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ON ACCURATE DOMINATION IN GRAPHS

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Abstract

A dominating set of a graph G is a subset $D \subseteq V_G$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. The accurate domination number of G, denoted by $\gamma_{\rm a}(G)$, is the cardinality of a smallest set D that is a dominating set of G and no $|D|$-element subset of $V_G \setminus D$ is a dominating set of G. We study graphs for which the accurate domination number is equal to the domination number. In particular, all trees G for which $\gamma_{\rm a}(G) = \gamma(G)$ are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph.

Keywords: domination number, accurate domination number, tree, corona.

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1. Introduction and Notation

We generally follow the notation and terminology of [1] and [9]. Let $G = (V_G, E_G)$ be a graph with vertex set V_G of order $n(G) = |V_G|$ and edge set E_G of size $m(G) = |E_G|$. If v is a vertex of G, then the *open neighborhood* of v is the set $N_G(v) = \{u \in V_G : uv \in E_G\}$, while the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For a subset X of V_G and a vertex x in X, the set $\mathrm{pn}_G(x, X) = \{v \in V_G : N_G[v] \cap X = \{x\}\}$ is called the *X-private neighborhood* of the vertex x, and it consists of those vertices of $N_G[x]$ which are not adjacent to any vertex in $X \setminus \{x\}$; that is, $\mathrm{pn}_G(x, X) = N_G[x] \setminus N_G[X \setminus \{x\}]$. The *degree* $d_G(v)$ of a vertex v in G is the number of vertices in $N_G(v)$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* The set of leaves of a graph G is denoted by L_G, while the set of support vertices by S_G. For a set $S \subseteq V_G$, the subgraph induced by S is denoted by $G[S]$, while the subgraph induced by $V_G \setminus S$ is denoted by $G - S$. Thus the graph $G - S$ is obtained from G by deleting the vertices in S and all edges incident with S. Let $\kappa(G)$ denote the number of components of G.

A *dominating set* of a graph G is a subset D of V_G such that every vertex not in D is adjacent to at least one vertex in D, that is, $N_G(x) \cap D \neq \emptyset$ for every $x \in V_G \setminus D$. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G. An *accurate dominating set* of G is a dominating set D of G such that no $|D|$-element subset of $V_G \setminus D$ is a dominating set of G. The *accurate domination number* of G, denoted by $\gamma_{\rm a}(G)$, is the cardinality of a smallest accurate dominating set of G. We call a dominating set of G of cardinality $\gamma(G)$ a *γ-set of G*, and an accurate dominating set of G of cardinality $\gamma_{\rm a}(G)$ a *$\gamma_{\rm a}$-set of G*. Since every accurate dominating set of G is a dominating set of G, we note that $\gamma(G) \leq \gamma_{\rm a}(G)$. The accurate domination in graphs was introduced by Kulli and Kattimani [11], and further studied in a number of papers (see, for example, [3, 6, 7, 10, 12–14, 16, 17]). A comprehensive survey of concepts and results on domination in graphs can be found in [9].

We denote the path and cycle on n vertices by P_n and C_n, respectively. We denote by K_n the *complete graph* on n vertices, and by $K_{m,n}$ the *complete bipartite graph* with partite sets of size m and n. The accurate domination numbers of some common graphs are given by the following formulas.

Observation 1. *The following holds.*

(a) *For* $n \geq 1$, $\gamma_{\rm a}(K_n) = \lfloor \frac{n}{2} \rfloor + 1$ *and* $\gamma_{\rm a}(K_{n,n}) = n + 1$.

(b) *For* $n > m \geq 1$, $\gamma_{\rm a}(K_{m,n}) = m$.

(c) *For* $n \geq 3$, $\gamma_{\rm a}(C_n) = \lfloor \frac{n}{3} \rfloor - \lfloor \frac{3}{n} \rfloor + 2$.

(d) *For* $n \geq 1$, $\gamma_{\rm a}(P_n) = \lceil \frac{n}{3} \rceil$ *unless* $n \in \{2, 4\}$ *when* $\gamma_{\rm a}(P_n) = \lceil \frac{n}{3} \rceil + 1$ (see Corollary 6).

In this paper we study graphs for which the accurate domination number is equal to the domination number. In particular, all trees G for which $\gamma_{\rm a}(G) = \gamma(G)$ are characterized. Furthermore, we compare the accurate domination number with the domination number of different coronas of a graph. Throughout the paper, we use the symbol $\mathcal{A}_\gamma(G)$ (respectively, $\mathcal{A}_{\gamma_{\rm a}}(G)$) to denote the set of all minimum dominating sets (respectively, minimum accurate dominating sets) of G.

2. Graphs with $\gamma_{\rm a}$ Equal to γ

We are interested in determining the structure of graphs for which the accurate domination number is equal to the domination number. The question about such graphs has been stated in [12]. We begin with the following general property of the graphs G for which $\gamma_{\rm a}(G) = \gamma(G)$.

Lemma 2. *Let G be a graph. Then $\gamma_{\rm a}(G) = \gamma(G)$ if and only if there exists a set $D \in \mathcal{A}_\gamma(G)$ such that $D \cap D' \neq \emptyset$ for every set $D' \in \mathcal{A}_\gamma(G)$.*

Proof. First assume that $\gamma_{\rm a}(G) = \gamma(G)$, and let D be a minimum accurate dominating set of G. Since D is a dominating set of G and $|D| = \gamma_{\rm a}(G) = \gamma(G)$, we note that $D \in \mathcal{A}_\gamma(G)$. Now let D' be an arbitrary minimum dominating set of G. If $D \cap D' = \emptyset$, then $D' \subseteq V_G \setminus D$, implying that D' would be a $|D|$-element dominating set of G, contradicting the fact that D is an accurate dominating set of G. Hence, $D \cap D' \neq \emptyset$.

Now assume that there exists a set $D \in \mathcal{A}_\gamma(G)$ such that $D \cap D' \neq \emptyset$ for every set $D' \in \mathcal{A}_\gamma(G)$. Then, D is an accurate dominating set of G, implying that $\gamma_{\rm a}(G) \leq |D| = \gamma(G) \leq \gamma_{\rm a}(G)$. Consequently, we must have equality throughout this inequality chain, and so $\gamma_{\rm a}(G) = \gamma(G)$. ∎

It follows from Lemma 2 that if G is a disconnected graph, then $\gamma_{\rm a}(G) = \gamma(G)$ if and only if $\gamma_{\rm a}(H) = \gamma(H)$ for at least one component H of G. In particular, if G has an isolated vertex, then $\gamma_{\rm a}(G) = \gamma(G)$. It also follows from Lemma 2 that for a graph G, $\gamma_{\rm a}(G) = \gamma(G)$ if G has one of the following properties: (1) G has a unique minimum dominating set (see, for example, [4] or [8] for some characterizations of such graphs); (2) G has a vertex which belongs to every minimum dominating set of G (see [15]); (3) G has a vertex adjacent to at least two leaves. Consequently, there is no forbidden subgraph characterization for the class of graphs G for which $\gamma_{\rm a}(G) = \gamma(G)$, as for any graph H, we can add an isolated vertex (or two leaves to one vertex of H), and in this way form a graph H' for which $\gamma_{\rm a}(H') = \gamma(H')$.

The *corona* $F \circ K_1$ of a graph F is the graph formed from F by adding a new vertex v' and edge vv' for each vertex $v \in V(F)$. A graph G is said to be a *corona graph* if $G = F \circ K_1$ for some connected graph F. We note that each vertex of a corona graph G is a leaf or it is adjacent to exactly one leaf of G. Recall that we denote the set of all leaves in a graph G by L_G, and set of support vertices in G by S_G.

Lemma 3. *If G is a corona graph, then $\gamma_{\rm a}(G) > \gamma(G)$.*

Proof. Assume that G is a corona graph. If $G = K_1 \circ K_1$, then $G = K_2$ and $\gamma_{\rm a}(G) = 2$ and $\gamma(G) = 1$. Hence, we may assume that $G = F \circ K_1$ for some connected graph F of order $n(F) \geq 2$. If $v \in V_G \setminus L_G$, then let $\overline{v}$ denote the unique leaf-neighbor of v in G. Now let D be an arbitrary minimum dominating set of G, and so $D \in \mathcal{A}_\gamma(G)$. Then, $|D \cap \{v, \overline{v}\}| = 1$ for every $v \in V_G \setminus L_G$. Consequently, D and its complement $V_G \setminus D$ are minimum dominating sets of G. Thus, D is not an accurate dominating set of G. This is true for every minimum dominating set of G, implying that $\gamma_{\rm a}(G) > \gamma(G)$. ■

Lemma 4. *If T is a tree of order at least three, then there exists a set $D \in \mathcal{A}_\gamma(T)$ such that the following hold.*

(a) $S_T \subseteq D$.

(b) $N_T(v) \subseteq V_T \setminus D$ *or* $|{\rm pn}_T(v, D)| \geq 2$ *for every* $v \in D \setminus S_T$.

Proof. Let T be a tree of order $n(T) \geq 3$. Among all minimum dominating sets of T, let $D \in \mathcal{A}_\gamma(T)$ be chosen that

(1) D contains as many support vertices as possible.

(2) Subject to (1), the number of components $\kappa(T[D])$ is as large as possible.

If the set D contains a leaf v of T, then we can simply replace v in D with the support vertex adjacent to v to produce a new minimum dominating set with more support vertices than D, a contradiction. Hence, the set D contains no leaves, implying that $S_T \subseteq D$. Suppose, next, that there exists a vertex v in D that is not a support vertex of T and such that $N_T(v) \not\subseteq V_T \setminus D$. Thus, v has at least one neighbor in D; that is, $N_T(v) \cap D \neq \emptyset$. By the minimality of the set D, we therefore note that ${\rm pn}_T(v, D) \neq \emptyset$. If $|{\rm pn}_T(v, D)| = 1$, say ${\rm pn}_T(v, D) = \{u\}$, then letting $D' = (D \setminus \{v\}) \cup \{u\}$, the set $D' \in \mathcal{A}_\gamma(T)$ and satisfies $S_T \subseteq D \setminus \{v\} \subset D'$ and $\kappa(T[D']) > \kappa(T[D])$, which contradicts the choice of D. Hence, if $v \in D$ is not a support vertex of T and $N_T(v) \not\subseteq V_T \setminus D$, then $|{\rm pn}_T(v, D)| \geq 2$. ■

We are now in a position to present the following equivalent characterizations of trees for which the accurate domination number is equal to the domination number.

Theorem 5. *If T is a tree of order at least two, then the following statements are equivalent.*

(1) *T is not a corona graph.*

(2) *There exists a set $D \in \mathcal{A}_\gamma(T)$ such that $\kappa(T - D) > |D|$.*

(3) $\gamma_{\rm a}(T) = \gamma(T)$.

(4) *There exists a set $D \in \mathcal{A}_\gamma(T)$ such that $D \cap D' \neq \emptyset$ for every $D' \in \mathcal{A}_\gamma(T)$.*

Proof. The statements (3) and (4) are equivalent by Lemma 2. The implication (3) $\Rightarrow$ (1) follows from Lemma 3. To prove the implication (2) $\Rightarrow$ (3), let us assume that $D \in \mathcal{A}_\gamma(T)$ and $\kappa(T - D) > |D|$. Thus, $\gamma(T - D) \geq \kappa(T - D) > |D| = \gamma(T)$. This proves that D is an accurate dominating set of T, and therefore $\gamma_{\rm a}(T) = \gamma(T)$.

Thus it suffices to prove that (1) implies (2). The proof is by induction on the order of a tree. The implication (1) $\Rightarrow$ (2) is obvious for trees of order two, three, and four. Thus assume that T is a tree of order at least five and T is not a corona graph. Let $D \in \mathcal{A}_\gamma(T)$ and assume that $S_T \subseteq D$. Since T is not a corona graph, the tree T has a vertex which is neither a leaf nor adjacent to exactly one leaf. We consider two cases, depending on whether $d_T(v) \geq 3$ for some vertex $v \in S_T$ or $d_T(v) = 2$ for every vertex $v \in S_T$.

Case 1. $d_T(v) \geq 3$ for some $v \in S_T$. Let v' be a leaf of T adjacent to v. Let T' be a component of $T - \{v, v'\}$. Now let T_1 and T_2 be the subtrees of T induced on the vertex sets $V_{T'} \cup \{v, v'\}$ and $V_T \setminus V_{T'}$, respectively. We note that both trees T_1 and T_2 have order strictly less than $n(T)$. Further, $V(T_1) \cap V(T_2) = \{v, v'\}$, $E(T_1) \cap E(T_2) = \{vv'\}$, and at least one of T_1 and T_2, say T_1, is not a corona graph. Applying the induction hypothesis to T_1, there exists a set $D_1 \in \mathcal{A}_\gamma(T_1)$ such that $\kappa(T_1 - D_1) > |D_1|$. If T_2 is a corona graph, then choosing D_2 to be the set of support vertices in T_2 we note that $D_2 \in \mathcal{A}_\gamma(T_2)$ and $\kappa(T_2 - D_2) = |D_2|$. If T_2 is not a corona graph, then applying the induction hypothesis to T_2, there exists a set $D_2 \in \mathcal{A}_\gamma(T_2)$ such that $\kappa(T_2 - D_2) > |D_2|$. In both cases, there exists a set $D_2 \in \mathcal{A}_\gamma(T_2)$ such that $\kappa(T_2 - D_2) \geq |D_2|$. We may assume that all support vertices of T_1 and T_2 are in D_1 and D_2, respectively. Thus, $v \in D_1 \cap D_2$, the union $D_1 \cup D_2$ is a γ-set of T, and $\kappa(T - (D_1 \cup D_2)) = \kappa(T_1 - D_1) + \kappa(T_2 - D_2) - 1 > |D_1| + |D_2| - 1 = |D_1 \cup D_2|$.

Case 2. $d_T(v) = 2$ for every $v \in S_T$. We distinguish two subcases, depending on whether $D \setminus S_T \neq \emptyset$ or $D \setminus S_T = \emptyset$.

Case 2.1. $D \setminus S_T \neq \emptyset$. Let v be an arbitrary vertex belonging to $D \setminus S_T$. It follows from the second part of Lemma 4 that there are two vertices v_1 and v_2 belonging to $N_T(v) \setminus D$. Let R be the tree obtained from T by adding a new vertex v' and the edge vv'. We note that D is a minimum dominating set of R and $S_R \subseteq D$. Let R' be the component of $R - \{v, v'\}$ containing v_1. Now let R_1 and R_2 be the subtrees of R induced by the vertex sets $V_{R'} \cup \{v, v'\}$ and $V_R \setminus V_{R'}$, respectively. We note that both trees R_1 and R_2 have order strictly less than $n(T)$. Further, $V(R_1) \cap V(R_2) = \{v, v'\}$, $E(R_1) \cap E(R_2) = \{vv'\}$, and neither R_1 nor R_2 is a corona graph. By the induction hypothesis, there exists a set $D_1 \in \mathcal{A}_\gamma(R_1)$ and a set $D_2 \in \mathcal{A}_\gamma(R_2)$ such that $\kappa(R_1 - D_1) > |D_1|$ and $\kappa(R_2 - D_2) > |D_2|$. We may assume that all support vertices of R_1 and R_2 are in D_1 and D_2, respectively. Thus, $v \in D_1 \cap D_2$, the union $D_1 \cup D_2$ is a γ-set of R, and

$$\begin{aligned}\kappa(T - (D_1 \cup D_2)) &= \kappa(R - (D_1 \cup D_2)) - 1 = (\kappa(R_1 - D_1) + \kappa(R_2 - D_2) - 1) - 1 \\ &= (\kappa(R_1 - D_1) - |D_1| + \kappa(R_2 - D_2) - |D_2|) - 2 + |D_1| + |D_2| \\ &\geq |D_1| + |D_2| = |D_1 \cup D_2| + 1 > |D_1 \cup D_2|.\end{aligned}$$

Case 2.2. $D \setminus S_T = \emptyset$. In this case, we note that $D = S_T$. Let v be an arbitrary vertex belonging to D and assume that $N_T(v) = \{u, w\}$, where $u \in L_T$. If $w \in L_T$, then $T = K_{1,2}$, contradicting the assumption that $n(T) \geq 5$. If $w \in S_T$, then $T = P_4 = K_2 \circ K_1$, contradicting the assumption that T is not a corona graph (and the assumption that $n(T) \geq 5$). Therefore, $w \in V_T \setminus (L_T \cup S_T)$. Thus, $V_T \setminus (L_T \cup S_T)$ is nonempty and $T - D$ has $|D|$ one-element components induced by leaves of T and at least one component induced by $V_T \setminus (L_T \cup S_T)$. Consequently, $\kappa(T - D) \geq |D| + 1 > |D|$. This completes the proof of Theorem 5. ∎

The equivalence of the statements (1) and (3) of Theorem 5 shows that the trees T for which $\gamma_{\rm a}(T) = \gamma(T)$ are easy to recognize. From Theorem 5 and from the well-known fact that $\gamma(P_n) = \lceil n/3 \rceil$ for every positive integer n, we also immediately have the following corollary which provides a slight improvement on Proposition 3 in [12].

Corollary 6. *For* $n \geq 1$, $\gamma_{\rm a}(P_n) = \gamma(P_n) = \lceil n/3 \rceil$ *if and only if* $n \in \mathbb{N} \setminus \{2, 4\}$.

3. Domination of General Coronas of a Graph

Let G be a graph, and let $\mathcal{F} = \{F_v \colon v \in V_G\}$ be a family of nonempty graphs indexed by the vertices of G. By $G \circ \mathcal{F}$ we denote the graph with vertex set

$$V_{G \circ \mathcal{F}} = (V_G \times \{0\}) \cup \bigcup_{v \in V_G} \left(\{v\} \times V_{F_v}\right)$$

and edge set determined by open neighborhoods defined in such a way that

$$N_{G \circ \mathcal{F}}((v, 0)) = (N_G(v) \times \{0\}) \cup \left(\{v\} \times V_{F_v}\right)$$

for every $v \in V_G$, and

$$N_{G \circ \mathcal{F}}((v, x)) = \{(v, 0)\} \cup \left(\{v\} \times N_{F_v}(x)\right)$$

if $v \in V_G$ and $x \in V_{F_v}$. The graph $G \circ \mathcal{F}$ is said to be the $\mathcal{F}$-*corona* of G. Informally, $G \circ \mathcal{F}$ is the graph obtained by taking a disjoint copy of G and all the graphs of $\mathcal{F}$ with additional edges joining each vertex v of G to every vertex in the copy of F_v. If all the graphs of the family $\mathcal{F}$ are isomorphic to one and the same graph F (as it was defined by Frucht and Harary [5]), then we simply write $G \circ F$ instead of $G \circ \mathcal{F}$. Recall that a graph G is said to be a *corona graph* if $G = F \circ K_1$ for some connected graph F.

The 2-*subdivided graph* $S_2(G)$ of a graph G is the graph with vertex set

$$V_{S_2(G)} = V_G \cup \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\}$$

and the adjacency is defined in such a way that

$$N_{S_2(G)}(x) = \{(x, xy) \colon y \in N_G(x)\}$$

if $x \in V_G \subseteq V_{S_2(G)}$, while

$$N_{S_2(G)}((x, xy)) = \{x\} \cup \{(y, xy)\}$$

if $(x, xy) \in \bigcup_{vu \in E_G} \{(v, vu), (u, vu)\} \subseteq V_{S_2(G)}$. Less formally, $S_2(G)$ is the graph obtained from G by subdividing every edge with two new vertices; that is, by replacing edges vu of G with disjoint paths $(v, (v, vu), (u, vu), u)$.

For a graph G and a family $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_G\}$, where $\mathcal{P}(v)$ is a partition of the neighborhood $N_G(v)$ of the vertex v, by $G \circ \mathcal{P}$ we denote the graph with vertex set

$$V_{G \circ \mathcal{P}} = (V_G \times \{1\}) \cup \bigcup_{v \in V_G} (\{v\} \times \mathcal{P}(v))$$

and edge set

$$E_{G \circ \mathcal{P}} = \bigcup_{v \in V_G} \{(v,1)(v,A) \colon A \in \mathcal{P}(v)\} \cup \bigcup_{uv \in E_G} \{(v,A)(u,B) \colon (u \in A) \wedge (v \in B)\}.$$

The graph $G \circ \mathcal{P}$ is called the $\mathcal{P}$-*corona* of G and was defined by Dettlaff *et al.* in [2]. It follows from this definition that if $\mathcal{P}(v) = \{N_G(v)\}$ for every $v \in V_G$, then $G \circ \mathcal{P}$ is isomorphic to the corona $G \circ K_1$. On the other hand, if $\mathcal{P}(v) = \{\{u\} \colon u \in N_G(v)\}$ for every $v \in V_G$, then $G \circ \mathcal{P}$ is isomorphic to the 2-subdivided graph $S_2(G)$ of G. Examples of $G \circ K_1$, $G \circ \mathcal{F}$, $G \circ \mathcal{P}$, and $S_2(G)$ are shown in Figure 1. In this case G is the graph $(K_2 \cup K_1) + K_1$ with vertex set $V_G = \{v, u, w, z\}$ and edge set $E_G = \{vu, vw, uw, wz\}$, where the family $\mathcal{F}$ consists of the graphs $F_v = F_w = K_1$, $F_z = K_2$, and $F_u = K_2 \cup K_1$, while $\mathcal{P} = \{\mathcal{P}(x) \colon x \in V_G\}$ is the family in which $\mathcal{P}(v) = \{\{u, w\}\}$, $\mathcal{P}(u) = \{\{v\}, \{w\}\}$, $\mathcal{P}(w) = \{\{u, v\}, \{z\}\}$, and $\mathcal{P}(z) = \{\{w\}\}$.

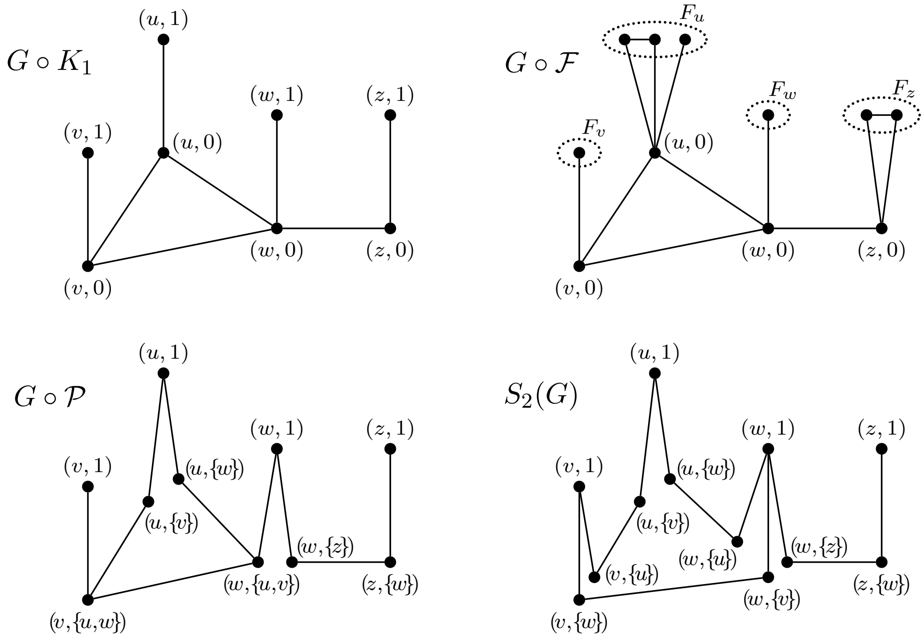

Figure 1. Coronas of $G = (K_2 \cup K_1) + K_1$.

We now study relations between the domination number and the accurate domination number of different coronas of a graph. Our first theorem specifies when these two numbers are equal for the $\mathcal{F}$-corona $G \circ \mathcal{F}$ of a graph G and a family $\mathcal{F}$ of nonempty graphs indexed by the vertices of G.

Theorem 7. *If G is a graph and $\mathcal{F} = \{F_v : v \in V_G\}$ is a family of nonempty graphs indexed by the vertices of G, then the following holds.*

(1) $\gamma(G \circ \mathcal{F}) = |V_G|$.

(2) $\gamma_{\rm a}(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$ *if and only if* $\gamma(F_v) > 1$ *for some vertex v of G.*

(3) $|V_G| \le \gamma_{\rm a}(G \circ \mathcal{F}) \le |V_G| + \min\{|V_{F_v}| : v \in V_G\}$.

Proof. (1) It is obvious that $V_G \times \{0\}$ is a minimum dominating set of $G \circ \mathcal{F}$ and therefore $\gamma(G \circ \mathcal{F}) = |V_G \times \{0\}| = |V_G|$.

(2) If $\gamma(F_v) > 1$ for some vertex v of G, then

$$\gamma(G\circ\mathcal{F}-(V_G\times\{0\})) = \sum_{v\in V_G} \gamma((G\circ\mathcal{F})[\{v\}\times V_{F_v}]) = \sum_{v\in V_G} \gamma(F_v) > |V_G| = |V_G\times\{0\}|$$

and this proves that no subset of $V_{G\circ\mathcal{F}} \setminus (V_G \times \{0\})$ of cardinality $|V_G \times \{0\}|$ is a dominating set of $G \circ \mathcal{F}$. Consequently $V_G \times \{0\}$ is a minimum accurate dominating set of $G \circ \mathcal{F}$ and therefore $\gamma_{\rm a}(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$.

Assume now that G and $\mathcal{F}$ are such that $\gamma_{\rm a}(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F})$. We claim that $\gamma(F_v) > 1$ for some vertex v of G. Suppose, contrary to our claim, that $\gamma(F_v) = 1$ for every vertex v of G. Then the set $U_v = \{x \in V_{F_v} : N_{F_v}[x] = V_{F_v}\}$, the set of universal vertices of F_v, is nonempty for every $v \in V_G$. Now, let D be any minimum dominating set of $G \circ \mathcal{F}$. Then, $|D| = \gamma(G \circ \mathcal{F}) = |V_G \times \{0\}| = |V_G|$, $|D \cap (\{(v, 0)\} \cup (\{v\} \times U_v))| = 1$, and the set $(\{(v, 0)\} \cup (\{v\} \times U_v)) \backslash D$ is nonempty for every $v \in V_G$. Now, if $\overline{D}$ is a system of representatives of the family $\{(\{(v, 0)\} \cup (\{v\} \times U_v)) \backslash D \colon v \in V_G\}$, then $\overline{D}$ is a minimum dominating set of $G \circ \mathcal{F}$. Since $\overline{D}$ and D are disjoint, D is not an accurate dominating set of $G \circ \mathcal{F}$. Consequently, no minimum dominating set of $G \circ \mathcal{F}$ is an accurate dominating set and therefore $\gamma(G \circ \mathcal{F}) < \gamma_{\rm a}(G \circ \mathcal{F})$, a contradiction.

(3) The lower bound is obvious as $|V_G| = \gamma(G \circ \mathcal{F}) \le \gamma_{\rm a}(G \circ \mathcal{F})$. Since $(V_G \times \{0\}) \cup (\{v\} \times V_{F_v})$ is an accurate dominating set of $G \circ \mathcal{F}$ (for every $v \in V_G$), we also have the inequality $\gamma_{\rm a}(G \circ \mathcal{F}) \le |V_G| + \min\{|V_{F_v}| \colon v \in V_G\}$. This completes the proof of Theorem 7. ∎

As a consequence of Theorem 7, we have the following result.

Corollary 8. *If G is a graph, then $\gamma_{\rm a}(G \circ K_1) = \gamma(G \circ K_1) + 1 = |V_G| + 1$.*

Proof. Since $\gamma(K_1) = 1$, it follows from Theorem 7 that $\gamma_{\rm a}(G \circ K_1) \ge \gamma(G \circ K_1) + 1 = |V_G| + 1$. On the other hand, the set $(V_G \times \{0\}) \cup \{(v, 1)\}$ is an accurate dominating set of $G \circ K_1$ and therefore $\gamma_{\rm a}(G \circ K_1) \le |(V_G \times \{0\}) \cup \{(v, 1)\}| = |V_G| + 1$. Consequently, $\gamma_{\rm a}(G \circ K_1) = \gamma(G \circ K_1) = |V_G| + 1$. ∎

From Theorem 7 we know that $\gamma_{\rm a}(G \circ \mathcal{F}) = \gamma(G \circ \mathcal{F}) = |V_G|$ if and only if the family $\mathcal{F}$ is such that $\gamma(F_v) > 1$ for some $F_v \in \mathcal{F}$, but we do not know any general formula for $\gamma_{\rm a}(G \circ \mathcal{F})$ if $\gamma(F_v) = 1$ for every $F_v \in \mathcal{F}$. The following theorem shows a formula for the domination number and general bounds for the accurate domination number of a $\mathcal{P}$-corona of a graph.

Theorem 9. *If G is a graph and $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_G\}$ is a family of partitions of the vertex neighborhoods of G, then the following holds.*

(1) $\gamma(G \circ \mathcal{P}) = |V_G|$.

(2) $\gamma_{\rm a}(G \circ \mathcal{P}) \ge |V_G|$.

(3) $\gamma_{\rm a}(G \circ \mathcal{P}) \le |V_G| + \min\big\{\min\{|\mathcal{P}(v)| \colon v \in V_G\}, 1 + \min\big\{|A| \colon A \in \bigcup_{v \in V_G} \mathcal{P}(v)\big\}\big\}$.

Proof. It follows from the definition of $G \circ \mathcal{P}$ that $V_G \times \{1\}$ is a dominating set of $G \circ \mathcal{P}$, and therefore $\gamma(G \circ \mathcal{P}) \le |V_G \times \{1\}| = |V_G|$. On the other hand, let $D \in \mathcal{A}_\gamma(G \circ \mathcal{P})$. Then $D \cap N_{G \circ \mathcal{P}}[(v, 1)] \neq \emptyset$ for every $v \in V_G$, and, since the sets $N_{G \circ \mathcal{P}}[(v, 1)]$ form a partition of $V_{G \circ \mathcal{P}}$, we have

$$\gamma(G \circ \mathcal{P}) = |D| = \left| \bigcup_{v \in V_G} (D \cap N_{G \circ \mathcal{P}}[(v, 1)]) \right| = \sum_{v \in V_G} |D \cap N_{G \circ \mathcal{P}}[(v, 1)]| \ge |V_G|.$$

Consequently, we have $|V_G| = \gamma(G \circ \mathcal{P}) \le \gamma_{\mathrm{a}}(G \circ \mathcal{P})$, which proves (1) and (2).

From the definition of $G \circ \mathcal{P}$ it also follows that each of the sets $(V_G \times \{1\}) \cup N_{G\circ\mathcal{P}}[(v,1)]$ (for every $v \in V_G$) and $(V_G \times \{1\}) \cup N_{G\circ\mathcal{P}}[(v,A)]$ (for every $v \in V_G$ and $A \in \mathcal{P}(v)$) is an accurate dominating set of $G \circ \mathcal{P}$. Hence,

$$\begin{aligned} |(V_G \times \{1\}) \cup N_{G\circ\mathcal{P}}[(v,1)]| &= |V_G \times \{1\}| + |N_{G\circ\mathcal{P}}((v,1))| = |V_G| + |\mathcal{P}(v)| \\ &\ge |V_G| + \min\{|\mathcal{P}(v)| \colon v \in V_G\} \ge \gamma_{\mathrm{a}}(G \circ \mathcal{P}), \end{aligned}$$

and similarly

$$\begin{aligned} |(V_G \times \{1\}) \cup N_{G\circ\mathcal{P}}[(v,A)]| &= |(V_G \times \{1\}) \cup \{(v,1)\} \cup N_{G\circ\mathcal{P}}((v,A))| \\ &= |V_G| + 1 + |A| \\ &\ge |V_G| + 1 + \min\{|A| \colon A \in \textstyle\bigcup_{v\in V_G} \mathcal{P}(v)\}. \end{aligned}$$

Therefore,

$$\gamma_{\mathrm{a}}(G \circ \mathcal{P}) \le |V_G| + \min\left\{ \min\{|\mathcal{P}(v)| \colon v \in V_G\}, 1 + \min\left\{ |A| \colon A \in \bigcup_{v\in V_G} \mathcal{P}(v) \right\} \right\}.$$

This completes the proof of Theorem 9. ∎

We do not know all the pairs $(G, \mathcal{P})$ achieving equality in the upper bound for the accurate domination number of a $\mathcal{P}$-corona of a graph, but Theorem 10 and Corollaries 11 and 12 show that the bounds in Theorem 9 are best possible. The next theorem also shows that the domination number and the accurate domination number of a 2-subdivided graph are easy to compute.

Theorem 10. *If G is a connected graph, then the following holds.*

(1) $\gamma(S_2(G)) = |V_G|$.

(2) $|V_G| \le \gamma_{\mathrm{a}}(S_2(G)) \le |V_G| + 2$.

(3) $\gamma_{\mathrm{a}}(S_2(G)) = \begin{cases} |V_G| + 2, & \text{if } G \text{ is a cycle,} \\ |V_G| + 1, & \text{if } G = K_2, \\ |V_G|, & \text{otherwise.} \end{cases}$

Proof. The statement (1) follows from Theorem 9 (1).

(2) The inequalities $|V_G| \le \gamma_{\mathrm{a}}(S_2(G)) \le |V_G| + 2$ are obvious if $G = K_1$. Thus assume that G is a connected graph of order at least two. Let u and v be adjacent vertices of G. Then, $V_G \cup \{(v,vu),(u,vu)\}$ is an accurate dominating set of $S_2(G)$ and we have $|V_G| = \gamma(S_2(G)) \le \gamma_{\mathrm{a}}(S_2(G)) \le |V_G \cup \{(v,vu),(u,vu)\}| = |V_G| + 2$.

(3) The connectivity of G implies that there are three cases to consider.

Case 1. $|E_G| > |V_G|$. In this case $S_2(G) - V_G$ has $|E_G|$ components and therefore no $|V_G|$-element subset of $V_{S_2(G)} \setminus V_G$ dominates $S_2(G)$. Hence, V_G is an accurate dominating set of $S_2(G)$ and $\gamma_{\rm a}(S_2(G)) = |V_G|$.

Case 2. $|E_G| = |V_G|$. In this case, G is a unicyclic graph. First, if G is a cycle, say $G = C_n$, then $S_2(G) = C_{3n}$ and $\gamma_{\rm a}(S_2(G)) = \gamma_{\rm a}(C_{3n}) = n + 2 = |V_G| + 2$ (see Proposition 3 in [12]). Thus assume that G is a unicyclic graph which is not a cycle. Then G has a leaf, say v. Now, if u is the only neighbor of v, then $(V_G \setminus \{v\}) \cup \{(v, vu)\}$ is a minimum dominating set of $S_2(G)$. Since $S_2(G) - ((V_G \setminus \{v\}) \cup \{(v, vu)\})$ has $|V_G| + 1$ components, $(V_G \setminus \{v\}) \cup \{(v, vu)\}$ is a minimum accurate dominating set of $S_2(G)$ and $\gamma_{\rm a}(S_2(G)) = |(V_G \setminus \{v\}) \cup \{(v, vu)\}| = |V_G|$.

Case 3. $|E_G| = |V_G| - 1$. In this case, G is a tree. Now, if $G = K_1$, then $S_2(G) = K_1$ and $\gamma_{\rm a}(S_2(G)) = \gamma_{\rm a}(K_1) = 1 = |V_G|$. If $G = K_2$, then $S_2(G) = P_4$ and $\gamma_{\rm a}(S_2(G)) = \gamma_{\rm a}(P_4) = 3 = 2 + 1 = |V_G| + 1$. Finally, if G is a tree of order at least three, then the tree $S_2(G)$ is not a corona graph and by (1) and Theorem 5 we have $\gamma_{\rm a}(S_2(G)) = \gamma(S_2(G)) = |V_G|$. ■

As a consequence of Theorem 10, we have the following results.

Corollary 11. *If T is a tree and $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_T\}$ is a family of partitions of the vertex neighborhoods of T, then*

$$\gamma_{\rm a}(T \circ \mathcal{P}) = \begin{cases} |V_T| + 1, & \text{if } |\mathcal{P}(v)| = 1 \text{ for every } v \in V_T, \\ |V_T|, & \text{if } |\mathcal{P}(v)| > 1 \text{ for some } v \in V_T. \end{cases}$$

Proof. If $|\mathcal{P}(v)| = 1$ for every $v \in V_T$, then $T \circ \mathcal{P} = T \circ K_1$ and the result follows from Corollary 8. If $|\mathcal{P}(v)| > 1$ for some $v \in V_T$, then the tree $T \circ \mathcal{P}$ is not a corona and the result follows from Theorem 5 and Theorem 9 (1). ■

Corollary 12. *For $n \geq 3$, if $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_{C_n}\}$ is a family of partitions of the vertex neighborhoods of C_n, then*

$$\gamma_{\rm a}(C_n \circ \mathcal{P}) = \begin{cases} n + 1, & \text{if } |\mathcal{P}(v)| = 1 \text{ for every } v \in V_{C_n}, \\ n + 2, & \text{if } |\mathcal{P}(v)| = 2 \text{ for every } v \in V_{C_n}, \\ n, & \text{otherwise.} \end{cases}$$

Proof. If $|\mathcal{P}(v)| = 1$ for every $v \in V_{C_n}$, then $C_n \circ \mathcal{P} = C_n \circ K_1$. Thus, by Theorem 8, we have $\gamma_{\rm a}(C_n \circ \mathcal{P}) = \gamma_{\rm a}(C_n \circ K_1) = \gamma(C_n \circ K_1) = |V_{C_n}| + 1 = n + 1$.

If $|\mathcal{P}(v)| > 1$ (and therefore $|\mathcal{P}(v)| = 2$) for every $v \in V_{C_n}$, then $C_n \circ \mathcal{P} = S_2(C_n) = C_{3n}$. Now, since $\gamma_{\rm a}(C_{3n}) = n + 2$ (as it was observed in [12]), we have $\gamma_{\rm a}(C_n \circ \mathcal{P}) = \gamma_{\rm a}(C_{3n}) = n + 2$.

Finally assume that there are vertices u and v in C_n such that $|\mathcal{P}(v)| = 1$ and $|\mathcal{P}(u)| = 2$. Then the sets

$$V^1_{C_n} = \{x \in V_{C_n} \colon |\mathcal{P}(x)| = 1\} \quad \text{and} \quad V^2_{C_n} = \{y \in V_{C_n} \colon |\mathcal{P}(y)| = 2\}$$

form a partition of V_{C_n}. Without loss of generality we may assume that $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_\ell, \ldots, z_1, z_2, \ldots, z_p, t_1, t_2, \ldots, t_q$ are the consecutive vertices of C_n, where

$$x_1, x_2, \ldots, x_k \in V^1_{C_n}, y_1, y_2, \ldots, y_\ell \in V^2_{C_n}, \ldots, z_1, z_2, \ldots, z_p \in V^1_{C_n},$$
$$t_1, t_2, \ldots, t_q \in V^2_{C_n},$$

and $k + \ell + \cdots + p + q = n$. It is easy to observe that $D = \{(x_i, N_{C_n}(x_i)) \colon i = 1, \ldots, k\} \cup \{(y_j, 1) \colon j = 1, \ldots, \ell\} \cup \cdots \cup \{(z_i, N_{C_n}(z_i)) \colon i = 1, \ldots, p\} \cup \{(t_j, 1) \colon j = 1, \ldots, q\}$ is a dominating set of $C_n \circ \mathcal{P}$. Since the set D is of cardinality $n = |V_{C_n}|$ and $n = \gamma(C_n \circ \mathcal{P})$ (by Theorem 9 (1)), D is a minimum dominating set of $C_n \circ \mathcal{P}$. In addition, since $C_n \circ \mathcal{P} - D$ has $k + (2 + (\ell - 1)) + \cdots + p + (2 + (q - 1)) > k + \ell + \cdots + p + q = n$ components, that is, since $\kappa(C_n \circ \mathcal{P} - D) > n$, no n-element subset of $V_{C_n \circ \mathcal{P}} \setminus D$ is a dominating set of $C_n \circ \mathcal{P}$. Thus, D is an accurate dominating set of $C_n \circ \mathcal{P}$ and therefore $\gamma(C_n \circ \mathcal{P}) = n$. ■

4. Closing Open Problems

We close with the following list of open problems that we have yet to settle.

Problem 13. Find a formula for the accurate domination number $\gamma_a(G \circ \mathcal{F})$ of the $\mathcal{F}$-corona of a graph G depending only on the family $\mathcal{F} = \{F_v \colon v \in V_G\}$ such that $\gamma(F_v) = 1$ for every $v \in V_G$.

Problem 14. Characterize the graphs G and the families $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_G\}$ for which $\gamma_a(G \circ \mathcal{P}) = |V_G| + \min\big\{\min\{|\mathcal{P}(v)| \colon v \in V_G\}, 1 + \min\big\{|A| \colon A \in \bigcup_{v \in V_G} \mathcal{P}(v)\big\}\big\}$.

Problem 15. It is a natural question to ask if there exists a nonnegative integer k such that $\gamma_a(G \circ \mathcal{P}) \le |V_G| + k$ for every graph G and every family $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_G\}$ of partitions of the vertex neighborhoods of G.

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