assigns a list $L(e)$ of possible colors to each edge e of G. If G has a proper edge-coloring φ such that $\varphi(e) \in L(e)$ for each edge e of G, then we say that G is *edge-L-colorable* or φ is an *edge-L-coloring* of G. The graph G is *edge-k-choosable* if it is edge-L-colorable for every edge assignment L satisfying $|L(e)| \geq k$ for all $e \in E(G)$. The *list edge chromatic number* $\chi'_{list}(G)$ of G is the smallest k such that G is edge-k-choosable.

On the list edge coloring of a graph, there is a celebrated conjecture known as the list edge coloring conjecture, which was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris (see [8, 13]).

Conjecture 1 [9]**.** *If G is a multigraph, then $\chi'_{list}(G) = \chi'(G)$.*

The conjecture has been proved for a few classes of graphs, such as graphs with $\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [4], outerplanar graphs [19], bipartite multigraphs [4, 7], complete graphs of odd order [9]. Vizing [15] proposed a weaker conjecture than Conjecture 1.

Conjecture 2 [9]**.** *Every graph G is edge-$(\Delta(G) + 1)$-choosable.*

Harris [10] showed that $\chi'_{list}(G) \leq 2\Delta(G) - 2$ if G is a graph with $\Delta(G) \geq 3$. This implies Conjecture 2 for the case $\Delta(G) = 3$. Juvan *et al.* [14] settled the case for $\Delta(G) = 4$ in 1999. And there are some other special cases of Conjecture 2 which have been confirmed, such as complete graphs [8], graphs with girth at least $8\Delta(G)(ln\Delta(G)(G) + 1.1)$ [15], planar graphs with $\Delta(G) \geq 8$ [1], and planar graphs with $\Delta(G) \neq 5$ and without intersecting 3-cycles [20]. Suppose that G is a planar graph without k-cycles for some fixed integer $3 \leq k \leq 6$. Then it was proved that Conjecture 2 holds if G satisfies one of the four following conditions:

(i) either $k = 3$ or $k = 4$ and $\Delta(G) \neq 5$ [22],

(ii) $k = 4$ [17],

(iii) $k = 5$ [20],

(iv) $k = 6$ and $\Delta(G) \neq 5$ [18].

Other related known results on this topic can be found in [5, 11, 12, 16].

Cai [6] proved that if G is a planar graph without chordal 6-cycles, then G is edge-k-choosable, where $k = \max\{8, \Delta(G) + 1\}$. In this paper, we will strengthen this result and obtain that if G is a planar graph and each 6-cycle of G contains at most one chord, then $\chi'_{list}(G) \leq \max\{7, \Delta(G)+1\}$ and $\chi'_{list}(G) \leq \max\{9, \Delta(G)\}$.

2. Main Results and Their Proofs

In the section, we always assume that all graphs are planar graphs that have been embedded in the plane and G is a planar graph without 6-cycles with two chords. We use $d_G(x)$, or simply $d(x)$, to denote the degree of a vertex x in G. For $f \in F(G)$, if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk, then we write $f = u_1u_2 \cdots u_nu_1$. The degree of a f, denoted by $d(f)$, is the number of edges incident with f, where each cut-edge is counted twice. We denote by $\delta(f)$ the minimum degree of vertices incident with the face f. A vertex (face) x is called to be a k-vertex (k-face), k^+-vertex (k^+-face) and k^--vertex (k^--face), if $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, respectively. $f_i(v)$ is the number of i-faces incident with v for each $v \in V(G)$.

First, we give some properties on G.

Lemma 3. *If v is a 5^+-vertex of G, then $f_3(v) \leq \left\lfloor \frac{3}{4}d(v) \right\rfloor$.*

Proof. Since G contains no 6-cycles with two chords, v is not incident with four consecutive 3-faces. So $f_3(v) \leq \left\lfloor \frac{3}{4}d(v) \right\rfloor$. ∎

Lemma 4. *Let u be a 4-vertex of G.*

(1) *If $f_3(u) = 3$, then $f_4(u) = 0$, that is, u is incident with a 5^+-face.*

(2) *If $f_3(u) = 2$, then $f_4(u) \leq 1$.*

Proof. Let neighbors of u be u_1, u_2, u_3, u_4 and faces incident with u be f_1, f_2, f_3, f_4 in the clockwise order, where f_1 is incident with u_1, u_2.

(1) Without loss of generality, we assume that f_1, f_2, f_3 are 3-faces. If f_4 is a 4-face uu_1vu_4u, then the 6-cycle $uu_2u_3u_4vu_1u$ contains two chords uu_3 and uu_4, a contradiction. So $d(f_4) \geq 5$, that is, $f_{5^+}(u) = 1$.

(2) Suppose that two 3-faces incident with u are not adjacent, without loss of generality, we assume that f_1, f_3 are 3-faces. If f_2 is a 4-face uu_2vu_3u, then the 6-cycle $uu_1u_2vu_3u_4u$ contains two chords uu_2 and uu_3, a contradiction. So $d(f_2) \geq 5$. By the same argument, we have $d(f_4) \geq 5$.

Suppose that two 3-faces incident with u are adjacent, without loss of generality, we assume that f_1, f_2 are 3-faces. If f_3 is a 4-face uu_3vu_4u, then we must have $v = u_1$. Since $d(u) = 4$, $d(u_4) \geq 5$. Thus if f_4 is a 4-face uu_1wu_4u, then we also have $w = u_3$, it is impossible. So $d(f_4) \geq 5$. By the same argument, if $d(f_4) = 4$, then $d(f_3) \geq 5$. Hence $f_4(u) \leq 1$. ∎

Lemma 5. *G satisfies at least one of the following conditions.*

(1) *G has an edge uv with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$.*

(2) *G has an even cycle $C = v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$.*

(3) *G has a 6-vertex u with five neighbors v, w, x, y, z such that $d(v) = d(y) = 3$ and $vw, xy, yz \in E(G)$ (see Figure 1).*

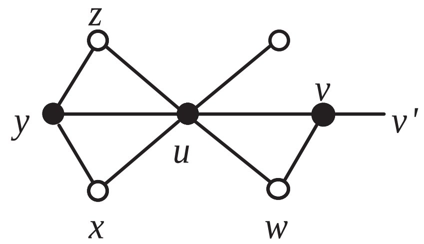

Figure 1. The subgraph for Lemma 5(3).

Proof. Let G be a minimal counterexample to the lemma. It is easy to check that G is connected. By the choice of G, we have the following observations.

(P1) For any edge uv, $d(u) + d(v) \geq \max\{9, \Delta(G) + 3\}$ by (1). Then $\delta(G) \geq 3$ and all neighbors of a i-vertex must be $(9 - i)^+$-vertices, where $i = 3, 4$ or 5.

(P2) Let G_3 be the subgraph induced by the edges incident with 3-vertices of G. Then G_3 is a forest.

By (P1), every two 3-vertices are not adjacent, and it follows that G_3 is a bipartite subgraph. By (2), G_3 contains no even cycles. So G_3 is a forest and (P2) holds. Let V_1 be the set of 3-vertices of G. Thus for any component of G_3, we select a vertex $u \notin V_1$ as a root of the tree. Then every 3-vertex has exactly two children. If $uv \in E(G_3)$, $u \in V_1$ and v is a child of u, then v is called a 3-*master* of u. Note that each 3-vertex has exactly two 3-masters and each vertex of degree at least 6 can be the 3-master of at most one 3-vertex.

According to the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ of a planar graph G, we have

$$\sum_{v\in V(G)} (3d(v)-10) + \sum_{f\in F(G)} (2d(f)-10) = -10(|V(G)|-|E(G)|+|F(G)|) = -20 < 0.$$

Now we define the initial weight function on $V(G) \cup F(G)$ by letting $w(x) = 3d(x) - 10$ for any $x \in V(G)$ and $w(x) = 2d(x) - 10$ for any $x \in F(G)$. Thus the total sum of weights is the negative number -20. We use the following rules to redistribute the initial charge that leads to a new charge $w'(x)$.

R1. Every 3-vertex v receives $\frac{1}{2}$ from each of its 3-masters.

R2. Let $f = uu'vv'u$ be a 4-face in G with $d(u) \leq \min\{d(u'), d(v), d(v')\}$. If $d(u) \geq 4$, then f receives $\frac{1}{2}$ from each of its incident vertices. Otherwise, f receives nothing from u, receives $\frac{1}{2}$ from v, $\frac{3}{4}$ from u' and $\frac{3}{4}$ from v'.

R3. Let f be a 3-face incident with a 4^+-vertex v. Then f receives a from v.

R3.1. If $d(v) = 4$, then

$$a = \begin{cases} \frac{1}{2} & \text{if } f_{4^-}(v) = 4 \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located in the middle} \\ & \text{of three consecutive 3-faces incident with } v, \\ \frac{3}{4} & \text{if } f_3(v) = 2 \text{ and } f_4(v) = 1, \text{ or if } f_3(v) = 3 \text{ and } f \text{ is located} \\ & \text{in one side of three consecutive 3-faces incident with } v, \\ 1 & \text{otherwise.} \end{cases}$$

R3.2. If $d(v) = 5$, then

$$a = \begin{cases} \frac{3}{2} & \text{if } f_3(v) = 3 \text{ and one of the following conditions holds:} \\ & \text{(i) } f_4(v) = 1, \\ & \text{(ii) } f_4(v) = 0 \text{ and } f \text{ is located in the middle of three} \\ & \quad \text{consecutive 3-faces incident with } v, \\ & \text{(iii) two faces adjacent to } f \text{ at } v \text{ are } 5^+\text{-faces.} \\ \frac{7}{4} & \text{otherwise.} \end{cases}$$

R3.3. If $d(v) \geq 6$, then

$$a = \begin{cases} \frac{3}{2} & \text{if } f \text{ is adjacent to two non-adjacent } (3, 6, 6^+)\text{-faces at } v \\ & \text{and } d(v) = 6, \\ & \text{if } f \text{ is incident with a 3-vertex,} \\ \frac{7}{4} & \text{otherwise.} \end{cases}$$

In the following, we will check that $w'(x) \geq 0$ for all elements $x \in V(G) \cup F(G)$ to obtain the following obvious contradiction.

$$0 \leq \sum_{v \in V \cup F} w'(x) = \sum_{v \in V \cup F} w(x) = -20.$$

First, we consider the final charge of any face f. If $d(f) \geq 5$, then it retains its initial charge and it follows that $w'(f) = w(f) = 2d(f) - 10 \geq 0$. Suppose that $d(f) = 4$. Then $w(f) = 8 - 10 = -2$. If $\delta(f) = 3$, then $w'(f) = w(f) + \frac{3}{4} + \frac{1}{2} + \frac{3}{4} = 0$ by R2. Otherwise $w'(f) = w(f) + 4 \times \frac{1}{2} = 0$. So $w'(f) \geq 0$ if $d(f) = 4$.

Suppose that $d(f) = 3$. Then $w(f) = 6 - 10 = -4$. If $\delta(f) = 3$, then f is incident with two 6^+-vertices by $(P1)$ and it follows that $w'(f) = w(f) + 2 + 2 = 0$ by R3.3. If $\delta(f) \geq 5$, then f receives at least $\frac{3}{2}$ from each of its incident vertices by R3.2 and R3.3, so $w'(f) \geq w(f) + 3 \times \frac{3}{2} > 0$. In the following, we assume that $\delta(f) = 4$. Let f be a 3-face $uvwu$ such that $d(u) = 4$. Then $d(v) \geq 5$ and $d(w) \geq 5$ by $(P1)$. According to R3.1, we consider the following three cases.

Case 1. f receives $\frac{1}{2}$ from u, that is, $f_{4^-}(u) = 4$ or $f_3(u) = 3$ and f is located in the middle of three consecutive 3-faces incident with u.

It suffices to check that f receives at least $\frac{7}{4}$ from each of v and w. Thus $w'(f) \geq w(f) + \frac{1}{2} + \frac{7}{4} + \frac{7}{4} = 0$, a contradiction.

Subcase 1.1. $f_{4^-}(u) = 4$, that is, u is incident with four faces of degree at most 4. Then $f_3(u) = 4$ or $f_3(u) = 1$ by Lemma 4. If $f_3(u) = 1$, then all faces adjacent to f are 4^+-faces, and it follows from R3.2 and R3.3 that f receives at least $\frac{7}{4}$ from v, w respectively. If $f_3(u) = 4$, then any 3-face incident with u must be adjacent to a 5^+-face and it follows from R3.2 and R3.3 that f receives at least $\frac{7}{4}$ from v, w respectively.

Subcase 1.2. $f_3(u) = 3$ and f is located in the middle of three consecutive 3-faces incident with u. If $d(v) \geq 6$, then two faces adjacent to f at v are not $(3, 6, 6^+)$-faces (since $d(u) = 4$ and uv is incident with two $(4, 5^+, 6^+)$-faces) and it follows from R3.3 that f receives at least $\frac{7}{4}$ from v. Suppose that $d(v) = 5$. Let five faces incident with v be $f, f_1, \ldots, f_4$ in clockwise order, where uv is incident with f and f_1(see Figure 2). Then $d(f_4) \geq 5$ since G contains no 6-cycles with two chords. If $f_3(v) = 3$, then $f_4(v) = 0$, and f is not located in the middle of three consecutive 3-faces incident with v(since $d(f_4) \geq 5$), and only one face adjacent to f at v is a 5^+-face (since $d(f_1) = 3$). So f receives at least $\frac{7}{4}$ from v by R3.2. By symmetry, f receives at least $\frac{7}{4}$ from w.

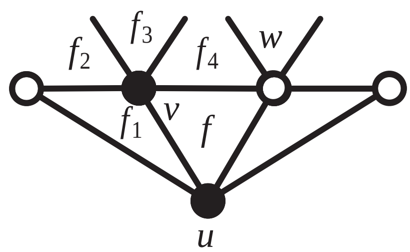

Figure 2. $d(u) = 4$, $f_3(u) = 3$ and f is located in the middle of three consecutive 3-faces incident with u.

Case 2. f receives $\frac{3}{4}$ from u. Then $f_3(u) = 2$ and $f_4(u) = 1$, or $f_3(v) = 3$ and f is located in the one side of these 3-faces by R3.1. Suppose that $f_3(u) = 2$ and $f_4(u) = 1$. Then the induced subgraph of u and its neighbors must be isomorphic to a configuration as Figure 3, where $w = x$ or $w = y$. If vx is incident with two 3-faces $uvxu$ and $vxx'v$, then the 6-cycle $xx'vyzux$ contains two chords uv and uy, a contradiction. If vx is incident with a 4-face $vxx'x''v$, then the 6-cycle $xx'x''vyux$ contains two chords uv and xv, a contradiction, too. So vx is incident with a 5^+-face. By the same argument, vy is incident with a 5^+-face, too. By R3.2 and R3.3, f receives at least $\frac{7}{4}$ from v, at least $\frac{3}{2}$ from w. So $w'(f) \geq w(f) + \frac{3}{4} + \frac{3}{2} + \frac{7}{4} = 0$ by R3.

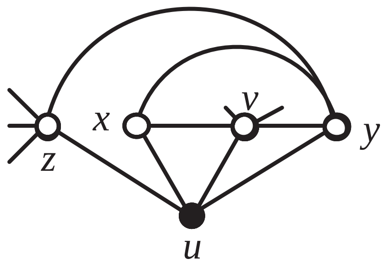

Figure 3. $w = x$ or $w = y$.

Suppose that u is incident with three 3-faces and f is located in the one side of these 3-faces. Then u is incident with a 5^+-face by Lemma 4. Without loss of generality, we assume that uv is incident with two 3-faces. By the similar arguments with Subcase 1.2, v sends at least $\frac{7}{4}$ to f. So $w'(f) \geq w(f) + \frac{3}{4} + \frac{3}{2} + \frac{7}{4} = 0$.

Case 3. f receives 1 from u. Since $d(v) \geq 5$, v sends at least $\frac{3}{2}$ to f by R3.2 and R3.3. Similarly, w sends at least $\frac{3}{2}$ to f. So $w'(f) \geq w(f) + 1 + \frac{3}{2} + \frac{3}{2} = 0$.

Till now, we have checked that $w'(f) \geq 0$ for any face $f \in F(G)$. Next, we begin to check the new charge of all vertices of G. Let v be a vertex of G. If $d(v) = 3$, then $w'(v) \geq w(v) + 2 \times \frac{1}{2} = 0$ by R1 since v has exactly two 3-masters. Suppose that $d(v) = 4$. If $f_{4^-}(v) \leq 2$, then $w'(v) = w(v) - 2 \times 1 = 0$ by R3.1. If $f_{4^-}(v) = 4$, then $w'(v) = w(v) - 4 \times \frac{1}{2} = 0$ by R3.1. If $f_{4^-}(v) = 3$, then $f_3(v) = 3$ and $f_4(v) = 0$, or $f_4(v) = 1$ and $f_3(v) = 2$ by Lemma 4. So $w'(v) \geq w(v) - \frac{1}{2} - 2 \times \frac{3}{4} = 0$.

Suppose that $d(v) = 5$. Then $w(v) = 15 - 10 = 5$ and $f_3(v) \leq 3$ by Lemma 3. If $f_3(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times \frac{7}{4} - 3 \times \frac{1}{2} = 0$ by R2 and R3.2. Suppose that $f_3(v) = 3$. If $f_4(v) = 1$, then $f_{5^+}(v) \geq 1$ and it follows that $w'(v) \geq w(v) - 3 \times \frac{3}{2} - \frac{1}{2} = 0$ by R2 and R3.2. Otherwise $f_{5^+}(v) = 2$ and it follows that $w'(v) \geq w(v) - 2 \times \frac{7}{4} - \frac{3}{2} = 0$ by R3.2.

Suppose that $d(v) = 6$. Then $w(v) = 18 - 10 = 8$ and $f_3(v) \leq \lfloor \frac{3}{4} \times 6 \rfloor = 4$ by Lemma 3. It follows from $(P2)$ that it may be the 3-master of some 3-vertex u, that is, v needs to send at most $\frac{1}{2}$ to its neighbors by R1. If $f_3(v) \leq 2$, then $w'(v) \geq w(v) - 2 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0$ by R1–R3. If $f_3(v) = 3$, then $f_{5^+}(v) \geq 1$ and it follows that $w'(v) \geq w(v) - 3 \times 2 - 2 \times \frac{3}{4} - \frac{1}{2} = 0$. Suppose that $f_3(v) = 4$. Then $f_4(v) = 0$. If v is incident with at most two $(3, 6, 6^+)$-faces, then $w'(v) \geq w(v) - 2 \times 2 - 2 \times \frac{7}{4} - \frac{1}{2} = 0$. Otherwise, v is incident with three $(3, 6, 6^+)$-faces by $(P2)$ and (3) of the lemma, and v is incident with three consecutive 3-faces in which the middle 3-face is incident with two non-adjacent $(3, 6, 6^+)$-faces. So $w'(v) \geq w(v) - 3 \times 2 - \frac{3}{2} - \frac{1}{2} = 0$ by R1 and R3.3.

Suppose that $d(v) = 7$. Then $f_3(v) \leq 5$ by Lemma 3. If $f_3(v) = 5$, then

$f_4(v) = 0$ and $w'(v) \geq w(v) - 5 \times 2 - \frac{1}{2} > 0$. If $f_3(v) = 4$, then $f_4(v) \leq 1$ and $w'(v) \geq w(v) - 4 \times 2 - \frac{3}{4} - \frac{1}{2} > 0$. If $f_3(v) \leq 3$, then $w'(v) \geq w(v) - 3 \times 2 - 4 \times \frac{3}{4} - \frac{1}{2} > 0$.

If $d(v) \geq 8$, then $f_3(v) \leq \left\lfloor \frac{3d(v)}{4} \right\rfloor$ by Lemma 3, and it follows that $w'(v) \geq w(v) - 2 \times \left\lfloor \frac{3d(v)}{4} \right\rfloor - \frac{3}{4}\left(d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor\right) - \frac{1}{2} = \frac{21(d(v)-8)}{16} \geq 0$.

Hence, we complete the proof of Lemma 5. ■

Theorem 6. *G is edge-k-choosable, where* $k = \max\{7, \Delta(G) + 1\}$.

Proof. Let G be a minimal counterexample to the theorem. Then there is an edge assignment L with $|L(e)| \geq k$ for all $e \in E(G)$, where $k = \max\{7, \Delta(G)+1\}$, such that G is not edge-L-colorable. By Lemma 5, we consider three cases as follows.

Case 1. G contains an edge uv with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$. Let $G' = G - uv$. Then G' has an edge-L-coloring ψ. Since there exist at most $\max\{6, \Delta(G)\}$ edges adjacent to uv and $|L(uv)| \geq \max\{7, \Delta(G) + 1\}$, we can color uv with some color from $L(uv)$ that was not used by ψ on the edges adjacent to uv. It is easy to show that any edge-L-coloring of G' can be extended to an edge-L-coloring of G. This contradicts the choice of the graph G.

Case 2. G contains an even cycle $C = v_1v_2 \cdots v_{2n}v_1$ with $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 3$. Let G' be the subgraph of G obtained by deleting the edges of C. Then G' has an edge-L-coloring ψ. We define an edge assignment L' of C such that $L'(e) = L(e)\backslash\{\psi(e')|\ e' \in E(G')$ is adjacent to e in $G\}$ for each $e \in E(C)$. It is easy to see that $L'(e) \geq 2$ for each $e \in E(C)$. It is showed in [3] that any even cycle is edge-2-choosable. So C is edge-L'-colorable and it follows that G is edge-L-colorable, a contradiction.

Case 3. G has a 6-vertex u with five neighbors v, w, x, y, z such that $d(v) = d(y) = 3$ and $vw, xy, yz \in E(G)$. Let $v' \in N(v)\backslash\{u, w\}$. According to Case 1, we assume that $d(v_1) + d(v_2) \geq \max\{9, \Delta(G) + 3\}$ for any edge $v_1v_2 \in E(G)$. Since $d(u) + d(v) = 6 + 3$, $\Delta(G) = 6$ and $d(w) = d(x) = d(z) = d(v') = 6$. Without loss of generality, we consider the worst case that $|L(e)| = 7$ for all $e \in E(G)$. By minimality of G, $G' = G - \{y, v\}$ has an edge-L-coloring ψ. For each $e \in E(G)$, let $L'(e) = L(e)\backslash\{\psi(e')|e' \in E(G')$ is adjacent to e in $G\}$.

If $|L'(xy)| \geq 3$, then we can color vv', vw,vu, yu, yz and xy successively to obtain an edge-L-coloring of G, a contradiction. So $|L'(xy)| = 2$. By the same argument, we have $|L'(yz)| = |L'(vw)| = |L'(vv')| = 2$. If $|L'(uy)| \geq 4$, then we can color vv', vw,vu, xy, yz and uy successively, a contradiction. So $|L'(uy)| = 3$. By the same argument, we have $|L'(uv)| = 3$. Hence $|L'(xy)| = |L'(yz)| = |L'(vw)| = |L'(vv')| = 2$ and $|L'(uy)| = |L'(uv)| = 3$.

If $L'(xy) \neq L'(yz)$, without loss of generality, we assume that there is a color $a \in L'(xy) \backslash L'(yz)$, then we color xy with a firstly, and then color vv', vw, vu, yu and yz successively, a contradiction. So $L'(xy) = L'(yz)$. By the same argument, we have $L'(vw) = L'(vv')$.

Without loss of generality, we assume that $\psi(ux) = 1$, $\psi(uz) = 2$, $\psi(uw) = 3$, $L'(xy) = L'(yz) = \{\alpha, \beta\}$. Then $1 \in L(xy)$ and $2 \in L(yz)$ for otherwise $|L'(xy)| \geq 3$ or $|L'(yz)| \geq 3$. Thus the colors $1, 2, \alpha, \beta$ are all distinct. At the same time, we have that $L'(ux) \subseteq \{1, 2, 3\}$ for otherwise we can recolor ux with a color in $L'(ux) \backslash \{1, 2, 3\}$, color xy with 1, and color vv', vw,vu, yu and yz successively to obtain an edge-L-coloring of G, a contradiction. By the same argument, we have $L'(uz) \subseteq \{1, 2, 3\}$ and $L'(uw) \subseteq \{1, 2, 3\}$. So $L'(ux) \cup L'(uz) \cup L'(uw) = \{1, 2, 3\}$.

Now if $1 \in L'(uz)$ and $2 \in L'(ux)$, that is, $\{1, 2\} \subseteq L'(uz) \cap L'(ux)$, then we recolor ux with 2, and uz with 1 to obtain a contradiction. So $\{1, 2\} \not\subseteq L'(uz) \cap L'(ux)$. Similarly, we have $\{1, 3\} \not\subseteq L'(ux) \cap L'(uw)$ and $\{2, 3\} \not\subseteq L'(uz) \cap L'(uw)$. These three results imply that $|L'(ux)| = |L'(uz)| = |L'(uw)| = 2$. Let $a \in L'(ux) \backslash \{1\}$, $b \in L'(uz) \backslash \{2\}$ and $c \in L'(uw) \backslash \{3\}$. Then $\{a, b, c\} = \{1, 2, 3\}$. Thus we recolor ux with a, uz with b and uw with c to obtain a final contradiction.

This completes the proof of Theorem 6. ■

According to the theorem, it is easy to obtain the following corollary.

Corollary 7. *If* $\Delta(G) \geq 6$, *then* $\chi'_{list}(G) \leq \Delta(G) + 1$.

The following result is about edge-Δ-choosable of embedded planar graphs without 6-cycles with two chords.

Theorem 8. *G is edge-k-choosable if* $k = \max\{9, \Delta(G)\}$.

This theorem implies that if G is a planar graph G with $\Delta(G) \geq 9$ and every 6-cycle of G contains at most one chord, then G is edge-Δ-choosable.

Proof. Suppose that there is an edge assignment L with $|L(e)| \geq k$ for all $e \in E(G)$ such that G is not edge-L-colorable, but all subgraphs of G are edge-L-colorable.

Lemma 9 [4]**.** *The graph G has the following properties.*

(1) *G is connected and* $\delta(G) \geq 2$.

(2) *G contains no edges uv with* $d(u) + d(v) \leq 10$.

(3) *G contains no 2-alternating cycles, that is, G does not contain an even cycle* $C = v_1v_2 \cdots v_{2n}v_1$ *with* $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$.

Suppose G_2 be the subgraph induced by the edges incident with the 2-vertices of G. By Lemma 9(2), any two 2-vertices are not adjacent in G, so G_2 does not contain any odd cycle. By Lemma 9(3), G_2 contains no even cycle. So G_2 is a forest. It follows that G_2 contains a matching M such that all 2-vertices in G_2 are saturated. If $uv \in M$ and $d(u) = 2$, then v is called the 2-*master* of u. It is easy to see that each 2-vertex has one exactly 2-master and each 9^+-vertex can be the 2-master of at most one 2-vertex.

Lemma 10 [21]**.** *Let $X = \{x \in V(G) \mid d_G(x) \leq 3\}$ and $Y = \bigcup_{x \in X} N(x)$. If $X \neq \emptyset$, then there exists a bipartite subgraph M' of G with partite sets X and Y such that $d_{M'}(x) = 1$ for any $x \in X$ and $d_{M'}(y) \leq 2$ for any $y \in Y$. Here, we call w the* 3-master *of u if $uw \in M'$ and $u \in X$.*

Now we use the method of redistribution of charge in order to obtain a contradiction. We assign an "initial charge" $c(x)$ to each element $x \in V(G) \cup F(G)$, where $c(x) = 3d(x) - 10$ if $x \in V(G)$ and $c(x) = 2d(x) - 10$ if $x \in F(G)$. Then

$$\sum_{x \in V(G) \cup F(G)} c(x) = \sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) < 0. \tag{1}$$

Our discharging rules are defined as follows.

R1. Let v be a 2-vertex. If v is incident with a 3-face and a 6^+-face f, then v receives 2 from f and 2 from its 2-master. Otherwise, v receives 2 from its 2-master and 2 from its 3-master.

R2. Every 3-vertex v receives 1 from its 3-master.

R3. Let f be a 3-face and v be a 4^+-vertex incident with f. Then f receives a from v, where

$$a = \begin{cases} \frac{1}{2} & \text{if } d(v) = 4, \\ \frac{3}{2} & \text{if } 5 \leq d(v) \leq 6, \\ \frac{7}{4} & \text{if } d(v) = 7, \\ 2 & \text{if } d(v) \geq 8. \end{cases}$$

R4. Let f be a 4-face incident with a 4^+-vertex v. Then f receives a from v, where

$$a = \begin{cases} \frac{1}{2} & \text{if } 4 \leq d(v) \leq 5, \\ \frac{3}{4} & \text{if } 6 \leq d(v) \leq 7, \\ 1 & \text{if } 8 \leq d(v). \end{cases}$$

Let $c'(x)$ be the final charge on $x \in V(G) \cup F(G)$. Then $\sum_{x \in V(G) \cup F(G)} c'(x) = \sum_{x \in V(G) \cup F(G)} c(x) < 0$. In the following, we will check that $c'(x) \geq 0$ for all $x \in V(G) \cup F(G)$ to get a contradiction.

Let f be a face of G. If $d(f) \geq 6$, then f is incident with at most $(d(f) - 5)$ 2-vertices each of which is incident with a 3-face, and it follows that $c'(f) \geq c(f) - 2(d(f) - 5) = 0$. If $d(f) = 5$, then f retains its initial charge and we have

$c'(f) = c(f) = 2d(f) - 10 \geq 0$. Suppose that $d(f) = 4$. If $\delta(f) \leq 3$, then f is incident with at least two 8^+-vertices by Lemma 9(2) and it follows from R4 that $c'(f) \geq c(f) + 2 \times 1 = 0$. Otherwise $c'(f) \geq c(f) + 2 \times \frac{1}{2} + 2 \times \frac{3}{4} > 0$. Suppose that $d(f) = 3$. If $\delta(f) \leq 3$, then f is incident with two 8^+-vertices by Lemma 9(2) and it follows from R3 that $c'(f) = c(f) + 2 + 2 = 0$. If $\delta(f) = 4$, then f is incident with two 7^+-vertices by Lemma 9(2). Note that any 4-vertex sends at least $\frac{1}{2}$ to each of its incident 3-face. So $c'(f) \geq c(f) + \frac{1}{2} + 2 \times \frac{7}{4} = 0$. If $\delta(f) \geq 5$, then $c'(f) \geq c(f) + 3 \times \frac{3}{2} > 0$. So $c'(f) \geq 0$ if $d(f) = 3$.

Let v be a vertex of G. If $d(v) = 2$, then $c'(v) = c(v) + 2 + 2 = 0$ by R1. If $d(v) = 3$, then $c'(v) = c(v) + 1 = 0$ by R2. If $d(v) = 4$, then $c'(v) \geq c(v) - \frac{1}{2} \times 4 = 0$ by R3 and R4. Suppose that $d(v) = 5$. Then $c(v) = 15 - 10 = 5$ and $f_3(v) \leq 3$ by Lemma 3. If $f_3(v) = 3$, then $f_4(v) \leq 1$ and it follows from R3 and R4 that $c'(v) \geq c(v) - 3 \times \frac{3}{2} - 1 \times \frac{1}{2} = 0$. If $f_3(v) \leq 2$, then $c'(v) \geq c(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{2} \geq 0$ by R3 and R4. If $d(v) = 6$, then $f_3(v) \leq 4$ by Lemma 3 and we have $c'(v) \geq c(v) - 4 \times \frac{3}{2} - 2 \times \frac{3}{4} > 0$. If $d(v) = 7$, then $f_3(v) \leq 5$ and we have $c'(v) \geq c(v) - 5 \times \frac{7}{4} - 2 \times \frac{3}{4} > 0$. Suppose that $d(v) = 8$. Then $f_3(v) \leq 6$ by Lemma 3, and it may be the 3-master of two 3-vertices by Lemma 10. If $f_3(v) = 6$, then $f_4(v) = 0$ and it follows that $c'(v) \geq c(v) - 6 \times 2 - 2 = 0$. If $f_3(v) = 5$, then $f_4(v) \leq 1$ and it follows that $c'(v) \geq c(v) - 5 \times 2 - 1 - 2 > 0$. If $f_3(v) \leq 4$, then $c'(f) \geq c(v) - 4 \times 2 - 4 \times 1 - 2 = 0$ by R3 and R4. So $c'(v) \geq 0$ if $d(v) = 8$.

Now we assume that $d(v) \geq 9$. By Lemmas 9 and 10, v may be the 3-master of two 3^--vertices and the 2-master of a 2-vertex, that is, v sends at most 5 to its incident 3^--vertices. Suppose that $d(v) = 9$. Then $f_3(v) \leq 6$. If $f_3(v) \leq 3$, then $c'(v) \geq c(v) - 3 \times 2 - 6 \times 1 - 5 = 0$. If $f_3(v) = 4$, then $f_4(v) \leq 4$ and $c'(v) \geq c(v) - 4 \times 2 - 4 \times 1 - 5 = 0$. If $f_3(v) = 5$, then $f_4(v) \leq 2$ and $c'(v) \geq c(v) - 5 \times 2 - 2 \times 1 - 5 = 0$. For $f_3(v) = 6$, we have $f_4(v) \leq 1$. If $f_4(v) = 0$, then $c'(v) \geq c(v) - 6 \times 2 - 5 = 0$. Otherwise, v and its neighbors must induce a configuration isomorphic to Figure 4. Thus, if $d(w) = 2$ or $d(x) = 2$, then f_1 is a 6^+-face. If $d(y) = 2$ or $d(z) = 2$, then f_2 is a 6^+-face. By R1, v sends at most 2 to its adjacent 2-vertices. By R2, v sends at most 2×1 to its adjacent 3-vertices. So $c'(v) \geq c(v) - 6 \times 2 - 1 - 4 = 0$.

Suppose that $d(v) = 10$. Then $f_3(v) \leq 7$. If $f_3(v) = 7$, then $f_4(v) \leq 1$ and it follows that $c'(v) \geq c(v) - 7 \times 2 - 1 - 5 = 0$. If $f_3(v) = 6$, then $f_4(v) \leq 2$ and it follows that $c'(v) \geq c(v) - 6 \times 2 - 2 \times 1 - 5 > 0$. If $f_3(v) \leq 5$, then $c'(v) \geq c(v) - 5 \times 2 - 5 \times 1 - 5 = 0$. Suppose that $d(v) = 11$. Then $c(v) = 3 \times 11 - 10 = 22$ and $f_3(v) \leq 8$. If $7 \leq f_3(v) \leq 8$, then $f_4(v) \leq 1$ and it follows that $c'(v) \geq 22 - 8 \times 2 - 1 - 5 = 0$. If $f_3(v) \leq 6$, then $c'(v) \geq 22 - 6 \times 2 - 5 \times 1 - 5 = 0$. If $d(v) \geq 12$, then $c'(v) \geq c(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor \times 2 - \left(d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor \right) \times 1 - 5 = 2d(v) - \left\lfloor \frac{3d(v)}{4} \right\rfloor - 15 \geq 0$.

Till now, we have checked that $c'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This contradiction completes the proof of Theorem 8. ■

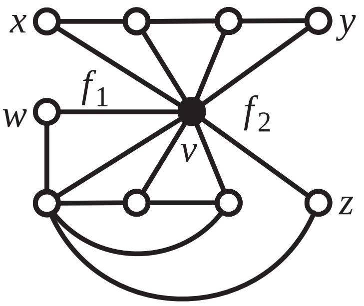

Figure 4. $d(v) = 9$, $f_3(v) = 6$ and $f_4(v) = 1$.

References

[1] M. Bonamy, *Planar graphs with* $\Delta \geq 8$ *are* $(\Delta+1)$*-edge-choosable*, SIAM J. Discrete Math. **29** (2015) 1735–1763.
doi:10.1137/130927449

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York, 1976).

[3] O.V. Borodin, *An extension of Kotzig's theorem and the list edge coloring of planar graphs*, Mat. Zametki **48** (1990) 22–48.

[4] O.V. Borodin, A.V. Kostochka and D.R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B **71** (1997) 184–204.
doi:10.1006/jctb.1997.1780

[5] J.S. Cai, J.F. Hou, X. Zhang and G.Z. Liu, *Edge-choosability of planar graphs without non-induced* 5*-cycles*, Inform. Process. Lett. **109** (2009) 343–346.
doi:10.1016/j.ipl.2008.12.001

[6] J.S. Cai, *List edge coloring of planar graphs without non-induced* 6*-cycles*, Graphs Combin. **31** (2015) 827–832.
doi:10.1007/s00373-014-1420-6

[7] F. Galvin, *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B **63** (1995) 153–158.
doi:10.1006/jctb.1995.1011

[8] R. Häggkvist and J. Janssen, *New bounds on the list-chromatic index of the complete graph and other simple graphs*, Combin. Probab. Comput. **6** (1997) 295–313.
doi:10.1017/S0963548397002927

[9] R. Häggkvist and A. Chetwynd, *Some upper bounds on the total and list chromatic numbers of multigraphs*, J. Graph Theory **16** (1992) 503–516.
doi:10.1002/jgt.3190160510

[10] A.J. Harris, Problems and conjectures in extrema graph theory, Ph.D. Dissertation (Cambridge University, UK, 1984).

[11] J.F. Hou, G.Z. Liu and J.S. Cai, *List edge and list total colorings of planar graphs without 4-cycles*, Theoret. Comput. Sci. **369** (2006) 250–255. doi:10.1016/j.tcs.2006.08.043

[12] J.F. Hou, G.Z. Liu and J.S. Cai, *Edge-choosability of planar graphs without adjacent triangles or without 7-cycles*, Discrete Math. **309** (2009) 77–84. doi:10.1016/j.disc.2007.12.046

[13] T.R. Jensen and B. Toft, Graph Coloring Problems (Wiley, New York, 1995).

[14] M. Juvan, B. Mohar and R. Škrekovski, *Graphs of degree 4 are 5-choosable*, J. Graph Theory **32** (1999) 250–262. doi:10.1002/(SICI)1097-0118(199911)32:3⟨250::AID-JGT5⟩3.0.CO;2-R

[15] A.V. Kostochka, *List edge chromatic number of graphs with large girth*, Discrete Math. **101** (1992) 189-201. doi:10.1016/0012-365X(92)90602-C

[16] B. Liu, J.F. Hou and G.Z. Liu, *List edge and list total colorings of planar graphs without short cycles*, Inform. Process. Lett. **108** (2008) 347–351. doi:10.1016/j.ipl.2008.07.003

[17] Y. Shen, G. Zheng, W. He and Y. Zhao, *Structural properties and edge choosability of planar graphs without 4-cycles*, Discrete Math. **308** (2008) 5789–5794. doi:10.1016/j.disc.2007.09.048

[18] W.F. Wang and K.W. Lih, *Structural properties and edge choosability of planar graphs without 6-cycles*, Combin. Probab. Comput. **10** (2001) 267–276.

[19] W.F. Wang and K.W. Lih, *Choosability, edge choosability and total choosability of outerplanar graphs*, European J. Combin. **22** (2001) 71–78. doi:10.1006/eujc.2000.0430

[20] W.F. Wang and K.W. Lih, *Choosability and edge choosability of planar graphs without five cycles*, Appl. Math. Lett. **15** (2002) 561–565. doi:10.1016/S0893-9659(02)80007-6

[21] J.L. Wu and P. Wang, *List-edge and list-total colorings of graphs embedded on hyperbolic surfaces*, Discrete Math. **308** (2008) 6210–6215. doi:10.1016/j.disc.2007.11.044

[22] L. Zhang and B. Wu, *Edge choosability of planar graphs without small cycles*, Discrete Math. **283** (2004) 289–293. doi:10.1016/j.disc.2004.01.001

Received 22 May 2017
Revised 10 September 2018
Accepted 10 September 2018