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ON LOCAL ANTIMAGIC CHROMATIC NUMBER OF CYCLE-RELATED JOIN GRAPHS

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Abstract

An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \ldots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, several sufficient conditions for $\chi_{la}(H) \leq \chi_{la}(G)$ are obtained, where H is obtained from G with a certain edge deleted or added. We then determined the exact value of the local antimagic chromatic number of many cycle-related join graphs.

Keywords: local antimagic labeling, local antimagic chromatic number, cycle, join graphs.

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1. Introduction

A connected graph $G = (V, E)$ is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i.e., a bijection $f : E \to \{1, \dots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$ (with e ranging over all the edges incident to u) has the property that any two adjacent vertices have distinct induced vertex labels (see [1, 2]). Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of u, for short, if no ambiguity occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of f. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) : f$ is a local antimagic labeling of $G\}$.

Let $O_n = \overline{K_n}$ be the empty graph of order $n \geq 1$. For any graph G, the join graph $H = G \vee O_n$ is defined by $V(H) = V(G) \cup \{v_j : 1 \leq j \leq n\}$ and $E(H) = E(G) \cup \{uv_j : u \in V(G), 1 \leq j \leq n\}$. In [1, Theorem 2.16], it was claimed that for any G with order $n \geq 4$,

$$\chi_{la}(G) + 1 \leq \chi_{la}(G \vee O_2) \leq \begin{cases} \chi_{la}(G) + 1 & \text{if } n \text{ is even,} \\ \chi_{la}(G) + 2 & \text{if } n \text{ is odd.} \end{cases}$$

In [4], Lau *et al.* showed that there exists a graph G order $n \geq 3$ such that $\chi_{la}(G \vee O_2) - \chi_{la}(G) = 3 - n \leq 0$. This implies that the above lower bound is invalid. They then showed that $\chi_{la}(G + O_n) \geq \chi(G) + 1$ and the bound is sharp. Several sufficient conditions for the following conjecture to hold were also given.

Conjecture 1.1. *For $n \geq 1$, $\chi_{la}(G \vee O_n) \geq \chi_{la}(G) + 1$ if and only if $\chi(G) = \chi_{la}(G)$.*

Let $G - e$ (or $G + e$) be the graph G with an edge e deleted (or added). As a natural extension, we have obtained in this paper several sufficient conditions for $\chi_{la}(G - e) \leq \chi_{la}(G)$ (or $\chi_{la}(G + e) \leq \chi_{la}(G)$). We then determine the exact value of the local antimagic chromatic number of many cycle related join graphs. We shall use the notation $[a, b] = \{c \in \mathbb{Z} : a \leq c \leq b\}$, for integers $a \leq b$. Unless stated otherwise, all graphs considered in this paper are simple, undirected, connected and of order at least 3. Thus $\chi_{la}(G) \geq 2$ for any graph G. Interested readers may refer to Yu *et al.* [7] for local antimagic labeling of subcubic graphs without isolated edges.

For $m, n \geq 2$, it is well known that a magic (m, n)-rectangle exists if and only if $m \equiv n \pmod 2$ and $(m, n) \neq (2, 2)$ (see [3,6]). Let $a_{i,j}$ be the (i, j)-entry of a magic (m, n)-rectangle with row constant $n(mn + 1)/2$ and column constant $m(mn + 1)/2$.

2. Bounds on Graphs with an Edge Deleted or Added

Observe that K_t, $t \geq 3$, is a complete t-partite graph with $\chi_{la}(K_t) = t$. The contrapositive of the following lemma gives a sufficient condition for a bipartite graph G to have $\chi_{la}(G) \geq 3$.

Lemma 2.1. *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the numbers of vertices of colors x and y, respectively. Then G is a bipartite graph whose sizes of parts are X and Y with $X > Y$, and*

$$xX = yY = \frac{q(q+1)}{2}. \tag{1}$$

Proof. Clearly G is bipartite. Each edge is incident with one vertex of color x and one vertex of color y. Hence we have the equation (1). Since $x < y$, $X > Y$. This completes the proof. ■

Lemma 2.2. *Suppose G is a d-regular graph of size q. If f is a local antimagic labeling of G, then $g = q + 1 - f$ is also a local antimagic labeling of G with $c(f) = c(g)$. Moreover, suppose $c(f) = \chi_{la}(G)$ and if $f(uv) = 1$ or $f(uv) = q$, then $\chi_{la}(G - uv) \leq \chi_{la}(G)$.*

Proof. Let $x, y \in V(G)$. Here, $g^+(x) = d(q+1) - f^+(x)$ and $g^+(y) = d(q+1) - f^+(y)$. Therefore, $f^+(x) = f^+(y)$ if and only if $g^+(x) = g^+(y)$. Thus, g is also a local antimagic labeling of G with $c(g) = c(f)$.

If $f(uv) = q$, then we may consider $g = q+1-f$. So without loss of generality, we may assume that $f(uv) = 1$. Define $h : E(G - uv) \to [1, |E(G)| - 1]$ such that $h(e) = f(e) - 1$ for $e \neq uv$. So, $h^+(x) = f^+(x) - d$ for each vertex x of $G - uv$. Therefore, $f^+(x) = f^+(y)$ if and only if $h^+(x) = h^+(y)$. Thus, h is also a local antimagic labeling of G with $c(h) = c(f)$. Consequently, $\chi_{la}(G - uv) \leq \chi_{la}(G)$. ■

Note that if G is a regular edge-transitive graph, then $\chi_{la}(G - e) \leq \chi_{la}(G)$.

Lemma 2.3. *Suppose G is a graph of size q and f is a local antimagic labeling of G. For any $x, y \in V(G)$, if*

(i) *$f^+(x) = f^+(y)$ implies that* $\deg(x) = \deg(y)$, *and*

(ii) *$f^+(x) \neq f^+(y)$ implies that* $(q+1)(\deg(x) - \deg(y)) \neq f^+(x) - f^+(y)$,

then $g = q + 1 - f$ is also a local antimagic labeling of G with $c(f) = c(g)$.

Proof. For any $x, y \in V(G)$, we have $g^+(x) = \deg(x)(q+1) - f^+(x)$ and $g^+(y) = \deg(y)(q+1) - f^+(y)$. Here $g^+(x) - g^+(y) = (q+1)(\deg(x) - \deg(y)) - (f^+(x) - f^+(y))$. If $f^+(x) = f^+(y)$, then condition (i) implies that $g^+(x) = g^+(y)$. If $f^+(x) \neq f^+(y)$, then condition (ii) implies that $g^+(x) \neq g^+(y)$. Thus, g is also a local antimagic labeling of G with $c(g) = c(f)$. ■

For $t \geq 2$, consider the following conditions for a graph G.

(i) $\chi_{la}(G) = t$ and f is a local antimagic labeling of G that induces a t-independent partition $\bigcup_{i=1}^{t} V_i$ of $V(G)$.

(ii) For each $x \in V_k$, $1 \leq k \leq t$, $\deg(x) = d_k$ satisfying $f^+(x) - d_a \neq f^+(y) - d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a \neq b \leq t$.

(iii) There exist two non-adjacent vertices u, v with $u \in V_i, v \in V_j$ for some $1 \leq i \neq j \leq t$ such that

  (a) $|V_i| = |V_j| = 1$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$; or

  (b) $|V_i| = 1$, $|V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$ except that $\deg(v) = d_j - 1$; or

  (c) $|V_i| \geq 2$, $|V_j| \geq 2$ and $\deg(x) = d_k$ for $x \in V_k$, $1 \leq k \leq t$ except that $\deg(u) = d_i - 1$, $\deg(v) = d_j - 1$,

  each satisfying $f^+(x) + d_a \neq f^+(y) + d_b$, where $x \in V_a$ and $y \in V_b$ for $1 \leq a \neq b \leq t$.

Lemma 2.4. *Let H be obtained from G with an edge e deleted. If G satisfies conditions* (i) *and* (ii) *and $f(e) = 1$, then $\chi(H) \leq \chi_{la}(H) \leq t$.*

Proof. By definition, we have the lower bound. Define $g : E(H) \to [1, |E(H)|]$ such that $g(e') = f(e') - 1$ for each $e' \in E(H)$. Observe that g is a bijection with $g^+(x) = f^+(x) - d_k$ for each $x \in V_k$, $1 \leq k \leq t$. Thus, $g^+(x) = g^+(y)$ if and only if $x, y \in V_k$, $1 \leq k \leq t$. Therefore, g is a local antimagic labeling of H with $c(g) = c(f)$. Thus, $\chi_{la}(H) \leq t$. ■

Lemma 2.5. *Suppose $uv \notin E(G)$. Let H be obtained from G with an edge uv added. If G satisfies conditions* (i) *and* (iii), *then $\chi(H) \leq \chi_{la}(H) \leq t$.*

Proof. By definition, we have the lower bound. Define $g : E(H) \to [1, |E(H)|]$ such that $g(uv) = 1$ and $g(e) = f(e)+1$ for $e \in E(G)$. Observe that g is a bijection with $g^+(x) = f^+(x) + d_k$ for each $x \in V_k$, $1 \leq k \leq t$. Thus, $g^+(x) = g^+(y)$ if and only if $x, y \in V_k$, $1 \leq k \leq t$. Therefore, g is a local antimagic labeling of H with $c(g) = c(f)$. Thus, $\chi_{la}(H) \leq t$. ■

In [1, Theorem 2.11], the authors showed that for any two distinct integers $m, n \geq 2$, $\chi_{la}(K_{m,n}) = 2$ if and only if $m \equiv n \pmod 2$. Let $K^-_{m,n}$ be the graph $K_{m,n}$ with an edge deleted. From the proof of [1, Theorem 2.11] and by Lemma 2.4, the following result is obvious.

Corollary 2.6. *For any two distinct integers $m, n \geq 2$ and $m \equiv n \pmod 2$, $\chi_{la}(K^-_{m,n}) = 2$.*

3. Cycle-Related Join Graphs

Consider the join graph $C_m \vee O_n$ with $V(C_m) = \{u_i : 1 \le i \le m\}$, $V(O_n) = \{v_j : 1 \le j \le n\}$ and $E(C_m \vee O_n) = \{u_i u_{i+1} : 1 \le i \le m\} \cup \{u_i v_j : 1 \le i \le m, 1 \le j \le n\}$, where $u_{m+1} = u_1$. Let $e_i = u_i u_{i+1}$ for $1 \le i \le m$. So $e_m = u_m u_1$. We shall keep these notations in this section unless stated otherwise.

Theorem 3.1. *For odd $m, n \ge 3$, $\chi_{la}(C_m \vee O_n) = 4$.*

Proof. Define an edge labeling $f : E(C_m \vee O_n) \to [1, mn + m]$ such that $f(e_{2i-1}) = i$ $(1 \le i \le (m+1)/2)$ and $f(e_{2i}) = m + 1 - i$ $(1 \le i \le (m-1)/2)$ and that $f(u_i v_j)$ is the (i, j)-entry of a magic (m, n)-rectangle containing integers in $[m + 1, mn + m]$ with row sum constant $n(mn + 1)/2 + mn$ and column sum constant $m(mn + 1)/2 + m^2$. One can check that

(i) $f^+(v_j) = m(mn + 1)/2 + m^2$,

(ii) $f^+(u_1) = n(mn + 1)/2 + mn + (m + 3)/2$,

(iii) $f^+(u_i) = n(mn + 1)/2 + mn + m + 1$ for even i, and

(iv) $f^+(u_i) = n(mn + 1)/2 + mn + m + 2$ for odd $i \ge 3$.

Suppose $m \le n$. We have $m(mn+1)/2+m^2 < n(mn+1)/2+mn+(m+3)/2 < n(mn + 1)/2 + mn + m + 1 < n(mn + 1)/2 + mn + m + 2$. So, $\chi_{la}(G) \le 4$.

Suppose $m > n$. We have $m(mn + 1)/2 + m^2 = n(mn + 1)/2 + mn + (m - n)m + (m - n)(mn + 1)/2 > n(mn + 1)/2 + mn + m + 2$. So, $\chi_{la}(G) \le 4$.

Since $\chi_{la}(G) \ge \chi(G) = 4$, we have $\chi_{la}(G) = 4$. ■

Corollary 3.2. *For odd $m, n \ge 3$, if $H = (C_m \vee O_n) - e$ where $e \notin E(C_m)$, then $\chi_{la}(H) = 4$.*

Proof. Note that $G = C_m \vee O_n$ has size $mn + m$ and every vertex belonging to C_m (or O_n) has degree $n + 2$ (or m). Let f be the local antimagic labeling as defined in the proof of Theorem 3.1. We can check that f satisfies the conditions of Lemma 2.3. Therefore, $g = mn+m+1-f$ is also a local antimagic labeling of G with $c(g) = 4$ such that $g(e) = 1$ for an edge $e \notin E(C_m)$. It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we have $4 = \chi(H) \le \chi_{la}(H) \le 4$. Thus, the result holds. ■

Theorem 3.3. *For $m \ge 2$ and $n \ge 1$, $\chi_{la}(C_{2m} \vee O_{2n}) = 3$.*

Proof. Let $G = C_{2m} \vee O_{2n}$. Define an edge labeling $f : E(G) \to [1, 4mn + 2m]$ such that $f(e_h) = h$ for $1 \le h \le 2m$ and $f(u_h v_k)$ is given below, for $1 \le h \le 2m$ and $1 \le k \le 2n$.

We define $f(u_1 v_1) = 2m + 1$ and $f(u_{2i-1} v_1) = 4m - 2i + 3$ for $2 \le i \le m$. For $1 \le i \le m$, define

(i) $f(u_{2i-1}v_2) = 6m - 2i + 1$,

(ii) $f(u_{2i-1}v_{2j-1}) = 2m(j-1) + 2i$ and $f(u_{2i-1}v_{2j}) = 2m(2n+1-j) - 2i + 2$, for $2 \le j \le n$,

(iii) $f(u_{2i}v_1) = 2m(2n+1) - 2i + 2$ and $f(u_{2i}v_2) = 4mn - 2i + 2$,

(iv) $f(u_{2i}v_{2j-1}) = 2m(2n-j+3) - 2i + 1$ and $f(u_{2i}v_{2j}) = 2m(j+1) + 2i - 1$, for $2 \le j \le n$.

One may check that f is a bijection. Observe that

(i) $f(u_{2i-1}v_1) + f(u_{2i-1}v_2) = 10m - 4i + 4$ and $f(u_{2i}v_1) + f(u_{2i}v_2) = 8mn + 2m - 4i + 4$ for $1 \le i \le m$,

(ii) $f(u_{2i}v_{2j-1}) + f(u_{2i}v_{2j}) = 4m(n+2)$ for $1 \le i \le m$ and $2 \le j \le n$,

(iii) $f(u_{2i-1}v_{2j-1}) + f(u_{2i-1}v_{2j}) = 4mn + 2$ for $1 \le i \le m$ and $2 \le j \le n$.

Thus

$$\begin{aligned}
f^+(u_1) &= f(e_1) + f(e_{2m}) + f(u_1v_1) + f(u_1v_2) + \sum_{j=2}^{n}(4mn+2) \\
&= 4mn^2 - 4mn + 2n + 10m - 1; \\
f^+(u_{2i-1}) &= f(e_{2i-2}) + f(e_{2i-1}) + (10m - 4i + 4) + \sum_{j=2}^{n}(4mn+2) \\
&= (4i-3) + (10m - 4i + 4) + (4mn+2)(n-1) \\
&= 4mn^2 - 4mn + 2n + 10m - 1 \text{ if } 2 \le i \le m; \\
f^+(u_{2i}) &= f(e_{2i-1}) + f(e_{2i}) + (8mn + 2m - 4i + 4) + \sum_{j=2}^{n} 4m(n+2) \\
&= (8mn + 2m + 3) + 4m(n+2)(n-1) \\
&= 4mn^2 + 12mn - 6m + 3 \text{ if } 1 \le i \le m;
\end{aligned}$$

$$\begin{aligned}
f^+(v_1) &= (2m+1) + \sum_{i=2}^{m}(4m - 2i + 3) + \sum_{i=1}^{m}(4mn + 2m - 2i + 2) \\
&= 4m^2n + 4m^2 + m; \\
f^+(v_2) &= \sum_{i=1}^{m}(4mn + 6m - 4i + 3) = 4m^2n + 4m^2 + m; \\
f^+(v_k) &= \sum_{i=1}^{m}(4mn + 4m + 1) = 4m^2n + 4m^2 + m \text{ if } 3 \le k \le 2n.
\end{aligned}$$

Now, let $g_1 = f^+(u_{2i-1}) = 4mn^2 - 4mn + 2n + 10m - 1$, $g_2 = f^+(u_{2i}) = 4mn^2 + 12mn - 6m + 3$, and $g_3 = f^+(v_j) = 4m^2n + 4m^2 + m$. Clearly, $g_1 < g_2$.

Suppose $n \geq m$. We have $g_2 - g_3 = 4mn(n-m) + m(12n-4m-7) + 6 > 0$. Suppose $m > n$. $g_3 - g_2 = 4mn(m-n-2) + m(4m-4n+7) - 3$. When $m-n \geq 2$, clearly $g_3 > g_2$. For $m - n = 1$, $g_3 - g_2 = -4m^2 + 15m - 3 \neq 0$.

We now consider $g_3 - g_1 = 2n[2m(m-n) - 1] + m(4n+4m-9) + 1$. If $m \geq n$, then $g_3 - g_1 \geq 2n(m-1) + m(2n+4m-9) + 1 > 0$. Suppose $n > m$. Now $g_1 - g_3 = 4mn(n-m-2) + 4m(n-m) + 2n + 9m - 1 > 0$ when $n - m \geq 2$. When $n - m = 1$, $g_1 - g_3 = -4m^2 + 11m + 1 \neq 0$.

Thus, $\chi_{la}(G) \leq 3$. Since $\chi_{la}(G) \geq \chi(G) = 3$, we have $\chi_{la}(G) = 3$. ∎

Corollary 3.4. *For $m \geq 2$, $n \geq 1$, if $H = (C_{2m} \vee O_{2n}) - e$, then $\chi_{la}(H) = 3$, where e is an edge of $C_{2m} \vee O_{2n}$.*

Proof. Note that $G = C_{2m} \vee O_{2n}$ has size $4mn+2m$ where every vertex belonging to C_{2m} (or O_{2n}) has degree $2n+2$ (or $2m$). Let f be the local antimagic labeling as defined in the proof of Theorem 3.3. Suppose $e \in E(C_{2m})$. It is straightforward to check that f satisfies the conditions of Lemma 2.4. Thus, we have $3 = \chi(H) \leq \chi_{la}(H) \leq 3$. Suppose $e \notin E(C_{2m})$. We can check that f satisfies the conditions of Lemma 2.3. Therefore, $g = 4mn + 2m + 1 - f$ is also a local antimagic labeling of G with $c(g) = 3$ such that $g(e) = 1$. It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we have $3 = \chi(H) \leq \chi_{la}(H) \leq 3$. Thus, the result holds. ∎

Since for odd $m, n \geq 3$, $\chi_{la}(C_m \vee O_n) = \chi_{la}(C_m) + 1 = \chi(C_m) + 1$, and for even $n \geq 2$, $\chi_{la}(C_m \vee O_n) = \chi_{la}(C_m) = \chi(C_m) + 1$, Theorems 3.1 and 3.3 provide further evidence that Conjecture 1.1 holds.

Note that $C_m \vee O_1 = W_m$, the wheel graph of order $m + 1 \geq 4$. In [4, Theorem 3.1], the authors proved that $\chi_{la}(W_m) = 3$ if $m \equiv 0 \pmod 4$. In [1, Theorem 2.14], the authors proved that $\chi_{la}(W_m) = 3$ if $m \equiv 2 \pmod 4$, and $\chi_{la}(W_m) = 4$ if m is odd. We note that for $m \equiv 1 \pmod 4$, the defined local antimagic labeling f (or f_3 in the proof) has three errors that should be corrected as $f(v_iv) = (8m + 5 - i)/4$ for $i \equiv 1 \pmod 4, i \neq 1$; $f(v_iv) = (7m + 4 - i)/4$ for $i \equiv 3 \pmod 4$; and $f^+(v_i) = (11m + 13)/4$ for odd $i \neq 1$. Moreover, for $m \equiv 3 \pmod 4$, the induced vertex label for v_i, $i \neq 1$ is odd, should be $9(m + 1)/4$.

Theorem 3.5.

$$\chi_{la}(W_4 - e) = \begin{cases} 3 & \text{if } e \notin E(C_4), \\ 4 & \text{otherwise.} \end{cases}$$

Proof. The graph in Figure 1 shows that $W_4 - e$ admits a local antimagic labeling f with $c(f) = 3$ so that $\chi_{la}(W_4 - e) = 3$ if $e \notin E(C_4)$.

Suppose $e \in E(C_4)$. Without loss of generality we may assume that $e = u_4u_1$. Suppose there were a local antimagic labeling f of $W_4 - e$ with $c(f) = 3$. Then

$f^+(v_1) = c$, $f^+(u_1) = f^+(u_3) = a$ and $f^+(u_2) = f^+(u_4) = b$, where a, b, c are distinct.

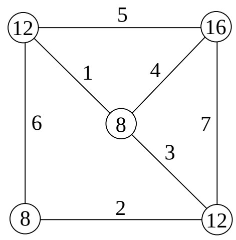


Figure 1. $W_4 - e$.

Clearly

$$(2) \qquad 28 = \sum_{i=1}^{7} i = 2a + f(v_1u_2) + f(v_1u_4) = 2b + f(v_1u_1) + f(v_1u_3).$$

Thus, $f(v_1u_2) \equiv f(v_1u_4) \pmod 2$ and $f(v_1u_1) \equiv f(v_1u_3) \pmod 2$.

It is easy to check that $\{f(u_1u_2), f(u_2u_3), f(u_3u_4)\} \neq \{2, 4, 6\}$. So we may assume that $f(v_1u_1)$ and $f(v_1u_3)$ are odd, and $f(v_1u_2)$ and $f(v_1u_4)$ are even. Under these conditions and from (2) we have $9 \leq a \leq 11$ and $8 \leq b \leq 12$.

1. Suppose $a = 9$. Then $f(v_1u_2) + f(v_1u_4) = 10$ and hence $\{f(v_1u_2), f(v_1u_4)\} = \{4, 6\}$. This implies that $f(u_1u_2) = 2$ and $f(v_1u_1) = 7$. If $f(v_1u_2) = 4$ and $f(v_1u_4) = 6$, then $f(u_2u_3) = f(u_3u_4)$ which is impossible. Thus $f(v_1u_2) = 6$ and $f(v_1u_4) = 4$. This implies that $9 \leq 2+6+f(u_2u_3) = b = 4+f(u_3u_4) \leq 9$. Hence $b = 9 = a$ which is a contradiction.
2. Suppose $a = 10$. We have $\{f(v_1u_1), f(u_1u_2)\} = \{3, 7\}$ and $\{f(v_1u_3), f(u_2u_3), f(u_3u_4)\} = \{1, 4, 5\}$. Since $f(v_1u_2)+f(v_1u_4) = 8$, $\{f(v_1u_2), f(v_1u_4)\} = \{2, 6\}$. Since $b \geq 8$, $f(v_1u_4) = 6$. Hence $f(v_1u_2) = 2$. Since $a \neq b$, $f(u_3u_4) = 5$ and hence $f(u_2u_3) = 4$. Now $f^+(u_2) \neq b = 11$, which is a contradiction.
3. Suppose $a = 11$. We have $f(v_1u_2) + f(v_1u_4) = 6$. This implies that $\{f(v_1u_2), f(v_1u_4)\} = \{2, 4\}$. Since 4 is occupied and $f(v_1u_1)+f(u_1u_2) = 11$, $f(v_1u_1) = 5$ and $f(u_1u_2) = 6$. Also we have $\{f(v_1u_3), f(u_2u_3), f(u_3u_4)\} = \{1, 3, 7\}$. Since $b \geq 8$, $f(u_3u_4) = 7$. Since $b \neq a$, $f(v_1u_4) = 2$. Now $b = 9$ and $f^+(u_2) \geq 10$ which yields a contradiction.

As a conclusion, $\chi_{la}(W_4 - e) \geq 4$. Note that from the discussion above, we have obtained a local antimagic labeling g for $W_4 - e$ with $c(g) = 4$. ■

Theorem 3.6. *Let e be an edge of W_m. For even $m \geq 6$, $\chi_{la}(W_m - e) = 3$.*

Proof. Consider $m = 6$. In Figure 2, we have the local antimagic labelings f with $c(f) = 3$ for the two cases of $W_6 - e$.

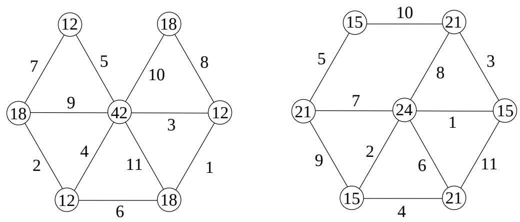


Figure 2. $W_6 - e$ with $c(f) = 3$.

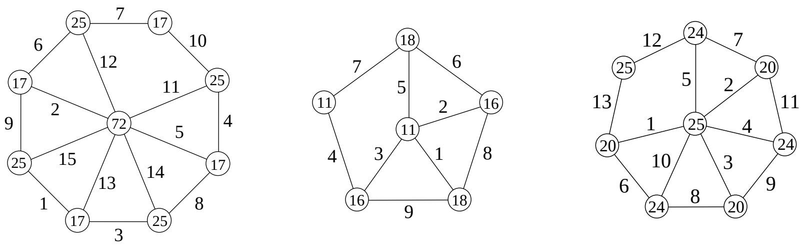
Thus, $\chi_{la}(W_6 - e) = 3$.

Consider $m \geq 8$. We have two cases.

Case (a) $e \in E(C_m)$. By [4, Theorem 3.1] and [1, Theorem 2.14] and the proofs, we have $\chi_{la}(W_m) = 3$ such that the corresponding local antimagic labeling f has $f(u_1u_2) = 1$. By symmetry we may let $e = u_1u_2$. By Lemma 2.4, we get $\chi_{la}(W_m - e) \leq 3$. Since $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$, $\chi_{la}(W_m - e) = 3$.

Case (b) $e \notin E(C_m)$. For $m = 8$, the graph in Figure 3(a) shows that $W_8 - e$ admits a local antimagic labeling g with $c(g) = 3$. Thus, $\chi_{la}(W_8 - e) = 3$.

Consider $m \geq 10$. By [4, Theorem 3.1] and [1, Theorem 2.14] and the proofs, we know that W_m admits a local antimagic labeling f with $f(v_1u_2) = 2m$ if $m \equiv 0 \pmod 4$, and $f(v_1u_4) = 2m$ if $m \equiv 2 \pmod 4$. By symmetry we may let $e = v_1u_2$ if $m \equiv 0 \pmod 4$, and $e = v_1u_4$ if $m \equiv 2 \pmod 4$. It is straightforward to check the conditions of Lemma 2.4. By Lemma 2.4, we get $\chi_{la}(W_m - e) \leq 3$. Since $\chi_{la}(W_m - e) \geq \chi(W_m - e) = 3$, $\chi_{la}(W_m - e) = 3$. ■



(a) $W_8 - v_1u_1$. (b) $W_5 - v_1u_5$. (c) $W_7 - v_1u_7$.

Figure 3. Some wheels with a spoke deleted.

Theorem 3.7. *Suppose $m \geq 3$ is odd. If $e \notin E(C_m)$, then*

$$\chi_{la}(W_m - e) = \begin{cases} 3 & \text{for } m = 3, 5, 7; \\ 4 & \text{otherwise.} \end{cases}$$

If $e \in E(C_m)$, then $3 \le \chi_{la}(W_m - e) \le 4$.

Proof. Suppose $e \notin E(C_m)$. Note that $\chi_{la}(W_m - e) \ge \chi(W_m - e) = 3$. Suppose the equality holds. Let $m = 2k+1$ and f is a local antimagic labeling of $W_{2k+1} - e$ with $c(f) = 3$. Without loss of generality, assume $e = v_1u_{2k+1}$. Thus, we must have $f^+(v_1) = f^+(u_{2k+1}) \ne f^+(u_1) = f^+(u_3) = \cdots = f^+(u_{2k-1}) \ne f^+(u_2) = f^+(u_4) = f^+(u_{2k})$. Thus, $k(2k+1) \le f^+(v_1) = f^+(u_{2k+1}) \le 8k+1$ giving us $1 \le k \le 3$. Thus, $\chi_{la}(W_m - e) \ge 4$ for $m \ge 9$. For $m = 3$, $W_3 - e \cong K_{1,1,2}$. The labeling is obvious. For $m = 5$, the labeling in Figure 3(b) shows that $\chi_{la}(W_5 - v_1u_5) = 3$. For $m = 7$, the labeling in Figure 3(c) shows that $\chi_{la}(W_7 - v_1u_7) = 3$.

Consider $m \ge 9$. By [1, Theorem 2.14] and the proof, we know that W_m admits a local antimagic labeling f with $c(f) = 4$. Moreover, $f(v_1u_5) = 2m$ if $m \equiv 1 \pmod 4$, and $f(v_1u_2) = 2m$ if $m \equiv 3 \pmod 4$. It is straightforward to check the conditions of Lemmas 2.3 and 2.4. By Lemma 2.3, we know W_m admits a local antimagic labeling g with $g(v_1u_5) = 1$ if $m \equiv 1 \pmod 4$, and $g(v_1u_2) = 1$ if $m \equiv 3 \pmod 4$. By Lemma 2.4, we get $\chi_{la}(W_m - e) = 4$.

Suppose $e \in E(C_m)$. By [1, Theorem 2.14] and the proof, together with Lemma 2.4, we know that $\chi_{la}(W_m - e) \le 4$. ■

Theorem 3.8. *For odd $m, n \ge 3$, $\chi_{la}(C_m \vee C_n) = 6$.*

Proof. Since $C_m \vee C_n$ and $C_n \vee C_m$ are isomorphic, we may assume that $n \le m$. Suppose $V(C_m \vee C_n) = V(C_m \vee O_n)$ and $E(C_m \vee C_n) = E(C_m \vee O_n) \cup \{e'_j = v_jv_{j+1} : 1 \le j \le n\}$ as in Theorem 3.1, where $v_{n+1} = v_1$. Let f be the local antimagic labeling of $C_m \vee O_n$ defined in the proof of Theorem 3.1. Define an edge labeling $g : E(C_m \vee C_n) \to [1, m+mn+n]$ such that $g(e) = f(e)$ for $e \in E(C_m \vee O_n)$ and $g(e'_j) = m + mn + f(e_j)$. One may check that g is a bijection. Moreover,

(i) $g^+(u_1) = g_1 = n(mn+1)/2 + mn + (m+3)/2$,
(ii) $g^+(u_i) = g_2 = n(mn+1)/2 + mn + m + 1$ for even i,
(iii) $g^+(u_i) = g_3 = n(mn+1)/2 + mn + m + 2$ for odd $i \ge 3$,
(iv) $g^+(v_1) = g_4 = m(mn+1)/2 + m^2 + 2(m+mn) + (n+3)/2$,
(v) $g^+(v_j) = g_5 = m(mn+1)/2 + m^2 + 2(m+mn) + n + 1$ for even j, and
(vi) $g^+(v_j) = g_6 = m(mn+1)/2 + m^2 + 2(m+mn) + n + 2$ for odd $j \ge 3$.

Clearly $g_k < g_{k+1}$ for $1 \le k \le 5$. Thus, $\chi_{la}(C_m \vee C_n) \le 6$. Since $\chi_{la}(C_m \vee C_n) \ge \chi(C_m \vee C_n) = \chi(C_m) + \chi(C_n) = 6$, we have $\chi_{la}(C_m \vee C_n) = 6$. ■

In [5], Haslegrave proved that every connected graph $G \ne K_2$ admits a local antimagic labeling which implies that $\chi_{la}(K_n) = n$ for all $n \ge 3$. We now consider the join graph $C_m \vee K_n$ with $V(C_m \vee K_n) = V(C_m \vee O_n)$ and

$E(C_m \vee K_n) = E(C_m \vee O_n) \cup \{v_iv_j : 1 \le i < j \le n\}$. In [1], the authors showed that $\chi_{la}(C_m \vee K_1) = 4$ for odd $m \ge 3$.

Theorem 3.9. *For odd* $m, n \ge 3$, $\chi_{la}(C_m \vee K_n) = n + 3$.

Proof. Let f be the local antimagic labeling of $C_m \vee O_n$ defined in the proof of Theorem 3.1. Let $h : E(K_n) \to [1, n(n-1)/2]$ be a local antimagic labeling of K_n. Note that $h^+(v_j)$ are distinct for $1 \le j \le n$. Define an edge labeling $g : E(C_m \vee K_n) \to [1, mn + m + n(n-1)/2]$ such that $g(e) = f(e)$ for $e \in E(C_m \vee O_n)$ and $g(e) = h(e) + mn + m$ for $e \in E(K_n)$. Note that $g^+(v_j) = f^+(v_j) + h^+(v_j) + (n-1)(mn+n)$. Since $f^+(v_j)$ are the same and $h^+(v_j)$ are distinct, $g^+(v_j)$ are distinct for $1 \le j \le n$.

Moreover,

(i) $g^+(u_1) = n(mn+1)/2 + mn + (m+3)/2$,

(ii) $g^+(u_i) = n(mn+1)/2 + mn + m + 1$ for even i,

(iii) $g^+(u_i) = n(mn+1)/2 + mn + m + 2$ for odd $i \ge 3$, and

(iv) $g^+(v_j) = f^+(v_j) + h^+(v_j) + (n-1)(mn+n) \ge m(mn+1)/2 + m^2 + (n-1)(nm+m) + n(n-1)/2$.

It is easy to show that $g^+(v_j) > g^+(u_i)$ for all $1 \le i \le m, 1 \le j \le n$. Thus, $\chi_{la}(C_m \vee K_n) \le n+3$. Since $\chi_{la}(C_m \vee K_n) \ge \chi(C_m \vee K_n) = n+3$, the theorem holds. ■

Theorem 3.10. *For* $m \ge 2, n \ge 1$, $\chi_{la}(C_{2m} \vee K_{2n}) = 2n + 2$.

Proof. Let f be the local antimagic labeling of $C_{2m} \vee O_{2n}$ defined in the proof of Theorem 3.3.

Suppose $n = 1$. Define an edge labeling $g : E(C_{2m} \vee K_2) \to [1, 6m+1]$ such that $g(e) = f(e)$ for $e \in E(C_{2m} \vee O_2)$ and $g(v_1v_2) = 6m+1$. We now swap the labels of $g(u_1v_1) = 2m+1$ and $g(u_1v_2) = 6m-1$ to get $g^+(u_{2i-1}) = 10m+1$ and $g^+(u_{2i}) = 10m+3$ for $1 \le i \le m$ and $g^+(v_1) = 8m^2 + 11m - 1$ and $g^+(v_2) = 8m^2 + 3m + 3$. Thus, $\chi_{la}(C_{2m} \vee K_2) \le 4$.

Now, consider $n \ge 2$. Let $h : E(K_{2n}) \to [1, n(2n-1)]$ be a local antimagic labeling of K_{2n}. Note that $h^+(v_j)$ are distinct for $1 \le j \le 2n$. Define an edge labeling $g : E(C_{2m} \vee K_{2n}) \to [1, 4mn + 2m + n(2n-1)]$ such that $g(e) = f(e)$ for $e \in E(C_{2m} \vee O_{2n})$ and $g(e) = h(e) + 4mn + 2m$ for $e \in E(K_{2n})$.

By the same argument as in the proof of Theorem 3.9, we obtain that $g^+(v_j)$ are distinct for $1 \le j \le 2n$.

From Theorem 3.3 we have $g^+(u_{2i}) = 4mn^2 - 4mn + 2n + 10m - 1 < g^+(u_{2i-1}) = 4mn^2 + 12mn - 6m + 3$ for $1 \le i \le m$. Moveover, $g^+(v_j) = f^+(v_j) + h^+(v_j) + (2n-1)(4mn+2m) \ge 4m^2n + 4m^2 + m + (2n-1)(4mn+2m) + n(2n-1)$ for each j. Clearly $g^+(v_j) > g^+(u_{2i-1})$ for $1 \le i \le m$ and $1 \le j \le 2n$.

Thus, $\chi_{la}(C_{2m} \vee K_{2n}) \le 2n+2$. Since $\chi_{la}(C_{2m} \vee K_{2n}) \ge \chi(C_{2m} \vee K_{2n}) = 2n+2$, the theorem holds. ■

Conjecture 3.11. *For $n \ge 2$, $\chi_{la}(G \vee K_n) \ge \chi_{la}(G) + n$ if and only if $\chi_{la}(G) = \chi(G)$.*

For $n \ge 2$, let M_{2n} be the Möbius ladder obtained from $C_{2n} = u_1u_2 \cdots u_nv_1 v_2 \cdots v_nu_1$ by adding the edges $u_iv_i, 1 \le i \le n$.

Theorem 3.12. *For odd $n \ge 3$, $\chi_{la}(M_{2n}) = 3$.*

Proof. Note that M_{2n} has size $3n$, and is bipartite with parts of the same size. Thus, by Lemma 2.1, $\chi_{la}(M_{2n}) \ge 3$.

Suppose $n = 3$, we get a local antimagic labeling by assigning the edges u_1u_2, u_2u_3, u_3v_1, v_1v_2, v_2v_3, v_3u_1, u_1v_1, u_2v_2, u_3v_3 by $1, 5, 4, 8, 6, 7, 3, 9, 2$, respectively. Clearly, the induced vertex coloring has three distinct colors, namely 11, 15, 23.

Suppose $n \ge 5$. Define a bijection $f : E(M_{2n}) \to [1, 3n]$ such that $f(u_1v_n) = \frac{3(n+1)}{2}$, $f(u_nv_1) = n$, $f(v_1v_2) = n+1$ and that

(i) $f(u_iu_{i+1}) = i$ for odd $i \in [1, n-2]$,

(ii) $f(u_iu_{i+1}) = \frac{3n+3-i}{2}$ for even $i \in [2, n-1]$,

(iii) $f(v_iv_{i+1}) = i$ for even $i \in [2, n-1]$,

(iv) $f(v_iv_{i+1}) = 2n - \frac{i-3}{2}$ for odd $i \in [3, n-2]$,

(v) $f(u_iv_i) = \frac{5n+2-i}{2}$ for odd $i \in [1, n]$,

(vi) $f(u_iv_i) = 3n + 1 - \frac{i}{2}$ for even $i \in [2, n-1]$.

One can verify that $f^+(u_i) = f^+(v_j) = \frac{9n+3}{2}$ for even $i \in [2, n-1]$ and odd $j \in [1, n]$; $f^+(u_i) = f^+(v_2) = 4n+3$ for odd $i \in [1, n]$ and $f^+(v_j) = 5n+3$ for even $j \in [4, n-1]$. Therefore, $\chi_{la}(M_{2n}) \le 3$. Hence, the theorem holds. ■

Corollary 3.13. *For odd $n \ge 3$, $\chi_{la}(M_{2n} - e) = 3$.*

Proof. By Lemma 2.1, we know that $\chi_{la}(M_{2n} - e) \ge 3$. Note that there are two possible graphs obtained by deleting an edge from M_{2n} (if $n > 3$), but using Lemma 2.2 with reference to the smallest label deals with one, and the largest label deals with the other. Therefore, we have $\chi_{la}(M_{2n} - e) \le 3$. Thus, $\chi_{la}(M_{2n} - e) = 3$. ■

Note that $M_4 = K_4$ with $\chi_{la}(M_4) = 4$.

Conjecture 3.14. *For even $n \ge 4$, $\chi_{la}(M_{2n}) = 4$.*

Theorem 3.15. *For $n \ge 1$, $\chi_{la}(M_6 \vee O_{2n}) = 3$.*

Proof. Let $V(M_6 \vee O_{2n}) = \{u_i : 1 \le i \le 6\} \cup \{v_j : 1 \le j \le 2n\}$ and $E(M_6 \vee O_{2n}) = \{u_iu_{i+1} : 1 \le i \le 5\} \cup \{u_1u_6, u_1u_4, u_2u_5, u_3u_6\} \cup \{u_iv_j : 1 \le i \le 6, 1 \le j \le 2n\}$. Define a bijection $g : E(M_6 \vee O_{2n}) \to [1, 12n+9]$ such that $g(u_1u_2) = 1$, $g(u_2u_3) = 3$, $g(u_3u_4) = 4$, $g(u_4u_5) = 2$, $g(u_5u_6) = 8$, $g(u_1u_6) = 5$, $g(u_1u_4) = 9$, $g(u_2u_5) = 7$, $g(u_3u_6) = 6$ and $g(u_iv_j) = f(u_iv_j) + 3$ for $1 \le i \le 6, 1 \le j \le 2n$, where f is the function as defined in the proof of Theorem 3.3 by taking $m = 3$.

One can easily check that $g^+(u_1) = 15 + \sum_{j=1}^{2n} f(u_1v_j) + 3(2n) = 12n^2 - 4n + 37$. Similarly, we get $g^+(u_3) = g^+(u_5) = g^+(u_1)$. Furthermore, for $i = 2, 4, 6$, we also have $g^+(u_i) = 12n^2 + 42n - 7$, whereas $g^+(v_j) = 36n + 57$ for $1 \le j \le 2n$. Clearly, g is a local antimagic labeling with $c(g) = 3$. Therefore, $\chi_{la}(M_6 \vee O_{2n}) \le 3$. Since M_6 is bipartite, we have $\chi_{la}(M_6 \vee O_{2n}) \ge \chi(M_6 \vee O_{2n}) = \chi(M_6) + \chi(O_{2n}) = 3$. Thus, $\chi_{la}(M_6 \vee O_{2n}) = 3$. ■

Corollary 3.16. *For $n \ge 1$, $\chi_{la}((M_6 \vee O_{2n}) - e) = 3$.*

Proof. Let $G = (M_6 \vee O_{2n}) - e$. We note that $\chi_{la}(G) \ge \chi(G) = 3$. Since M_6 is edge-transitive, we only need to consider (i) $e \notin E(M_6)$, and (ii) $e \in E(M_6)$.

In (i), it is straightforward to check the conditions of Lemma 2.3. By Lemma 2.3, we know $M_6 \vee O_{2n}$ admits a local antimagic labeling $h = 12n+10-g$ with $c(h) = c(g) = 3$, where g is as defined in the proof of Theorem 3.15. Now,

$$h^+(u_i) = \begin{cases} 12n^2 + 60n - 7 & \text{if } i = 1, 3, 5, \\ 12n^2 + 14n + 37 & \text{if } i = 2, 4, 6, \end{cases}$$

$h^+(v_j) = 36n + 3$ for $1 \le j \le 2n$, and $h(uv) = 1$ for an edge $uv \notin E(M_6)$. It is straightforward to check the condition of Lemma 2.4. By Lemma 2.4, we have $\chi_{la}(G) = 3$.

In (ii), it is straightforward to check the condition of Lemma 2.4. By Lemma 2.4, we have $\chi_{la}(G) = 3$. ■

For $m \ge 3$, $n \ge 1$, let $G(m, n)$ be the graph obtained from $C_m \vee O_n$ by deleting the edges u_mv_j, $1 \le j \le n$. We can also view $G(m, n)$ as the graph obtained from $C_{m-1} \vee O_n$ by subdividing one of the cycle edges. Note that $G(m, 1)$ is the graph W_m with a spoke deleted. By Theorems 3.5 and 3.6, we have $\chi_{la}(G(2m, 1)) = 3$ for $m \ge 2$. Moreover, by Theorem 3.7, we have determined the value of $\chi_{la}(G(2m+1, 1))$ for $m \ge 1$.

Theorem 3.17. *For $n \ge 1$, $\chi_{la}(G(4, n)) = 3$.*

Proof. When $n = 1$, we have proved the result in Theorem 3.5. So we may assume that $n \ge 2$. Since $\chi(G(4, n)) \ge 3$, it suffices to provide a local antimagic labeling f for $G(4, n)$ with $c(f) = 3$.

For $n = 4k - 1$, $k \geq 1$, the labeling matrix of $G(4, 3)$ under f is given below.

	u_1	u_2	u_3	u_4	v_1	v_2	v_3	$f^+(u_i)$
u_1	$*$	8	$*$	9	5	1	13	36
u_2	8	$*$	7	$*$	3	12	4	34
u_3	$*$	7	$*$	10	11	6	2	36
u_4	9	$*$	10	$*$	$*$	$*$	$*$	19
$f^+(v_j)$	$*$	$*$	$*$	$*$	19	19	19	

The following tables are the first 4 rows of the labeling matrix of $G(4, 4k-1)$ under f, where $k \geq 3$.

	u_1	u_2	u_3	u_4	v_1	v_2	$\cdots$	v_k	v_{k+1}	v_{k+2}	$\cdots$	v_{2k}
u_1	$*$	$10k+1$	$*$	$6k$	$8k$	$8k-1$	$\cdots$	$7k+1$	$9k$	$9k-1$	$\cdots$	$8k+1$
u_2	$10k+1$	$*$	$4k$	$*$	1	3	$\cdots$	$2k-1$	$2k+1$	$2k+3$	$\cdots$	$4k-1$
u_3	$*$	$4k$	$*$	$12k+1$	$10k$	$10k-1$	$\cdots$	$9k+1$	$7k$	$7k-1$	$\cdots$	$6k+1$
u_4	$6k$	$*$	$12k+1$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$

	v_{2k+1}	v_{2k+2}	$\cdots$	v_{3k-2}	v_{3k-1}	v_{3k}	$\cdots$	v_{4k-4}
u_1	$12k$	$12k-1$	$\cdots$	$11k+3$	$5k+1$	$5k$	$\cdots$	$4k+4$
u_2	2	4	$\cdots$	$2k-4$	$2k-2$	$2k$	$\cdots$	$4k-8$
u_3	$6k-1$	$6k-2$	$\cdots$	$5k+2$	$11k+2$	$11k+1$	$\cdots$	$10k+5$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$

	v_{4k-3}	v_{4k-2}	v_{4k-1}	$f^+(u_i)$
u_1	$4k-6$	$4k+2$	$4k+1$	$32k^2+k-10$
u_2	$10k+4$	$10k+3$	$10k+2$	$8k^2+16k+21$
u_3	$4k+3$	$4k-4$	$4k-2$	$32k^2+k-10$
u_4	$*$	$*$	$*$	$18k+1$

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k+1$, i.e., the v_j-column sum, for $1 \leq j \leq 4k-1$. This labeling can be applied to $k = 2$ (the block-columns for v_{2k+1} to v_{4k-4} do not appear). The following shows the assignment for $G(4, 7)$.

	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$f^+(u_i)$
u_1	$*$	21	$*$	12	16	15	18	17	2	10	9	120
u_2	21	$*$	8	$*$	1	3	5	7	24	23	22	114
u_3	$*$	8	$*$	25	20	19	14	13	11	4	6	120
u_4	12	$*$	25	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	37
$f^+(v_j)$	$*$	$*$	$*$	$*$	37	37	37	37	37	37	37	

For $n = 4k + 1$, $k \geq 1$, the labeling matrix for $G(4, 5)$ is given next.

	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	v_5	$f^+(u_i)$
u_1	$*$	4	$*$	16	10	9	8	11	13	71
u_2	4	$*$	6	$*$	1	3	17	12	15	58
u_3	$*$	6	$*$	14	19	18	5	7	2	71
u_4	16	$*$	14	$*$	$*$	$*$	$*$	$*$	$*$	30
$f^+(v_j)$	$*$	$*$	$*$	$*$	30	30	30	30	30	

Similarly, we show the first 4 rows of the labeling matrix of $G(4, 4k+1)$ under f, where $k \geq 3$.

	u_1	u_2	u_3	u_4	v_1	v_2	$\cdots$	v_{k-2}
u_1	$*$	$10k+6$	$*$	$12k+7$	$8k+4$	$8k+3$	$\cdots$	$7k+7$
u_2	$10k+6$	$*$	$4k+2$	$*$	1	3	$\cdots$	$2k-5$
u_3	$*$	$4k+2$	$*$	$6k+3$	$10k+5$	$10k+4$	$\cdots$	$9k+8$
u_4	$12k+7$	$*$	$6k+3$	$*$	$*$	$*$	$\cdots$	$*$

	v_{k-1}	v_k	$\cdots$	v_{2k-4}	v_{2k-3}	v_{2k-2}	v_{2k-1}	v_{2k}	v_{2k+1}
u_1	$9k+7$	$9k+6$	$\cdots$	$8k+10$	$6k+8$	$6k+7$	$6k+6$	$6k+5$	$4k+1$
u_2	$2k-3$	$2k-1$	$\cdots$	$4k-9$	$4k-7$	$4k-5$	$4k-3$	$4k-1$	$6k+4$
u_3	$7k+6$	$7k+5$	$\cdots$	$6k+9$	$8k+9$	$8k+8$	$8k+7$	$8k+6$	$8k+5$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$	$*$

	v_{2k+2}	v_{2k+3}	$\cdots$	v_{3k+1}	v_{3k+2}	v_{3k+3}	$\cdots$	v_{4k+1}	$f^+(u_i)$
u_1	$12k+6$	$12k+5$	$\cdots$	$11k+7$	$5k+2$	$5k+1$	$\cdots$	$4k+3$	$32k^2+41k+12$
u_2	2	4	$\cdots$	$2k$	$2k+2$	$2k+4$	$\cdots$	$4k$	$8k^2+22k+12$
u_3	$6k+2$	$6k+1$	$\cdots$	$5k+3$	$11k+6$	$11k+5$	$\cdots$	$10k+7$	$32k^2+41k+12$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$	$18k+10$

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k+10$, for $1 \leq j \leq 4k+1$. This labeling can be applied to $k = 2$ (the block-columns for v_1 to v_{2k-4} do not appear). The following shows the assignment for $G(4, 9)$.

	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	$f^+(u_i)$
u_1	$*$	26	$*$	31	20	19	18	17	9	30	29	12	11	222
u_2	26	$*$	10	$*$	1	3	5	7	16	2	4	6	8	88
u_3	$*$	10	$*$	15	25	24	23	22	21	14	13	28	27	222
u_4	31	$*$	15	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	46
$f^+(v_j)$	$*$	$*$	$*$	$*$	46	46	46	46	46	46	46	46	46	

For $n = 4k+2$, the following tables are the first 4 rows of the labeling matrix of $G(4, 4k+2)$ under f, where $k \geq 1$.

	u_1	u_2	u_3	u_4	v_1	v_2	$\cdots$	v_k
u_1	$*$	$8k+6$	$*$	$12k+9$	$10k+7$	$10k+6$	$\cdots$	$9k+8$
u_2	$8k+6$	$*$	$12k+10$	$*$	1	3	$\cdots$	$2k-1$
u_3	$*$	$12k+10$	$*$	$6k+4$	$8k+5$	$8k+4$	$\cdots$	$7k+6$
u_4	$12k+9$	$*$	$6k+4$	$*$	$*$	$*$	$\cdots$	$*$

	v_{k+1}	v_{k+2}	$\cdots$	v_{2k}	v_{2k+1}	v_{2k+2}	v_{2k+3}	$\cdots$	v_{3k+1}
u_1	$7k+5$	$7k+4$	$\cdots$	$6k+6$	$6k+5$	$12k+8$	$12k+7$	$\cdots$	$11k+9$
u_2	$2k+1$	$2k+3$	$\cdots$	$4k-1$	$4k+1$	2	4	$\cdots$	$2k$
u_3	$9k+7$	$9k+6$	$\cdots$	$8k+8$	$8k+7$	$6k+3$	$6k+2$	$\cdots$	$5k+4$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$\cdots$	$*$

	v_{3k+2}	v_{3k+3}	$\cdots$	v_{4k+1}	v_{4k+2}	$f^+(u_i)$
u_1	$5k+3$	$5k+2$	$\cdots$	$4k+4$	$4k+3$	$32k^2+55k+23$
u_2	$2k+2$	$2k+4$	$\cdots$	$4k$	$10k+8$	$8k^2+36k+25$
u_3	$11k+8$	$11k+7$	$\cdots$	$10k+9$	$4k+2$	$32k^2+55k+23$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$18k+13$

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k+13$, for $1 \le j \le 4k+2$. This labeling can be applied to $k = 0$. The following shows the assignment for $G(4,2)$.

	u_1	u_2	u_3	u_4	v_1	v_2	$f^+(u_i)$
u_1	$*$	6	$*$	9	5	3	23
u_2	6	$*$	10	$*$	1	8	25
u_3	$*$	10	$*$	4	7	2	23
u_4	9	$*$	4	$*$	$*$	$*$	13
$f^+(v_j)$	$*$	$*$	$*$	$*$	13	13	

For $n = 4k$, the following tables are the first 4 rows of the labeling matrix of $G(4,4k)$ under f, where $k \ge 2$.

	u_1	u_2	u_3	u_4	v_1	v_2	$\cdots$	v_{k-1}
u_1	$*$	$10k+3$	$*$	$12k+4$	$10k+2$	$10k+1$	$\cdots$	$9k+4$
u_2	$10k+3$	$*$	$6k+2$	$*$	1	3	$\cdots$	$2k-3$
u_3	$*$	$6k+2$	$*$	$6k+1$	$8k+2$	$8k+1$	$\cdots$	$7k+4$
u_4	$12k+4$	$*$	$6k+1$	$*$	$*$	$*$	$\cdots$	$*$

	v_k	v_{k+1}	$\cdots$	v_{2k-2}	v_{2k-1}	v_{2k}	v_{2k+1}	v_{2k+2}	$\cdots$	v_{3k-1}
u_1	$7k+3$	$7k+2$	$\cdots$	$6k+5$	$6k+4$	$6k+3$	$12k+3$	$12k+2$	$\cdots$	$11k+5$
u_2	$2k-1$	$2k+1$	$\cdots$	$4k-5$	$4k-3$	$4k-1$	2	4	$\cdots$	$2k-2$
u_3	$9k+3$	$9k+2$	$\cdots$	$8k+5$	$8k+4$	$8k+3$	$6k$	$6k-1$	$\cdots$	$5k+2$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$*$	$*$	$\cdots$	$*$

	v_{3k}	v_{3k+1}	$\cdots$	v_{4k-2}	v_{4k-1}	v_{4k}	$f^+(u_i)$
u_1	$5k+1$	$5k$	$\cdots$	$4k+3$	$4k+2$	$4k$	$32k^2+23k+3$
u_2	$2k$	$2k+2$	$\cdots$	$4k-4$	$4k-2$	$10k+4$	$8k^2+24k+9$
u_3	$11k+4$	$11k+3$	$\cdots$	$10k+6$	$10k+5$	$4k+1$	$32k^2+23k+3$
u_4	$*$	$*$	$\cdots$	$*$	$*$	$*$	$18k+5$

It is easy to check that $f^+(u_4) = f^+(v_j) = 18k+5$, for $1 \le j \le 4k$. Again, this labeling can be applied to $k = 1$. The following shows the assignment for

$G(4,4)$.

	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	$f^+(u_i)$
u_1	$*$	13	$*$	16	10	9	6	4	58
u_2	13	$*$	8	$*$	1	3	2	14	41
u_3	$*$	8	$*$	7	12	11	15	5	58
u_4	16	$*$	7	$*$	$*$	$*$	$*$	$*$	23
$f^+(v_j)$	$*$	$*$	$*$	$*$	23	23	23	23	

Since $f^+(u_1) = f^+(u_3) \neq f^+(u_2) \neq f^+(u_4) = f^+(v_j), 1 \leq j \leq n$, we have $c(f) = 3$. The proof is complete. ∎

Note that $P_3 \vee O_{n+1}$ can be obtained from $G(4, n)$ by adding the edge u_2u_4. By Lemma 2.5, the following is obtained.

Corollary 3.18. *If $G \equiv P_3 \vee O_{n+1}$, then $\chi_{la}(G) = 3$.*

Problem 3.19. Determine $\chi_{la}(P_m \vee O_n)$ for $m \geq 4, n \geq 2$.

Theorem 3.20. *For* (i) $m \geq 3$, $n \geq 4$, (ii) $m \geq 21$, $n = 3$, *and* (iii) $m \geq 4$, $n = 2$, $\chi_{la}(G(2m, 2n-1)) = 4$.

Proof. Note that $\chi_{la}(G(2m, 2n-1)) \geq \chi(G(2m, 2n-1)) = 3$. Suppose f is a local antimagic labeling of $G(2m, 2n-1)$ with $c(f) = 3$. We may have (I) $a = f^+(u_{2i-1}), 1 \leq i \leq m$; $b = f^+(v_j) = f^+(u_{2m}), 1 \leq j \leq 2n-1$; $c = f^+(u_{2i}), 1 \leq i < m$; or (II) $a = f^+(u_{2i-1}), 1 \leq i \leq m$; $b = f^+(v_j), 1 \leq j \leq 2n-1$; $c = f^+(u_{2i}), 1 \leq i \leq m$. Here a, b, c are distinct. Now, every v_j is adjacent to $2m-1$ vertices of C_{2m}.

For (I), $\sum_{j=1}^{2n-1} f^+(v_j) \geq 1 + 2 + \cdots + (2n-1)(2m-1) = (2n-1)(2m-1)(2mn-m-n+1)$. So,

$$(2m-1)(2mn-m-n+1) \leq b = f^+(u_{2m}) \leq 8mn - 4n + 1 \tag{3}$$

giving $n \leq \frac{(2m-1)(m-1)+1}{(2m-1)(2m-5)}$. By simple calculus, we have $n \leq \frac{11}{5}$. When $n = 2$, we get $m = 3$. This is not a case.

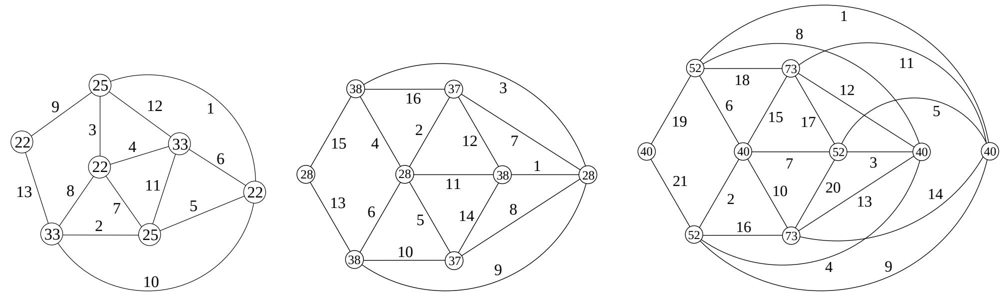
For (II), there are exactly $(2n-1)(m-1) + 2m - 2 = 2mn + m - 2n - 1$ edges incident to the vertices u_{2i} for $1 \leq i \leq m-1$. Each label of these edges contributes to the sum $\sum_{i=1}^{m-1} f^+(u_{2i})$ exactly once. Thus, $(m-1)c \geq \frac{1}{2}(2mn + m - 2n - 1)(2mn + m - 2n)$. Therefore, we will get

$$(2n+1)(2mn+m-2n) \leq 2c = 2f^+(u_{2m}) \leq 16mn - 8n + 2. \tag{4}$$

However, if $n \geq 5$ and $m \geq 3$, $(2n+1)(2mn+m-2n) \geq 11(2mn+m-2n) \geq 16mn + 18n + 11m - 22n = 16mn - 4n + 11m$, contradicting (4). When $n = 4$, we get $m = 2$, contradicting $m \geq 3$. When $n = 3$, we get $2 \leq m \leq 20$, contradicting $m \geq 21$. So, $\chi_{la}(G(2m, 2n-1)) \geq 4$ under each of the given condition.

Define $f : E(G(2m, 2n-1)) \to [1, 4mn-2n+1]$ such that $f(u_{2m}u_1) = (2m-1)(2n-1)+1$, $f(u_{2i}u_{2i+1}) = (2m-1)(2n-1)+i+1$ for $1 \leq i \leq m-1$, $f(u_{2i-1}u_{2i}) = (2m-1)(2n-1)+2m+1-i$ for $1 \leq i \leq m$ and $f(u_iv_j) = a_{i,j}$, $1 \leq i \leq 2m-1, 1 \leq j \leq 2n-1$, where $a_{i,j}$ is the (i,j)-entry of a $(2m-1, 2n-1)$-magic rectangle with constant row sum $(2n-1)(2mn-m-n+1)$ and constant column sum $(2m-1)(2mn-m-n+1)$. One may check that f is a bijection with $g_1 = f^+(v_j) = (2m-1)(2mn-m-n+1)$ for $1 \leq j \leq 2n-1$, $g_2 = f^+(u_{2i}) = (2n-1)(2mn-m-n+1)+2(2m-1)(2n-1)+2m+2 = (2n+3)(2mn-m-n+1)+2m$ for $1 \leq i \leq m-1$, $g_3 = f^+(u_{2i-1}) = (2n-1)(2mn-m-n+1)+2(2m-1)(2n-1)+2m+1 = (2n+3)(2mn-m-n+1)+2m-1$ for $1 \leq i \leq m$ and $g_4 = f^+(u_{2m}) = 2(2m-1)(2n-1)+m+2 = 4(2mn-m-n+1)+m$. Clearly, $g_2 > g_3 > g_4$. It is routine to verify that $g_1 \neq g_2, g_3, g_4$. Thus, $\chi_{la}(G(2m, 2n-1)) \leq 4$. The theorem holds. ∎

Example 3.21. The following are labelings that give $\chi_{la}(G(5,2)) = \chi_{la}(G(6,2)) = \chi_{la}(G(6,3)) = 3$.



Note that $G(5,2)$ and $G(6,2)$ are two graphs we have not considered before.

Problem 3.22. For $m \geq 5$, find $\chi_{la}(G(m,n))$ for $G(m,n)$ not being a graph in Theorem 3.20 and Example 3.21.

Little is known about bipartite graphs G with $\chi_{la}(G) = 2$ (see [1, Theorems 2.11 and 2.12]). For $m \geq 2, i \geq 1$, let $B(n_1, n_2, \ldots, n_m)$ be the union of K_{2,n_i} with bipartition (X_i, Y_i), where $X_i = \{x_{i-1}, x_i\}$, $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}\}$ and $x_m = x_0$.

It is known from [1, Theorem 2.8 and Theorem 2.12] that $\chi_{la}\big(B\big(1^{[m]}\big)\big) = \chi_{la}(C_{2m}) = 3$ and $\chi_{la}\big(B\big(n^{[2]}\big)\big) = \chi_{la}(K_{2,2n}) = 2$ for $n \geq 2$. The following theorem gives another family of bipartite graphs with χ_{la} equal to 2.

Theorem 3.23. *Suppose $m \geq 3$ and $n \geq 2$. We have $\chi_{la}\big(B\big(n^{[m]}\big)\big) = 2$ if n is even or both m and n are odd; $2 \leq \chi_{la}\big(B\big(n^{[m]}\big)\big) \leq 3$ for odd n and even m.*

Proof. First note that the edges in each $K_{2,n}$ are $x_{i-1}y_{i,j}$ and $x_iy_{i,j}$ for $1 \le i \le m, 1 \le j \le n$.

Suppose $n \ge 2$ is even. Define a bijection $f : E(G) \to [1, 2mn]$ such that

$$f(x_{i-1}y_{i,j}) = \begin{cases} (i-1)n + j & \text{for odd } j \in [1, n-1], \\ (2m - i + 1)n - j + 1 & \text{for even } j \in [2, n], \end{cases}$$

$$f(x_iy_{i,j}) = \begin{cases} (2m - i + 1)n - (j-1) & \text{for odd } j \in [1, n-1], \\ (i-1)n + j & \text{for even } j \in [2, n], \end{cases}$$

where $1 \le i \le m$.

Recall that $x_m = x_0$. It is easy to verify that $f^+(y_{i,j}) = 2mn + 1$ and $f^+(x_i) = 2mn^2 + n$ for $1 \le i \le m, 1 \le j \le n$. Hence, $\chi_{la}(G) \le 2$. Since $\chi_{la}(G) \ge \chi(G) = 2$, we have $\chi_{la}(G) = 2$ for even $n \ge 2$.

Suppose n is odd and m is odd. Let A be a magic (m, n)-rectangle. For $1 \le i \le m$, let $(f(x_iy_{i,1}), \dots, f(x_i, y_{i,n}))$ be the i-th row of A and let $f(x_{i-1}y_{i,j}) = 2mn + 1 - f(x_iy_{i,j})$ for $1 \le j \le n$. Clearly $f^+(y_{i,j}) = 2mn + 1$ for $1 \le i \le m$ and $1 \le j \le n$. Since the row sum of A is $n(mn+1)/2$, $f^+(x_i) = n(2mn+1)$ for $1 \le i \le m$. Here, $\chi_{la}(G) \le 2$ and hence $\chi_{la}(G) = 2$.

Suppose n is odd and m is even. Define a bijection $f : E(G) \to [1, 2mn]$

$$f(x_{i-1}y_{i,j}) = (i-1)n + j,$$
$$f(x_iy_{i,j}) = (2m - i + 1)n - j + 1,$$

where $1 \le i \le m$.

It is easy to verify that $f^+(y_{i,j}) = 2mn + 1$, $f^+(x_0) = n(mn + n + 1)$ and $f^+(x_i) = n(2mn + n + 1)$ for $1 \le i \le m - 1$. Thus, $\chi_{la}(G) \le 3$. ■

Example 3.24. The following is a local antimagic labeling according to the construction described in the proof above, which induces a 2-coloring for $B(3^{[3]})$.

$$A = \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}.$$

	$y_{1,1}$	$y_{1,2}$	$y_{1,3}$	$y_{2,1}$	$y_{2,2}$	$y_{2,3}$	$y_{3,1}$	$y_{3,2}$	$y_{3,3}$
x_1	2	7	6	10	14	18	*	*	*
x_2	*	*	*	9	5	1	15	16	11
x_3	17	12	13	*	*	*	4	3	8

It is clear that each row sum is 57 and each column sum is 19.

Example 3.25. The following is a local antimagic labeling inducing a 2-coloring for $B(3^{[4]})$.

	$y_{1,1}$	$y_{1,2}$	$y_{1,3}$	$y_{2,1}$	$y_{2,2}$	$y_{2,3}$	$y_{3,1}$	$y_{3,2}$	$y_{3,3}$	$y_{4,1}$	$y_{4,2}$	$y_{4,3}$
x_1	1	5	17	23	19	10	*	*	*	*	*	*
x_2	*	*	*	2	6	15	22	16	14	*	*	*
x_3	*	*	*	*	*	*	3	9	11	21	18	13
x_4	24	20	8	*	*	*	*	*	*	4	7	12

It is easy to see that the row sum is always 75 and the column sum is always 25.

Problem 3.26. Determine $\chi_{la}(B(n_1, n_2, \ldots, n_m))$ for $B(n_1, n_2, \ldots, n_m) \neq B(n^{[m]})$.

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