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DECOMPOSITION OF THE TENSOR PRODUCT OF COMPLETE GRAPHS INTO CYCLES OF LENGTHS 3 AND 6

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Abstract

By a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition of a graph G, we mean a partition of the edge set of G into α cycles of length 3 and β cycles of length 6. In this paper, necessary and sufficient conditions for the existence of a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition of $(K_m \times K_n)(\lambda)$, where $\times$ denotes the tensor product of graphs and λ is the multiplicity of the edges, is obtained. In fact, we prove that for $\lambda \geq 1$, $m, n \geq 3$ and $(m,n) \neq (3,3)$, a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition of $(K_m \times K_n)(\lambda)$ exists if and only if $\lambda(m-1)(n-1) \equiv 0 \pmod 2$ and $3\alpha + 6\beta = \frac{\lambda m(m-1)n(n-1)}{2}$.

Keywords: cycle decomposition, tensor product.

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1. Introduction

Throughout this paper, graphs are assumed to be loopless and finite. Let C_k denote the cycle of length k. The complete graph on n vertices is denoted by K_n. A graph G is said to be *H-decomposable* if the edge set $E(G)$ can be partitioned into $E_1, E_2, \ldots, E_k$ such that $\langle E_i \rangle \simeq H, 1 \leq i \leq k$. If a graph G can be decomposed into cycles of length k, then we say that G admits a *C_k-decomposition* and in this case we write $G = C_k \oplus C_k \oplus \cdots \oplus C_k$; also we write it as $C_k \mid G$. A graph G is said to be *$\{H_1, H_2\}$-decomposable* if the edge set of G can be partitioned into $E_1, E_2, \ldots, E_k$ such that $\langle E_i \rangle \simeq H_1$ or $\langle E_i \rangle \simeq H_2$, $1 \leq i \leq k$ and $H_1, H_2 \in \{\langle E_1 \rangle, \langle E_2 \rangle, \ldots, \langle E_k \rangle\}$. The graph obtained by replacing each edge of G by λ

parallel edges is denoted by $G(\lambda)$. For an integer k, kG denotes k disjoint copies of G. Definitions which are not given here can be found in [9].

For two simple graphs G_1 and G_2 their *tensor product*, denoted by $G_1 \times G_2$, has vertex set $V(G_1)\times V(G_2)$ in which $(x_1,y_1)(x_2,y_2)$ is an edge whenever x_1x_2 is an edge in G_1 and y_1y_2 is an edge in G_2, see Figure 1. Similarly, the *wreath product* of the graphs G_1 and G_2, denoted by $G_1 \circ G_2$, has vertex set $V(G_1)\times V(G_2)$ in which $(x_1,y_1)(x_2,y_2)$ is an edge whenever x_1x_2 is an edge in G_1 or, $x_1 = x_2$ and y_1y_2 is an edge in G_2, see Figure 2. Note that, $(G_1 \times G_2)(\lambda) \simeq G_1(\lambda)\times G_2 \simeq G_1 \times G_2(\lambda)$. Let $V(G) = \{x^1, x^2, \ldots, x^m\}$ and $V(H) = \{1,2,\ldots,n\}$. For $x^i \in V(G)$, $x^i \times V(H) = \{(x^i, j) \mid j \in \{1,2,\ldots,n\}\}$; we denote (x^i, j) by x^i_j. The set $X^i = \{x^i_1, x^i_2, \ldots, x^i_n\} = x^i \times V(H)$ is called the i^{th} *layer* (of vertices) or i^{th} partite set of $G\times H$ (respectively $G \circ H$), corresponding to the vertex $x^i, 1 \le i \le m$, of $V(G)$. Clearly, $K_m \circ \overline{K}_n$ is the complete m-partite graph in which each of its partite sets has n vertices. Further, $K_m \times K_n = K_m \circ \overline{K}_n - E(nK_m)$, where nK_m denotes n disjoint copies of K_m. As the tensor product is commutative, $K_m \times K_n \simeq K_n \times K_m$.

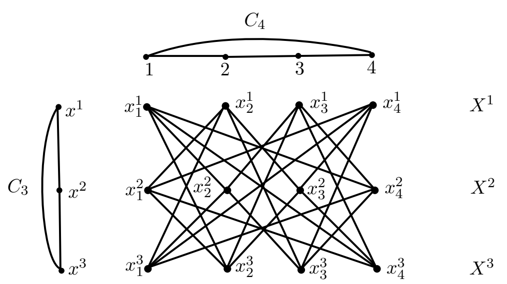


Figure 1. The graph $C_3 \times C_4$.

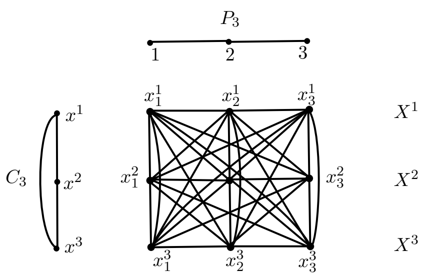


Figure 2. The graph $C_3 \circ P_3$.

In the study of group divisible designs, complete multipartite graphs $K_m \circ \overline{K}_n$ are decomposed into complete subgraphs; but in a modified group divisible design the graph $K_m \times K_n$ is decomposed into complete subgraphs, see [3–6, 24]. In [5], Assaf used modified group divisible designs to construct covering and packing designs, and group divisible designs with block size 5. Further, a C_p-decomposition, p a prime, of the graph $K_m \times K_n$ was used to find a C_p-decomposition of $K_m \circ \overline{K}_n$, see [25]. Moreover, a resolvable $2k$-cycle decomposition of $K_m \times K_n$ and a decomposition of $K_m \times K_n$ into closed trails of length k have been studied in [33, 34]. Besides that, Hamilton cycle decompositions of the graphs $K_m \times K_n, K_{m,m} \times K_n, K_{m,m} \times (K_r \circ \overline{K}_s)$ and $(K_m \circ \overline{K}_n) \times (K_r \circ \overline{K}_s)$ and the directed Hamilton cycle decompositions of the symmetric digraphs $(K_m \times K_n)^*$, $(K_{m,m} \times K_n)^*, (K_{m,m} \times (K_r \circ \overline{K}_s))^*$, $((K_m \times K_n) \times K_r)^*$, $((K_m \circ \overline{K}_n) \times K_r)^*$ and $((K_m \circ \overline{K}_n) \times (K_r \circ \overline{K}_s))^*$ are obtained in [8, 28–31, 35]. Hence $K_m \times K_n$ is proved to be an important proper spanning subgraph of the regular complete multipartite graph $K_m \circ \overline{K}_n$.

Decompositions of complete graphs into specified subgraphs have been studied for a long time. Decompositions of complete graphs into cycles are well-studied. Decompositions of graphs into fixed length cycles and varying length cycles are completely settled for the complete graphs K_n and the complete multigraphs $K_n(\lambda)$. In [1,21,36], it is proved that if n is odd and $k \mid \binom{n}{2}$, $3 \leq k \leq n$, then $C_k \mid K_n$. Further, if n is even and $k \mid \frac{n(n-2)}{2}$, $3 \leq k \leq n$, then $C_k \mid K_n - I$, where I is a perfect matching of K_n. Bryant *et al.* [13,14] completely settled the problem of decomposing $K_n(\lambda), \lambda \geq 1$ into cycles of varying lengths.

Chou *et al.* [16] obtained a necessary and sufficient condition for the existence of a decomposition of $K_{a,b}$ (respectively $K_{m,m} - I$, where $m \geq 3$ is odd and I denotes a perfect matching) into cycles of length $4, 6$ and 8. In [17], Chou and Fu considered a $\{C_4^r, C_{2t}^s\}$-decomposition of $K_{a,b}$ and $K_{m,m} - I$, where m is odd and I denotes a perfect matching. Later, Fu *et al.* [18] proved that the necessary conditions for the existence of a decomposition of $K_{m,m}$ (respectively $K_{m,m} - I$) into cycles of distinct lengths are sufficient whenever m is even (respectively odd) except $m = 4$. Recently, Asplund *et al.* [2] established a necessary and sufficient condition for the existence of a decomposition of $K_{a,b}(\lambda)$ into cycles of arbitrary lengths.

Billington *et al.* [12] proved the existence of a C_5-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$. Muthusamy and Shanmuga Vadivu [32] proved the existence of a C_{2k}-decomposition of $K_m \circ \overline{K}_n$. Very recently, irrespective of the parity of k, the authors of [15] actually solve the existence problem for a C_k-decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ whose cycle-set can be partitioned into 2-regular graphs containing all the vertices except those belonging to one part. A $\{C_4^\alpha, C_5^\beta\}$-decomposition of $K_m \circ \overline{K}_n$ was given by Fu [22]. Moreover, Bahmanian and Šajna [7] showed that if $K_m(\lambda n)$ has a decomposition into cycles of lengths $k_1, k_2, \ldots, k_t$ (plus a perfect matching if $\lambda n(m-1)$ is odd), then $(K_m \circ \overline{K}_n)(\lambda)$ has a decomposition into cycles of lengths $k_1 n, k_2 n, \ldots, k_t n$ (plus a perfect matching if $\lambda n(m-1)$ is odd).

Billington obtained necessary and sufficient conditions for the existence of a $\{C_3^\alpha, C_4^\beta\}$-decomposition of the graph $K_{a,b,c}$ $a \leq b \leq c$, see [10]. Ganesamurthy and Paulraja proved that the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of the graph $K_{a,b,c}, a \leq b \leq c$, see [19]. In [3], Assaf obtained a C_3-decomposition of $(K_m \times K_n)(\lambda)$. For $p \geq 5$, p a prime, existence of C_p-decompositions of $K_m \times K_n$ and $K_m \circ \overline{K}_n$ were proved by Manikandan and Paulraja [25–27]. Existence of a C_k-decomposition of $K_m \times K_n$ is not yet known for general k. In this paper, we obtain a necessary and sufficient condition for the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $(K_m \times K_n)(\lambda)$.

Besides other results, the following main theorem is proved.

Theorem 1. *For $\lambda \geq 1$, $m, n \geq 3$ and $(m,n) \neq (3,3)$, the graph $(K_m \times K_n)$ (λ) admits a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition if and only if* $\lambda(m-1)(n-1) \equiv 0 \pmod 2$ *and* $3\alpha + 6\beta = \frac{\lambda m(m-1)n(n-1)}{2}$.

2. Notation and Terminology

A *latin square* of order n, denoted by L_n, is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \dots, n\}$ such that each row and each column of the array contains each of the symbols in $\{1, 2, \dots n\}$ exactly once. As in [11], a cell (i, j) is termed "empty" if it contains no entry and "filled" otherwise. We represent a *partial latin square* L by a set of ordered triples (i, j, k), where entry k occurs in row i and column j. In this sense (i, j, k) is an element of L. For our convenience, we avoid, if necessary, drawing empty cells of a partial latin square. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i, $1 \leq i \leq n$. A latin square of order k is *cyclic* if the 1^{st} row entries are $a_1, a_2, a_3, \dots, a_k$, then the s^{th} row entries are $a_s, a_{s+1}, a_{s+2}, \dots, a_{s-1}$, in order.

Remark 2. Using a latin square, L_n, of order n, the complete tripartite graph $K_{n,n,n}$, $n \geq 2$, can be decomposed into C_3's as follows. Let the partite sets of $K_{n,n,n}$ be $\{x_1^i, x_2^i, x_3^i, \dots, x_n^i\}$, $1 \leq i \leq 3$. For the $(i,j)^{th}$ cell of L_n with entry k, there corresponds a 3-cycle (x_i^1, x_j^2, x_k^3) in $K_{n,n,n}$. Since L_n has n^2 cells, we obtain n^2 cycles of length 3 which decompose $K_{n,n,n}$. Further, if we consider an idempotent latin square L_n of order $n, n \geq 3$, then the non-diagonal cells of L_n give a C_3-decomposition of $K_3 \times K_n$, as $K_3 \times K_n = K_3 \circ \overline{K}_n - E(nK_3)$.

Remark 3. Consider a cyclic latin square C' of order $n \geq 3$ on the set $\{1, 2, \dots, n\}$, where n is an odd integer and the i^{th} row elements, in order, are $i, i+1, i+2, \dots, i-1$. Let $n = 2k+1$, $k \geq 1$. Now we rename the entries in C' by $j \to 1 + (j-1)k'$, where $k' = k+1$. The resulting latin square, I_n, is idempotent and commutative. Existence of an idempotent commutative latin square of order $2k+1$ is guaranteed in [23]. The entries in the cells in $T = \{(1,2), (2,3), \dots, (k-1,k), (k,1)\}$ is a transversal of I_n. We can extend the latin square I_n to I_{n+1}, $n+1 = 2k+2$, $k \geq 1$, using the method of stripping the transversal T of I_n, see [23]. The resulting latin square I_{n+1}, is idempotent, see Appendix. Then for any $n \geq 3$, we can obtain an idempotent latin square of order n.

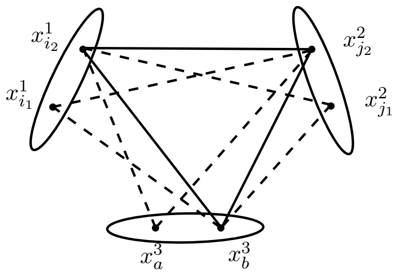
Remark 4. The edges of the triangles corresponding to the entries of each of the partial latin squares of Figure 3, define a graph isomorphic to $K_{2,2,2} - E(K_3)$ and it can be decomposed into three C_3's or, a C_3 and a C_6, see Figure 3, where r_{i_j} and c_{j_k} denote the row i_j and column j_k. Observe that in each case, in each of the three cells of the partial latin square, there are only two distinct symbols.

	c_{j_1}	c_{j_2}
r_{i_1}		b
r_{i_2}	b	a

	c_{j_1}	c_{j_2}
r_{i_1}	a	
r_{i_2}	b	a

	c_{j_1}	c_{j_2}
r_{i_1}	a	b
r_{i_2}		a

	c_{j_1}	c_{j_2}
r_{i_1}	a	b
r_{i_2}	b	



The subgraph of $K_{2,2,2}$ corresponding to the first partial latin square given above.
Normal edges induce a C_3 and broken edges induce a C_6.

Figure 3. $K_{2,2,2} - E(K_3) = C_3 \oplus C_6$, where $K_3 = \langle x_{i_1}^1, x_{j_1}^2, x_a^3 \rangle$.

1. An idempotent latin square of order n without its diagonal entries is denoted by $I_n - D$.
2. An ordered triple (i, j, k), stands for the $(i, j)^{th}$ entry of a latin square is k.
3. At some places, we write the entries of a partial latin square by ordered triples; for example, the three triples $(x_i, y_l, z), (x_k, y_j, z)$ and (x_k, y_l, w) represent the partial latin square

	c_{y_j}	c_{y_l}
r_{x_i}		z
r_{x_k}	z	w

where r_{x_i} represents the row x_i and similarly c_{y_j} represents the column y_j.

3. $\{C_3^\alpha, C_6^\beta\}$-Decomposition of $K_3 \times K_n$

In this section, we prove the existence of a decomposition of $K_3 \times K_n$ into α cycles of length 3 and β cycles of length 6.

The following lemma is a simple observation.

Lemma 5. *The graph $K_3 \times K_3$ cannot be decomposed into 4 copies of C_3 and a C_6.*

Proof. The proof is left to the reader. ■

Lemma 6. *For $(\alpha, \beta) \neq (4, 1)$, the graph $K_3 \times K_3$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. Let the vertex set of the three partite sets of $K_3 \times K_3$ be $\{x_1^i, x_2^i, x_3^i\}$, $1 \leq i \leq 3$. Observe that α is always even and the maximum value of α is 6.

(i) $(\alpha, \beta) = (6, 0)$. Consider the unique idempotent latin square I_3; the non-diagonal entries of I_3 give six edge disjoint copies of C_3, see Remark 2.

(ii) $(\alpha, \beta) = (2, 2)$. A required set of cycles are (x_1^1, x_3^2, x_2^3), (x_2^1, x_3^2, x_1^3), $(x_1^1, x_3^3, x_1^2, x_2^3, x_3^1, x_2^2)$ and $(x_2^1, x_3^3, x_2^2, x_1^3, x_3^1, x_1^2)$.

(iii) $(\alpha, \beta) = (0, 3)$. A set of three cycles of length 6 is $(x_1^1, x_2^2, x_3^1, x_1^2, x_2^1, x_3^2)$, $(x_1^1, x_2^3, x_3^1, x_1^3, x_2^1, x_3^3)$ and $(x_1^2, x_2^3, x_3^2, x_1^3, x_2^2, x_3^3)$. ∎

Lemma 7. *The graph $K_3 \times K_4$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. We consider only the possible values for α and β.

(i) $(\alpha, \beta) = (12, 0)$. The entries of the non-diagonal cells of an idempotent latin square I_4 give a C_3-decomposition of $K_3 \times K_4$, see Remark 2.

(ii) $(\alpha, \beta) \in \{(10, 1), (8, 2), (6, 3), (4, 4)\}$.

Consider the following partial latin square $I_4 - D$ of I_4.

	c_1	c_2	c_3	c_4
r_1		4	2	3
r_2	3		4	1
r_3	4	1		2
r_4	2	3	1	

The cells of $I_4 - D$ are partitioned into the following partial latin squares.

	c_2	c_3
r_1	4	2
r_2		4

	c_1	c_4
r_1		3
r_2	3	1

	c_1	c_4
r_3	4	2
r_4	2	

	c_2	c_3
r_3	1	
r_4	3	1

The edges of $K_3 \times K_4$ corresponding to each of these partial latin squares induces the subgraph isomorphic to $K_{2,2,2} - E(K_3)$, and it admits a decomposition consisting of three C_3's or, a C_3 and a C_6, see Figure 3. Depending on the value of α and β, we choose C_3's or, a C_3 and a C_6 corresponding to each of these partial latin squares to get a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_3 \times K_4$.

(iii) $(\alpha, \beta) \in \{(2, 5), (0, 6)\}$. The graph

$$\begin{aligned} K_3 \times K_4 &= K_3 \times (K_3 \oplus K_{1,3}) \\ &= K_3 \times K_3 \oplus K_3 \times K_{1,3} \\ &= K_3 \times K_3 \oplus K_3 \times K_2 \oplus K_3 \times K_2 \oplus K_3 \times K_2. \end{aligned}$$

As the graph $K_3 \times K_2 \simeq C_6$, and the graph $K_3 \times K_3$ has a $\{C_3^r, C_6^s\}$-decomposition for $(r, s) \neq (4, 1)$, we obtain a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_3 \times K_4$. ∎

Lemma 8. *The graph $K_3 \times K_n, 5 \leq n \leq 11,$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. If $(\alpha,\beta)=(n(n-1),0)$, then the required decomposition exists by Remark 2. So we suppose that $\beta \neq 0$. First we consider $1 \le \beta \le n-1$. Consider an $I_n - D$, where I_n is obtained as in Remark 3; the idempotent latin squares $I_n, 5 \le n \le 11$, are given in Appendix. We use $n-1$ partial latin squares, each having three cells, of $I_n - D, 5 \le n \le 11$, to obtain C_6's, $1 \le \beta \le n-1$; the three cells are chosen so that two cells are filled by a common symbol, (see Remark 4). According to our notation, each set of three triples in the following list of triples gives a partial latin square (of $I_n - D$) having three filled cells.

n = 5. $\{(r_1,c_3,2)(r_1,c_4,5)(r_2,c_3,5)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,1)\}, \{(r_3,c_1,2)(r_3,c_2,5)(r_4,c_1,5)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,1)\}$.

n = 6. $\{(r_1,c_2,6)(r_1,c_3,2)(r_2,c_3,6)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,1)\}, \{(r_3,c_1,2)(r_3,c_2,5)(r_4,c_1,5)\}, \{(r_4,c_2,3)(r_4,c_3,1)(r_5,c_2,1)\}, \{(r_5,c_3,4)(r_6,c_2,4)(r_6,c_3,5)\}$.

n = 7. $\{(r_1,c_3,2)(r_1,c_4,6)(r_2,c_3,6)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,7)\}, \{(r_1,c_6,7)(r_1,c_7,4)(r_2,c_6,4)\}, \{(r_3,c_1,2)(r_3,c_2,6)(r_4,c_1,6)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,7)\}, \{(r_6,c_1,7)(r_6,c_2,4)(r_7,c_1,4)\}$.

n = 8. $\{(r_1,c_2,8)(r_1,c_3,2)(r_2,c_3,8)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,7)\}, \{(r_1,c_6,7)(r_1,c_7,4)(r_2,c_6,4)\}, \{(r_3,c_1,2)(r_3,c_2,6)(r_4,c_1,6)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,7)\}, \{(r_6,c_2,4)(r_6,c_3,1)(r_7,c_2,1)\}, \{(r_7,c_3,5)(r_8,c_2,5)(r_8,c_3,6)\}$.

n = 9. $\{(r_1,c_3,2)(r_1,c_4,7)(r_2,c_3,7)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,8)\}, \{(r_1,c_6,8)(r_1,c_7,4)(r_2,c_6,4)\}, \{(r_1,c_8,9)(r_2,c_7,9)(r_2,c_8,5)\}, \{(r_3,c_1,2)(r_3,c_2,7)(r_4,c_1,7)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,8)\}, \{(r_6,c_1,8)(r_6,c_2,4)(r_7,c_1,4)\}, \{(r_7,c_2,9)(r_8,c_1,9)(r_8,c_2,5)\}$.

n = 10. $\{(r_1,c_2,10)(r_1,c_3,2)(r_2,c_3,10)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,8)\}, \{(r_1,c_6,8)(r_1,c_7,4)(r_2,c_6,4)\}, \{(r_1,c_8,9)(r_2,c_7,9)(r_2,c_8,5)\}, \{(r_3,c_1,2)(r_3,c_2,7)(r_4,c_1,7)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,8)\}, \{(r_6,c_1,8)(r_6,c_2,4)(r_7,c_1,4)\}, \{(r_7,c_2,9)(r_8,c_1,9)(r_8,c_2,5)\}, \{(r_9,c_2,1)(r_9,c_3,6)(r_{10},c_2,6)\}$.

n = 11. $\{(r_1,c_3,2)(r_1,c_4,8)(r_2,c_3,8)\}, \{(r_1,c_5,3)(r_2,c_4,3)(r_2,c_5,9)\}, \{(r_1,c_6,9)(r_1,c_7,4)(r_2,c_6,4)\}, \{(r_1,c_8,10)(r_2,c_7,10)(r_2,c_8,5)\}, \{(r_1,c_9,5)(r_1,c_{10},11)(r_2,c_9,11)\}, \{(r_3,c_1,2)(r_3,c_2,8)(r_4,c_1,8)\}, \{(r_4,c_2,3)(r_5,c_1,3)(r_5,c_2,9)\}, \{(r_6,c_1,9)(r_6,c_2,4)(r_7,c_1,4)\}, \{(r_7,c_2,10)(r_8,c_1,10)(r_8,c_2,5)\}, \{(r_9,c_1,5)(r_9,c_2,11)(r_{10},c_1,11)\}$.

Each of the subgraphs of $K_3 \times K_n$ corresponding to the above $n-1, 5 \le n \le 11$, partial latin squares is isomorphic to $K_{2,2,2} - E(K_3)$, see Figure 3, and it can be decomposed into C_3's or, a C_3 and a C_6 and hence $K_3 \times K_n$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition, when $(\alpha,\beta)=(n(n-1)-2i,i), 5 \le n \le 11, 1 \le i \le n-1$. The filled cells of $I_n - D, 5 \le n \le 11$, which are not covered by the above $n-1$ partial latin squares partition the remaining edges of $K_3 \times K_n$ into 3-cycles, by Remark 2.

Now we complete the proof by induction on $n, n \ge 5$, for $\beta \ge n$. For $n=5$, $K_3 \times K_5 = K_3 \times K_4 \oplus K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2$; we use Lemma 7 and the fact that

$K_3 \times K_2 \simeq C_6$ to complete the proof. The graph $K_3 \times K_{n+1} = K_3 \times (K_n \oplus K_{1,n}) = K_3 \times K_n \oplus K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2$. Now a required decomposition follows by induction applied to $K_3 \times K_n$ and the fact that $K_3 \times K_2 \simeq C_6$. ■

Lemma 9. *If $\beta \geq 4$, then the graph $K_3 \times (K_6 - e)$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. The graph
$$\begin{aligned} K_3 \times (K_6 - e) &= K_3 \times (K_5 \oplus K_{1,4}) \\ &= K_3 \times K_5 \oplus \underbrace{K_3 \times K_2 \oplus \cdots \oplus K_3 \times K_2}_{4-copies}. \end{aligned}$$
As $K_3 \times K_2 \simeq C_6$ and a $\{C_3^r, C_6^s\}$-decomposition of $K_3 \times K_5$ follows by Lemma 8, we have the desired result. ■

Lemma 10. *If $\beta = 2$, then the graph $K_3 \times (K_6 - e)$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. The graph
$$\begin{aligned} K_3 \times (K_6 - e) &= K_3 \times (K_3 \oplus K_3 \oplus K_3 \oplus K_3 \oplus K_2 \oplus K_2) \\ &= K_3 \times K_3 \oplus K_3 \times K_3 \oplus K_3 \times K_3 \oplus K_3 \times K_3 \\ &\quad \oplus K_3 \times K_2 \oplus K_3 \times K_2. \end{aligned}$$
As $K_3 \times K_2 \simeq C_6$, the result follows by Lemma 6. ■

Lemma 11. *If $\beta \neq 1$, then the graph $K_3 \times (K_7 - E(K_3))$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. The graph
$$\begin{aligned} K_3 \times (K_7 - E(K_3)) &= K_3 \times (\underbrace{K_3 \oplus K_3 \oplus \cdots \oplus K_3}_{6-copies}) \\ &= K_3 \times K_3 \oplus \cdots \oplus K_3 \times K_3 \end{aligned}$$
Now the result follows by Lemma 6. ■

Lemma 12. *The cells of the first two rows of $I_n - D$, where $n = 2k + 2$, can be partitioned into $\lfloor \frac{4k+2}{3} \rfloor$ partial latin squares, each of which is one of the form given in Figure 3, together with one or two filled cells depending on n.*

Proof. Let $n = 2k + 2$, $k \geq 1$. Obtain the idempotent latin square I_n and the partial latin square $I_n - D$, as in Remark 3. The entries of the first two rows of $I_n - D$ are shown in Figure 4, see Appendix for $I_n, 5 \leq n \leq 11$. We partition the cells of these two rows of I_{n-D} into $\lfloor \frac{4k+2}{3} \rfloor$ 3-subsets as shown in Figures 5, 6 and 7 according to $n \equiv 0, 2$ or 4 (mod 6), respectively. Each of the subsets has three filled cells having two distinct elements as shown in Remark 4.

	c_1	c_2	c_3	c_4	c_5	$\dots$	c_{2k-2}	c_{2k-1}	c_{2k}	c_{2k+1}	c_{2k+2}
r_1		$2k+2$	2	$k+3$	3	$\dots$	$2k$	k	$2k+1$	$k+1$	$k+2$
r_2	$k+2$		$2k+2$	3	$k+4$	$\dots$	k	$2k+1$	$k+1$	1	$k+3$

Figure 4. First two rows of $I_n - D$.

$n \equiv 0 \pmod 6$:

	c_1	c_2	c_3	c_4	c_5	c_6	$\ldots$	c_{6k-4}	c_{6k-3}	c_{6k-2}	c_{6k-1}	c_{6k}
r_1		$6k$	2	$3k+2$	3	$3k+3$	$\ldots$	$6k-2$	$3k-1$	$6k-1$	$3k$	$3k+1$
r_2	$3k+1$		$6k$	3	$3k+3$	4	$\ldots$	$3k-1$	$6k-1$	$3k$	1	$3k+2$

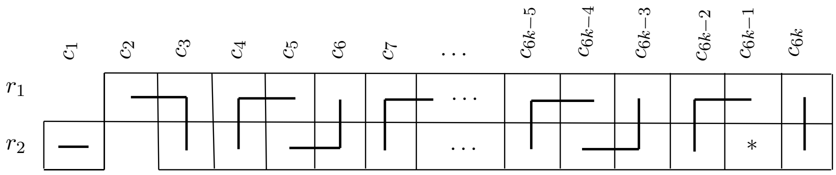


Figure 5. Except the cell with $*$, all other cells are partitioned into 3 cells as shown above, where the last column cells are combined with the first cell of the second row.

$n \equiv 2 \pmod 6$:

	c_1	c_2	c_3	c_4	c_5	$\ldots$	c_{6k-2}	c_{6k-1}	c_{6k}	c_{6k+1}	c_{6k+2}
r_1		$6k+2$	2	$3k+3$	3	$\ldots$	$6k$	$3k$	$6k+1$	$3k+1$	$3k+2$
r_2	$3k+2$		$6k+2$	3	$3k+4$	$\ldots$	$3k$	$6k+1$	$3k+1$	1	$3k+3$

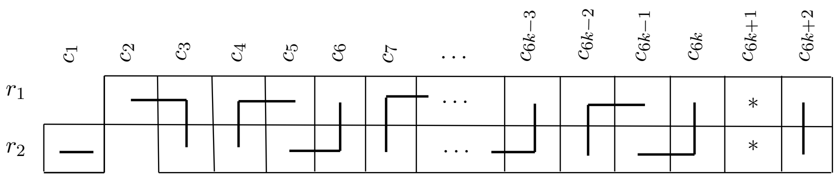


Figure 6. Except the two cells with $*$, all other cells are partitioned into 3 cells as shown above, where the last column cells are combined with the first cell of the second row.

$n \equiv 4 \pmod 6$:

	c_1	c_2	c_3	c_4	c_5	$\ldots$	c_{6k}	c_{6k+1}	c_{6k+2}	c_{6k+3}	c_{6k+4}
r_1		$6k+4$	2	$3k+4$	3	$\ldots$	$6k+2$	$3k+1$	$6k+3$	$3k+2$	$3k+3$
r_2	$3k+3$		$6k+4$	3	$3k+5$	$\ldots$	$3k+1$	$6k+3$	$3k+2$	1	$3k+4$

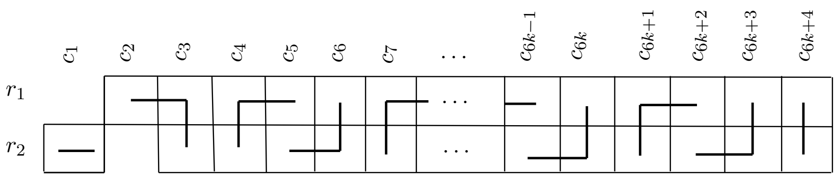


Figure 7. The two cells of the last column cells are combined with the first cell of the second row.

∎

We apply following theorem to prove Theorem 14.

Theorem 13 [19]. *Let $K_{a,b,c}$ be the complete tripartite graph with $a \leq b \leq c$ and let $K_{a,b,c} \neq K_{1,1,c}$, when $c \equiv 1 \pmod 6$ and $c > 1$. If $a \equiv b \equiv c \pmod 6$, then $K_{a,b,c}$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition for any $\alpha \equiv a \pmod 2$, with $0 \leq \alpha \leq ab$.*

Theorem 14. *The graph $K_3 \times K_n$, $n \geq 4$, admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. Since the graph $K_3 \times K_n$ has a C_3-decomposition, we assume that $\beta \geq 1$. Because of Lemmas 7 and 8, we assume that $n \geq 12$.

Case (i): $n \equiv 0 \pmod 4$. Let $n = 4k, k \geq 3$. The graph $K_3 \times K_n = K_3 \times (kK_4 \oplus K_k \circ \overline{K}_4) = k(K_3 \times K_4) \oplus K_3 \times (K_k \circ \overline{K}_4) = G_1 \oplus G_2$, where $G_1 = k(K_3 \times K_4)$ and $G_2 = K_3 \times (K_k \circ \overline{K}_4)$.

The graph $G_2 = K_3 \times (K_k \circ \overline{K}_4) = (K_3 \times K_k) \circ \overline{K}_4 = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \circ \overline{K}_4 = (K_{4,4,4} \oplus K_{4,4,4} \oplus \cdots \oplus K_{4,4,4})$, since $K_3 \mid K_3 \times K_n$. Now invoke Theorem 13 and Lemma 7 to the graphs $K_{4,4,4}$ and G_1, respectively, to complete the proof of this case.

Case (ii): $n \equiv 1 \pmod 4$. Let $n = 4k+1, k \geq 3$. The graph $K_3 \times K_n = K_3 \times (\underbrace{K_5 \oplus K_5 \oplus \cdots \oplus K_5}_{k-copies} \oplus K_k \circ \overline{K}_4) = (K_3 \times K_5) \oplus (K_3 \times K_5) \oplus \cdots \oplus (K_3 \times K_5) \oplus K_3 \times (K_k \circ \overline{K}_4) = G_1 \oplus G_2$, where $G_1 = (K_3 \times K_5) \oplus (K_3 \times K_5) \oplus \cdots \oplus (K_3 \times K_5)$ and $G_2 = K_3 \times (K_k \circ \overline{K}_4) = (K_3 \times K_k) \circ \overline{K}_4 = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \circ \overline{K}_4$. As in Case (i), G_2 is isomorphic to $K_{4,4,4} \oplus \cdots \oplus K_{4,4,4}$.

Now apply Theorem 13 and Lemma 8 to the graphs $K_{4,4,4}$ and G_1, respectively, to complete the proof of this case.

Case (iii): $n \equiv 2 \pmod 4$. Let $n = 4k+2$, $k \geq 3$. First we prove for the case $\beta < 2(k-1) = 2k-2$. Out of the $\lfloor \frac{8k+2}{3} \rfloor$ partial latin squares, each having 3 cells, described in Lemma 12, consider $2k-3$ partial latin squares. The edge induced subgraph of $K_3 \times K_n$, corresponding to each of these $2k-3$ partial latin squares admits three copies of C_3 or, a C_3 and a C_6 and the cells not covered by these partial latin squares, give a C_3-decomposition of the remaining subgraph of $K_3 \times K_n$. Thus we obtain a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $K_3 \times K_n$.

Next consider the case $\beta \geq 2(k-1)$. The graph $K_3 \times K_n = K_3 \times K_{4k+2} = K_3 \times (K_6 \oplus K_6 - e \oplus K_6 - e \oplus \cdots \oplus K_6 - e \oplus K_k \circ \overline{K}_4) = K_3 \times K_6 \oplus K_3 \times K_6 - e \oplus \cdots \oplus K_3 \times K_6 - e \oplus K_3 \times (K_k \circ \overline{K}_4) = G_1 \oplus G_2 \oplus G_3$, where $G_1 = K_3 \times K_6$, $G_2 = (K_3 \times K_6 - e) \oplus (K_3 \times K_6 - e) \oplus \cdots \oplus (K_3 \times K_6 - e)$ and $G_3 = K_3 \times (K_k \circ \overline{K}_4)$. The result follows by Lemmas 8, 9 and 10 as the graph G_3 is isomorphic to the graph G_2 considered in Case (i) above.

Case (iv): $n \equiv 3 \pmod 4$. Let $n = 4k+3, k \geq 3$. If $\beta = 1$, then consider the cells $\{(r_1, c_3, 2)(r_1, c_4, 2k+4)(r_2, c_3, 2k+4)\}$ of $I_{(4k+3)} - D$; the subgraph of

$K_3 \times K_n$ corresponding to these three cells is a C_3 and a C_6, and each of the remaining cells of $I_{4k+3} - D$ gives a C_3.

If $\beta \geq 2$, then $K_3 \times K_n = K_3 \times K_{4k+3} = K_3 \times (K_7 \oplus (K_7 - E(K_3)) \oplus \cdots \oplus (K_7 - E(K_3)) \oplus K_k \circ \overline{K}_4) = K_3 \times K_7 \oplus K_3 \times (K_7 - E(K_3)) \oplus \cdots \oplus K_3 \times (K_7 - E(K_3)) \oplus (K_3 \times (K_k \circ \overline{K}_4)) = G_1 \oplus G_2 \oplus G_3$, where $G_1 = K_3 \times K_7$, $G_2 = K_3 \times (K_7 - E(K_3)) \oplus \cdots \oplus K_3 \times (K_7 - E(K_3))$ and $G_3 = K_3 \times (K_k \circ \overline{K}_4)$. Now apply Lemma 8 to G_1 and Lemma 11 to G_2; the graph G_3 is isomorphic to the graph G_2 in Case (i). ∎

4. $\{C_3^\alpha, C_6^\beta\}$-DECOMPOSITION OF $(K_m \times K_n)(\lambda)$

In this section we prove the existence of a $\{C_3^\alpha, C_6^\beta\}$-decomposition of $(K_m \times K_n)(\lambda)$. We need some lemmas to prove the main theorem.

Lemma 15. *The graph $K_{1,3} \times K_5$ has a decomposition into ten C_6's.*

Proof. Let $V(K_{1,3}) = \{x^1, x^2, x^3, x^4\}$ with the center x^1 and $V(K_5) = \{1, 2, 3, 4, 5\}$. Let $V(K_{1,3} \times K_5) = \bigcup_{i=1}^4 X^i$, where X^i is as defined in the introduction. Let $C = (x_1^1, x_3^3, x_4^1, x_3^2, x_5^1, x_4^4)$ and $C' = (x_1^1, x_2^4, x_5^1, x_1^2, x_4^1, x_2^3)$. Then $\{C, \rho(C), \ldots, \rho^4(C), C', \rho(C'), \ldots, \rho^4(C')\}$ is a C_6-decomposition, where $\rho = (12345)$ and its powers are the permutations acting on the subscripts of the vertices of the cycles C and C', where $\rho(C)$ stands for $\left(x_{\rho(1)}^1, x_{\rho(3)}^3, x_{\rho(4)}^1, x_{\rho(3)}^2, x_{\rho(5)}^1, x_{\rho(4)}^4\right)$. ∎

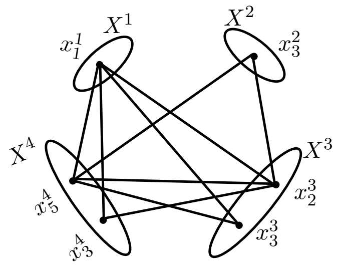
Assaf proved the existence of a C_3-decomposition of $(K_m \times K_n)\,(\lambda)$ whenever the obvious necessary conditions are satisfied, see [3]. The proof of it uses a C_3-decomposition of $K_4 \times K_5$; but the C_3-decomposition of $K_4 \times K_5$ given in Lemma 3.4 of [3] contains a typo. The next lemma contains a proof of C_3-decomposition of $K_4 \times K_5$.

Lemma 16. *The graph $K_4 \times K_5$ has a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

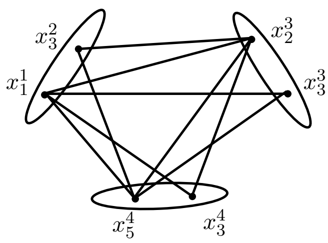
Proof. Let $V(K_4) = \{x^1, x^2, x^3, x^4\}$ and $V(K_5) = \{1, 2, 3, 4, 5\}$. Let vertex set of $K_4 \times K_5$ be as defined in Lemma 15. The eight cycles C^i, $1 \leq i \leq 8$, given below and $\rho, \rho^2, \rho^3, \rho^4$ applied to the subscripts of vertices of the C^i, which we denote by $\rho^j(C^i)$, decompose $K_4 \times K_5$ into 3-cycles, that is, $C^1, \rho(C^1), \ldots, \rho^4(C^1), C^2, \rho(C^2), \ldots, \rho^4(C^2), \ldots, C^8, \rho(C^8), \ldots, \rho^4(C^8)$ is a C_3-decomposition of $K_4 \times K_5$, where $\rho(C)$ is defined as in the previous lemma.

$$C^1 = (x_1^1, x_2^3, x_3^4) \qquad C^2 = (x_1^1, x_3^3, x_5^4) \qquad C^3 = (x_3^2, x_2^3, x_5^4)$$
$$C^4 = (x_1^1, x_2^2, x_5^3) \qquad C^5 = (x_2^2, x_4^3, x_3^4) \qquad C^6 = (x_1^1, x_3^2, x_4^3)$$
$$C^7 = (x_1^1, x_5^2, x_4^4) \qquad C^8 = (x_1^1, x_4^2, x_2^4).$$

First we consider the proof for the case $1 \le \beta \le 10$. Let $G_i = C^{3i-2} \cup C^{3i-1} \cup C^{3i}$, $1 \le i \le 2$, be the subgraph of $K_4 \times K_5$, where cycles $C^j, 1 \le j \le 8$, denote the above 3-cycles. Observe that the edge induced subgraph G_i, $1 \le i \le 2$, is isomorphic to $K_{2,2,2} - E(K_3)$, see Figure 8.



The edge induced subgraph G_1 of $K_4 \times K_5$.



A redrawing of the graph G_1 $G_1 \cong K_{2,2,2} - E(K_3)$.

Figure 8.

Let $\rho = (12345)$ be the permutation on $V(K_5) = \{1,2,3,4,5\}$. Allow $\rho, \rho^2, \rho^3, \rho^4$ to act on the subscripts of the vertices of $G_i, 1 \le i \le 2$, and $C^j, 7 \le j \le 8$, which we denote by $G_i, \rho(G_i), \rho^2(G_i), \rho^3(G_i), \rho^4(G_i), C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j)$, $1 \le i \le 2, 7 \le j \le 8$. For $i = 1,2$, $G_i, \rho(G_i), \rho^2(G_i), \rho^3(G_i), \rho^4(G_i)$ give ten copies of $K_{2,2,2} - E(K_3)$ and for $j = 7,8$, $C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j)$, give ten copies of C_3 in $K_4 \times K_5$. As each $K_{2,2,2} - E(K_3)$ is decomposable into three copies of C_3 or, a C_3 and a C_6, these ten copies of $K_{2,2,2} - E(K_3)$ give β cycles of length 6, where $1 \le \beta \le 10$ and the rest into C_3's.

Next we consider the proof for the case $\beta \ge 11$. As the graph $K_4 \times K_5 = (K_3 \oplus K_{1,3}) \times K_5 = K_3 \times K_5 \oplus K_{1,3} \times K_5$, the lemma follows by Lemmas 8 and 15. ■

Lemma 17. *The graph $K_6 \times K_5$ admits a $\{C_3^\alpha, C_6^\beta\}$-decomposition.*

Proof. Let $V(K_6) = \{x^1, x^2, \ldots, x^6\}$ and $V(K_5) = \{1,2,3,4,5\}$. A set of 20 base cycles for a C_3-decomposition of $K_6 \times K_5$ is given below.

$$
\begin{array}{lll}
C^1 = (x_1^1, x_4^3, x_2^6) & C^2 = (x_1^1, x_3^2, x_5^5) & C^3 = (x_3^2, x_1^3, x_2^6) \\
C^4 = (x_1^3, x_4^4, x_3^6) & C^5 = (x_2^1, x_4^4, x_5^6) & C^6 = (x_2^1, x_5^2, x_1^3) \\
C^7 = (x_2^2, x_4^3, x_5^4) & C^8 = (x_4^3, x_5^5, x_3^6) & C^9 = (x_5^4, x_4^5, x_3^6) \\
C^{10} = (x_1^1, x_3^3, x_2^4) & C^{11} = (x_2^1, x_3^3, x_5^5) & C^{12} = (x_1^2, x_2^4, x_5^5) \\
C^{13} = (x_3^1, x_2^4, x_4^5) & C^{14} = (x_1^2, x_5^3, x_4^5) & C^{15} = (x_5^3, x_2^4, x_3^5) \\
C^{16} = (x_2^1, x_3^2, x_5^4) & C^{17} = (x_2^1, x_4^5, x_1^6) & C^{18} = (x_4^1, x_3^2, x_1^6) \\
C^{19} = (x_1^2, x_2^5, x_3^6) & C^{20} = (x_4^2, x_3^4, x_5^6). &
\end{array}
$$

First we consider the proof for the case $\beta \leq 30$. Let $G_i = C^{3i-2} \cup C^{3i-1} \cup C^{3i}$, $1 \leq i \leq 6$; clearly the edge induced subgraph $G_i, 1 \leq i \leq 6$, of $K_6 \times K_5$, is isomorphic to $K_{2,2,2} - E(K_3)$.

Let $\rho = (12345)$ be a permutation on $V(K_5) = \{1, 2, 3, 4, 5\}$. Then $G_i, \rho(G_i)$, $\rho^2(G_i), \rho^3(G_i), \rho^4(G_i), C^j, \rho(C^j), \rho^2(C^j), \rho^3(C^j), \rho^4(C^j), 1 \leq i \leq 6, 19 \leq j \leq 20$, where $\rho^s(G_i)$ and $\rho^r(C^j)$ have the same meaning as in the proof of Lemma 16, give 30 copies of $K_{2,2,2} - E(K_3)$ and 10 copies of C_3 in $K_6 \times K_5$. Each copy of $K_{2,2,2} - E(K_3)$ is decomposable into C_3's or, a C_3 and a C_6 and using this decomposition of $K_{2,2,2} - E(K_3)$ suitably, we can achieve a required $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition of $K_6 \times K_5$, for $\beta \leq 30$.

Next let $\beta \geq 31$. Clearly, $K_6 \times K_5 = (K_4 \oplus K_3 \oplus K_{1,3} \oplus K_{1,3}) \times K_5 = (K_4 \times K_5) \oplus (K_3 \times K_5) \oplus (K_{1,3} \times K_5) \oplus (K_{1,3} \times K_5)$. By Lemmas 8, 15 and 16, the lemma follows. ■

We quote the following results to prove our main Theorem 1.

Theorem 18 [23]**.** (i) *If* $n \equiv 1$ or 3 (mod 6), *then* K_n *can be decomposed into cycles of length* 3.

(ii) *If* $n \equiv 5$ (mod 6), *then* K_n *can be decomposed into* K_3*'s and a* K_5.

Lemma 19 [20]**.** *If* $n \equiv 0$ or 1 (mod 3), *then* K_n *can be decomposed into* K_3*'s,* K_4*'s and* K_6*'s.*

Theorem 20 [20]**.** *Let* λ *and* $m \geq 3$ *be positive integers. There exists a* K_3*-decomposition of* $K_m(\lambda)$ *if and only if* $\lambda(m-1) \equiv 0$ (mod 2) *and* $\lambda m(m-1) \equiv 0$ (mod 6).

***Proof of Theorem* 1.** $\lambda = 1$. The proof of the necessity is obvious and we prove the sufficiency. If $m = 3$ or $n = 3$, then the result follows by Theorem 14. Since $(m, n) \neq (3, 3)$, we assume that m and n are at least 4. As m or n is odd and the tensor product is commutative, we assume that m is odd. Then $m \equiv 1, 3$ or 5 (mod 6). If $m \equiv 1$ or 3 (mod 6) then the graph

$K_m \times K_n = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \times K_n$, by Theorem 18,

$= K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n$.

Now by Theorem 14 the result follows. If $m \equiv 5$ (mod 6), let $m = 6k + 5$. Since $K_m = K_5 \oplus K_3 \oplus \cdots \oplus K_3$, by Theorem 18, $K_m \times K_n = K_5 \times K_n \oplus K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n$, $n \geq 4$. Because of Theorem 14, it is enough to show that the graph $K_5 \times K_n$ has a $\{C_3^{\alpha}, C_6^{\beta}\}$-decomposition. By the divisibility condition, $n \equiv 0$ or 1 (mod 3). Since $n \equiv 0$ or 1 (mod 3), K_n can be decomposed into K_3's, K_4's and K_6's, by Lemma 19. Then $K_5 \times K_n$ is the edge disjoint union of the graphs $K_5 \times K_3, K_5 \times K_4$ and $K_5 \times K_6$, and now apply Lemmas 8, 16 and 17 to complete the proof.

Next we consider the case $\lambda = 2$. By hypothesis, either $m \equiv 0$ or 1 (mod 3) or $n \equiv 0$ or 1 (mod 3). Without loss of generality, assume that $m \equiv 0$ or 1 (mod 3), as the tensor product is commutative. The graph

$$(K_m \times K_n)(2) \simeq K_m(2) \times K_n = (K_3 \oplus K_3 \oplus \cdots \oplus K_3) \times K_n, \text{ by Theorem 20}$$
$$= (K_3 \times K_n \oplus K_3 \times K_n \oplus \cdots \oplus K_3 \times K_n).$$

The result follows by Theorem 14. Now we consider the case $\lambda = 3$. As λ is odd, either m or n is odd; we assume that m is odd. $(K_m \times K_n)(3) \simeq K_m(3) \times K_n = (K_3 \oplus \cdots \oplus K_n) \times K_n$, by Theorem 20. Now apply Theorem 14, the result follows. The last case is $\lambda = 6$. Edge divisibility condition is satisfied for all m and n and again by applying Theorem 20, the desired result is obtained. This completes the proof. □

Appendix

I_5 :

	c_1	c_2	c_3	c_4	c_5
r_1	1	4	2	5	3
r_2	4	2	5	3	1
r_3	2	5	3	1	4
r_4	5	3	1	4	2
r_5	3	1	4	2	5

I_6 :

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	1	6	2	5	3	4
r_2	4	2	6	3	1	5
r_3	2	5	3	6	4	1
r_4	5	3	1	4	6	2
r_5	6	1	4	2	5	3
r_6	3	4	5	1	2	6

I_7 :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
r_1	1	5	2	6	3	7	4
r_2	5	2	6	3	7	4	1
r_3	2	6	3	7	4	1	5
r_4	6	3	7	4	1	5	2
r_5	3	7	4	1	5	2	6
r_6	7	4	1	5	2	6	3
r_7	4	1	5	2	6	3	7

I_8 :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
r_1	1	8	2	6	3	7	4	5
r_2	5	2	8	3	7	4	1	6
r_3	2	6	3	8	4	1	5	7
r_4	6	3	7	4	8	5	2	1
r_5	3	7	4	1	5	8	6	2
r_6	7	4	1	5	2	6	8	3
r_7	8	1	5	2	6	3	7	4
r_8	4	5	6	7	1	2	3	8

I_9 :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
r_1	1	6	2	7	3	8	4	9	5
r_2	6	2	7	3	8	4	9	5	1
r_3	2	7	3	8	4	9	5	1	6
r_4	7	3	8	4	9	5	1	6	2
r_5	3	8	4	9	5	1	6	2	7
r_6	8	4	9	5	1	6	2	7	3
r_7	4	9	5	1	6	2	7	3	8
r_8	9	5	1	6	2	7	3	8	4
r_9	5	1	6	2	7	3	8	4	9

I_{10} :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
r_1	1	10	2	7	3	8	4	9	5	6
r_2	6	2	10	3	8	4	9	5	1	7
r_3	2	7	3	10	4	9	5	1	6	8
r_4	7	3	8	4	10	5	1	6	2	9
r_5	3	8	4	9	5	10	6	2	7	1
r_6	8	4	9	5	1	6	10	7	3	2
r_7	4	9	5	1	6	2	7	10	8	3
r_8	9	5	1	6	2	7	3	8	10	4
r_9	10	1	6	2	7	3	8	4	9	5
r_{10}	5	6	7	8	9	1	2	3	4	10

I_{11} :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
r_1	1	7	2	8	3	9	4	10	5	11	6
r_2	7	2	8	3	9	4	10	5	11	6	1
r_3	2	8	3	9	4	10	5	11	6	1	7
r_4	8	3	9	4	10	5	11	6	1	7	2
r_5	3	9	4	10	5	11	6	1	7	2	8
r_6	9	4	10	5	11	6	1	7	2	8	3
r_7	4	10	5	11	6	1	7	2	8	3	9
r_8	10	5	11	6	1	7	2	8	3	9	4
r_9	5	11	6	1	7	2	8	3	9	4	10
r_{10}	11	6	1	7	2	8	3	9	4	10	5
r_{11}	6	1	7	2	8	3	9	4	10	5	11

Idempotent latin squares of orders $5, 6, \ldots, 11$ are given above.

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