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MINIMAL GRAPHS WITH RESPECT TO GEOMETRIC DISTANCE REALIZABILITY

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Abstract

A graph G is minimal non-unit-distance graph if there is no drawing of G in Euclidean plane having all edges of unit length, but, for each edge e of G, $G - e$ has such a drawing. We prove that, for infinitely many n, the number of non-isomorphic n-vertex minimal non-unit-distance graphs is at least exponential in n.

Keywords: unit-distance graph, odd-distance graph, Euclidean plane.

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1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges, and their drawings in the plane. By *drawing* $D = D(G)$ of a graph $G = (V, E)$ in the plane, we mean a function ϕ defined on $V \cup E$ which assigns each vertex $v \in V$ a point $\phi(v) \in \mathbb{R}^2$, and, each edge $uv \in E$ is mapped to a simple curve $\phi(uv) \subset \mathbb{R}^2$ with endpoints $\phi(u), \phi(v)$. It is usually assumed that ϕ is injective on V and, for each edge $uv \in E$ and each $w \in V, w \neq u, v$, $\phi(w) \notin \phi(uv)$ holds; a drawing which violates some of these two properties is called *degenerate*. When mentioning distances between points $\phi(x), \phi(y)$ (which correspond to vertices x, y of G) in the plane, we refer to the Euclidean distance (here denoted as $dist(x, y)$) unless specified otherwise. All graph-theoretic terms which are not defined here are used in accordance with [11].

To describe properties of graphs of a graph family $\mathcal{G}$, one aims to obtain a complete characterization of members of $\mathcal{G}$, usually in terms of forbidden subgraphs, induced subgraphs, minors or topological minors. A related source of information on properties of $\mathcal{G}$ is the set $\mathcal{M}(\mathcal{G})$ of *minimal non-$\mathcal{G}$*-graphs (that is, the graphs which do not belong to $\mathcal{G}$, but each of their proper subgraphs is in $\mathcal{G}$) and the function $f(\mathcal{G}, n)$ whose value is the number of non-isomorphic n-vertex graphs from $\mathcal{M}(\mathcal{G})$. The asymptotic character of $f(\mathcal{G}, n)$ may serve as an indirect indicator for 'tractability' of the family $\mathcal{G}$ in terms of efficient algorithmic recognition of good characterization. For example, Kuratowski's theorem [6] yields that the minimal non-planar graphs are exactly the subdivisions of K_5 or $K_{3,3}$, and the number of such graphs on n vertices is certainly polynomial in n (because it is bounded from above by the number of ways how to redistribute 2-valent vertices on the edges of K_5 or $K_{3,3}$). On the other hand, for 1-planar graphs (for which there exists a drawing in the plane such that each its edge is crossed at most once), it was shown in [5] that, for each $n \geq 63$, there exist at least $2^{\frac{n-54}{4}}$ non-isomorphic minimal non-1-planar graphs. Note that $f(\mathcal{G}, n)$ is not related to the number of distinct n-vertex graphs of $\mathcal{G}$, as this number is exponential for both planar and 1-planar graphs.

In this paper, we study the function $f(\mathcal{U}D, n)$ for the family $\mathcal{U}D$ of graphs defined by the property that, for each $G \in \mathcal{U}D$, there exists a non-degenerate drawing D of G in the Euclidean plane such that all edges of D are unit segments (D is further referred as *unit-distance drawing* of G). The family $\mathcal{U}D$ is a part of larger family of *unit-distance graphs* in $\mathbb{R}^2$: here the vertex set V is a subset of $\mathbb{R}^2$ and the edge set E is a subset of all pairs $\{x, y\},\ x, y \in V$ with $dist(x, y) = 1$ (see [1]). The latter family was widely studied mainly in connection with the (Hadwiger)-Nelson problem of determining the chromatic number of the plane (for its history and connections to other areas of mathematics, see the excellent monograph [9]). The problem of characterization of minimal non-unit-distance graphs was first mentioned in [2] and seems to be still open. It is easy to see that the family $\mathcal{U}D$ is not closed under taking minors, since any subdivision of $K_{2,3}$ (which is not a unit-distance graph) has an unit-distance drawing. In [4] and [10], it is proven that the problem of recognition of unit-distance graphs (and, more generally, the problem of determining the minimum dimension of Euclidean space in which a graph has unit-distance drawing) is NP-hard.

We prove the following result.

Theorem 1. *For infinitely many integers n, there exist at least exponentially many non-isomorphic n-vertex graphs from $\mathcal{M}(\mathcal{UD})$.*

2. The Proof

For the purpose of the subsequent proof, we briefly recall the equidistribution theorem.

Theorem 2 (Equidistribution theorem, [12]). *Let a be an irrational number. Then the sequence $\{n\cdot a - \lfloor n\cdot a\rfloor\}_{n=1}^{\infty}$ is uniformly distributed on the unit interval $(0,1)$.*

Claim 3. *The inequality*

$$(*) \qquad n-\sqrt{3}\cdot\left\lfloor\frac{n}{\sqrt{3}}\right\rfloor < \frac{2-\sqrt{3}}{2}$$

holds for infinitely many positive integers n.

Proof. The equidistribution theorem for $a=\frac{1}{\sqrt{3}}$ yields that the sequence $\left\{\frac{n}{\sqrt{3}} - \left\lfloor\frac{n}{\sqrt{3}}\right\rfloor\right\}_{n=1}^{\infty}$ is uniformly distributed on the interval $(0,1)$. Thus, the sequence $s=\left\{n-\sqrt{3}\cdot\left\lfloor\frac{n}{\sqrt{3}}\right\rfloor\right\}_{n=1}^{\infty}$ is uniformly distributed on the interval $\left(0,\sqrt{3}\right)$ as the elements of this sequence are exactly the elements of the previous sequence multiplied by $\sqrt{3}$. Our claim now follows from the fact that the terms on the right side of $(*)$ are positive and the uniform distribution of s. ■

From now on we will denote the set of all positive integers satisfying $(*)$ by $\mathcal{S}$ and, for a positive integer $n\in\mathcal{S}$, we define $N=\left\lfloor\frac{n}{\sqrt{3}}\right\rfloor$; note that $\min\mathcal{S}=7$.

We continue by defining, for any $n\in\mathcal{S}$, the set $\mathcal{G}_n$ of graphs $G_{n,S}$ in the following way: start with a 'snake' S_n consisting of $4n-1$ triangles (see Figure 1), denote its 2-valent vertices (the left and the right one, respectively) as v_0 and v_{2n}. Note that S_n has (up to mirror symmetry) a unique unit-distance drawing.

Let B_1 and B_2 be the graphs in Figure 2. We define the sequence $S=\{D_i\}_{i=1}^{2N}$, where each D_i is either a copy of B_1 or B_2, and the number of occurrences of B_1 and B_2 in S is equal. Concatenate the graphs of S by identifying vertices $y_i\in D_i$ and $x_{i+1}\in D_{i+1}$, thus forming a 'chain'. Finally, identify vertex $x_1\in D_1$ with vertex $v_0\in S_n$ and $y_{2N}\in D_{2N}$ with $v_{2n}\in S_n$ obtaining the graph $G_{n,S}$ (the resulting graph clearly depends on the choice of S).

Claim 4. *The number of non-isomorphic graphs in $\mathcal{G}_n$ is at least exponential in terms of number of their vertices.*

Proof. We have $\frac{1}{2}\binom{2N}{N}$ possible orderings of terms in S (the term $\frac{1}{2}$ prevents us from including mirror symmetry), which is at least $\frac{4^N}{2\cdot(2N+1)} > 2^N$ for $N\geq 5$. On the other hand, the number of vertices of any graph $G_{n,S}\in\mathcal{G}_n$ is $4n+1+$

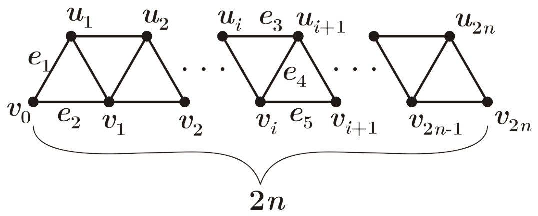

Figure 1. The 'snake' S_n.

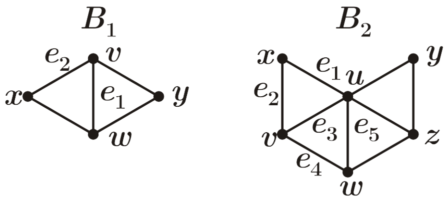

Figure 2. The blocks B_1 and B_2.

$8N$, which is linear in N. Thus the number of non-isomorphic graphs in $\mathcal{G}_n$ is exponential in the size of their vertex set (it is easy to see that for two sequences $S_1 = \{D_i\}_{i=1}^{2N}$ and $S_2 \neq S_1$, the graphs G_{n,S_1} and G_{n,S_2} are isomorphic only if $S_2 = \{D_{2N-i}\}_{i=1}^{2N}$, but then their drawings possess mirror symmetry). ∎

Claim 5. *Every graph $G \in \mathcal{G}_n$ is not a unit-distance graph.*

Proof. Assume that a unit distance drawing of G exists. There is a unique (up to mirror symmetry) unit-distance realization of the graph S_n. The distance between vertices v_0 and v_{2n} in this drawing is necessarily $2n$. Similarly, for $B_i, i \in \{1,2\}$, the unit-distance realization of B_i is also unique (up to mirror symmetry) and the distance between vertices x and y in this drawing is exactly $\sqrt{3}$. Thus, the distance between vertices x_1 and y_{2N} is at most $2N\sqrt{3} = 2\sqrt{3} \cdot \left\lfloor \frac{n}{\sqrt{3}} \right\rfloor < 2n$, the last inequality follows from the irrationality of $\sqrt{3}$. Notice that in G vertex x_1 is identified with vertex v_0 and vertex v_{2n} is identified with y_{2N}, but the distance between these vertices is either $2n$ (from the realisation of S_n) and, on the other hand, strictly less then $2n$ (from the properties of the drawings of blocks B_i, $i \in [1, 2N]$). This contradiction shows that G is not a unit-distance graph. ∎

Claim 6. *Every graph $G \in \mathcal{G}_n$ belongs to $\mathcal{M}(UD)$.*

Proof. We need to show that for every edge $e \in E(G)$, the graph $G - e$ belongs to $\mathcal{UD}$. This is done by case analysis.

First, assume that the removed edge e belongs to block D_i for some $i \in [1, 2N]$ and this block is isomorphic with B_1. Here we need to distinguish two cases (up to mirror symmetry) labeled with e_1, e_2 in Figure 2. In each of these cases the drawing of this block can be deformed in such a way that the distance between vertices x_i and y_i in this drawing of $D_i - e$ is arbitrarily close to 2. The drawings for each case are illustrated in Figure 3. For each $j \in [1, 2N], j \neq i$, the distance between vertices x_j and y_j is exactly $\sqrt{3}$. Thus the sum $\sum_{i=1}^{2N} dist(x_j, y_j)$ is arbitrary close to $(2N-1)\sqrt{3} + 2$. On the other hand, from $(*)$ it follows that $2n < (2N-1)\sqrt{3} + 2$, so there exists such a drawing that the considered sum is bigger than $2n$.

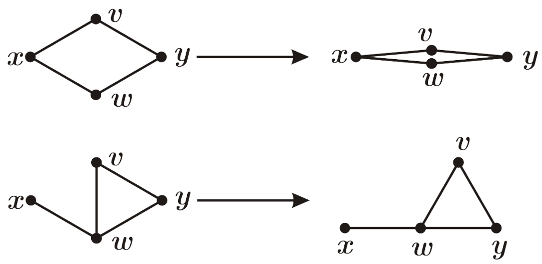

Figure 3. Deformation of B_1 without edge.

Next, assume that the removed edge e belongs to block D_i for some $i \in [1, 2N]$ and this block is isomorphic with B_2. In this case we need to distinguish 5 subcases (again, up to mirror symmetry) labelled e_i, $i \in [1, 5]$, in Figure 2. Again, in each of the subcases, the drawing of this block can be deformed in such a way that the distance between vertices x_i and y_i in this drawing of $D_i - e$ is arbitrarily close to 2. The deformation for each case is illustrated in Figure 4. Again, similarly to the previous case, it follows that $\sum_{i=1}^{2N} dist(x_j, y_j) > 2n$.

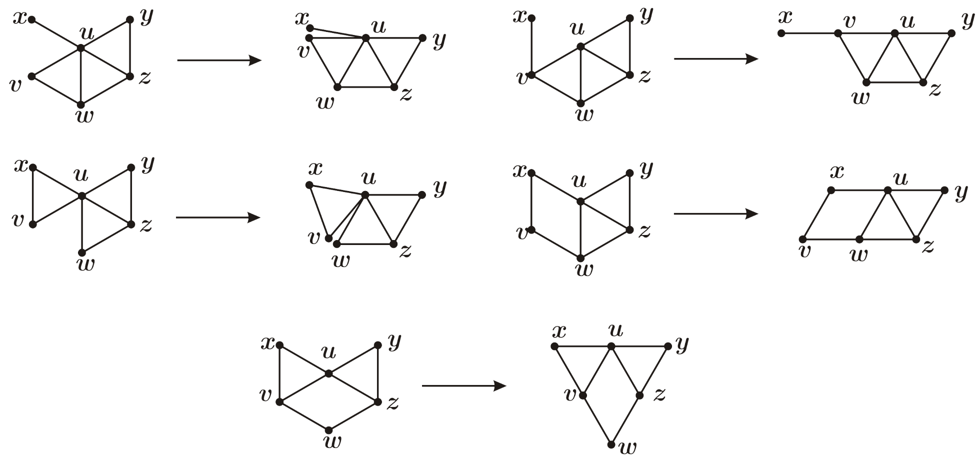

Figure 4. Deformation of B_2 without edge.

Finally, assume that the edge e belongs to S_n. Figure 5 shows all five cases how to deform $S_n - e$ to obtain a non-degenerate unit-distance drawing in which the Euclidean distance between v_0 and v_{2n} will be less than $2n$. The difference between $2n$ and $2N$ is less than $2 - \sqrt{3}$, so it suffices to decrease the distance between vertices v_0 and v_{2n} by any value bigger than $2 - \sqrt{3}$. This is obviously possible for the first case (vertex v_0 arbitrary close to v_2) and the second case (v_0 arbitrary close to v_1). In the third case, the worst possibility (in terms of distance between v_0 and v_{2n}) is to remove the edge u_1u_2, but v_0 could be rotated arbitrary close to vertex v_2, thus the distance between v_0 and v_{2n} would be arbitrary close to $2(n-1)$.

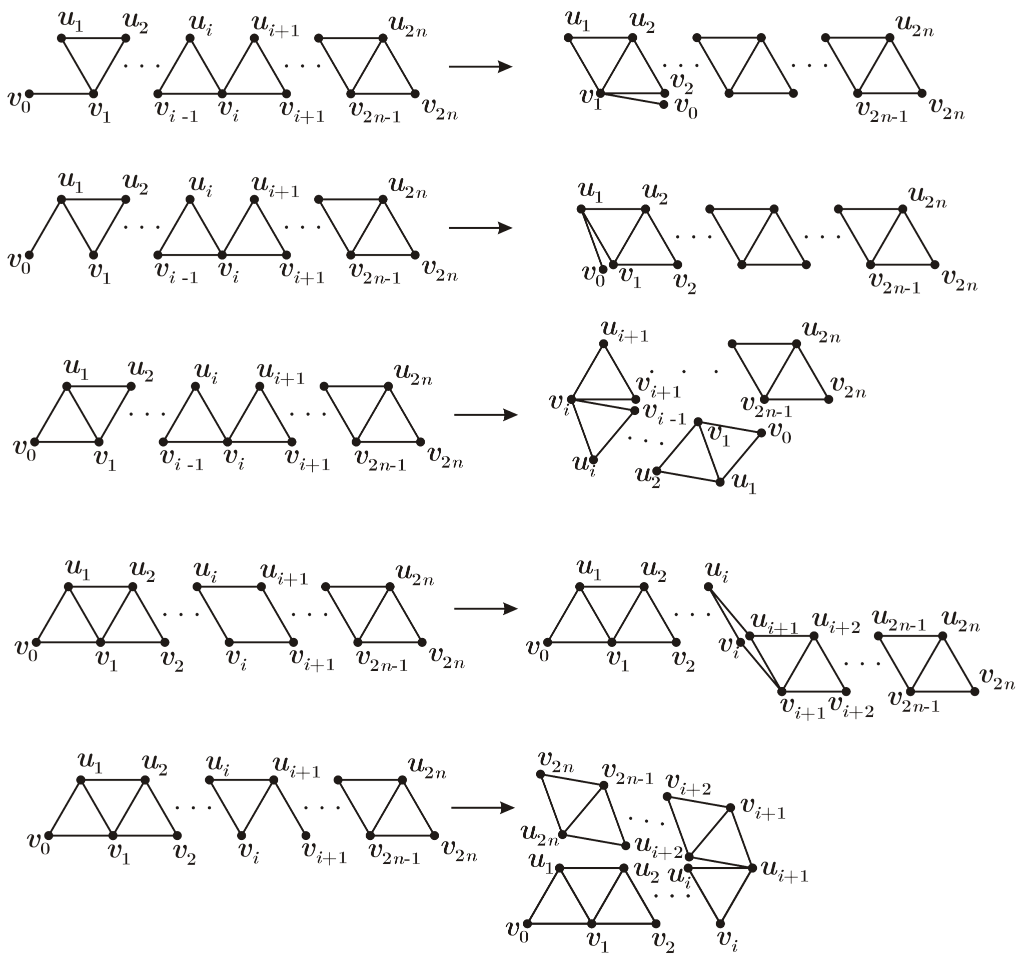

Figure 5. Deformation of S_n without edge.

After deforming the considered graph as illustrated for the fourth case, the distance between vertices v_0 and v_{2n} is arbitrary close to $\left((2n-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2\right)^{\frac{1}{2}}$ (consider the right-angled triangle where the line segment connecting v_0 and v_{2n} would form the hypotenuse), which converges to $2n-\frac{1}{2}$ as n increases (even for the smallest possible $n=7$ this distance is 13.52775 which is quite close to 13.5). It follows from $(*)$ that $2n-\frac{1}{2}<2N\sqrt{3}+2-\sqrt{3}-\frac{1}{2}<2N\sqrt{3}$.

In fifth case, the vertex v_0 could be rotated around the vertex u_{i+1} to be close to the line segment connecting vertices v_{i+1} and v_{2n}. This time the worst case would be to remove the edge v_1v_2, but the vertex v_0 would be close to vertex v_3 and the distance between v_0 and v_{2n} in this case would be close to $2n-3$.

In all five described cases, the distance between vertices v_0 and v_{2n} in a unit-distance drawing of S_n-e can be decreased at least by a factor arbitrary close to $\frac{1}{2}$, which is bigger than $2-\sqrt{3}$.

Now, it suffices to show that the chain of graphs corresponding to the sequence S can be attached to v_0 and v_{2n} to obtain a non-degenerate unit-distance drawing of G. This is implied by the following general auxiliary result.

Lemma 7. *Let $\{H_i\}_{i=1}^k$ be a sequence of graphs such that, for each $i\in[1,k]$ there exists a non-degenerate unit-distance drawing D_i of H_i containing two vertices $x_{i-1}^i, x_i^i\in V(H_i)$ having Euclidean distance $d_i>1$. Further, let D be a non-degenerate unit-distance drawing of a graph H containing vertices u,v with their Euclidean distance less than $\sum_{i=1}^k d_i$. Consider the graph G' obtained by identification of each vertex $x_i^i\in V(H_i)$ with vertex $x_i^{i+1}\in V(H_{i+1})$ and subsequent identification of u with x_0^1 and v with vertex x_k^k. Then there exists a non-degenerate unit-distance drawing of G'.*

Proof. For $i=1,\dots,k-1$, let x_i be the vertex resulting from the identification of x_i^i with x_i^{i+1}, and let $x_0=u, x_k=v$. Start with the placement of D_1 after identifying u with x_0^1. If we place the unit-distance drawing D_1 of H_1 in such a way that x_1 in H_1 lies on the line segment connecting points x_0 and x_k, then the Euclidean distance between x_1 and x_k will be strictly less than $\sum_{i=2}^k d_i$. Then there exists, by continuity of Euclidean distance, $\varepsilon\in\mathbb{R}$ such that when rotating the drawing of H_i by an angle $\alpha\in(-\varepsilon,\varepsilon)$ around the point x_0, the Euclidean distance between x_1 and x_k is still strictly less than $\sum_{i=2}^k d_i$.

We have to show that the placement of H_1 can be arranged in a way that the unit-distance drawing obtained is not degenerate. First, let us consider the case when a vertex $w\in V(D_1)$ lies on an edge $e\in E(D)$. The feasible positions for w are determined by the angle $\alpha\in(-\varepsilon,\varepsilon)$ and form a circular arc. An edge $e\in E(D)$ can intersect this arc in at most two points, so there are a finite number of forbidden positions for w. Now, consider the case when $w\in V(D)$ and $e\in E(D_1)$. We argue in the same way, but this time we fix the unit-distance drawing of H_1 and rotate the drawing D by angle α around the vertex x_0. As H_1 and H are finite graphs, an appropriate α can be chosen such that all vertices of $V(D_1) \cup V(D)$ are distinct and none of them appears in the interior of a line segment of $D_1 \cup D$.

The above described step can be repeated for all $i \in [1, k-3]$, however, when placing the drawing D_{k-2} of H_{k-2}, one has to be more careful, because the position of x_{k-2} determines the position of x_{k-1}. When preparing to place D_{k-2} the situation is that we have a point x_{k-3} whose distance from x_k is less than $d_{k-2} + d_{k-1} + d_k$. Again, if we place D_{k-2} such that x_{k-2} lies on the line segment determined by points x_{k-3} and x_k, the Euclidean distance of x_{k-2} and x_k is less than $d_{k-1} + d_k$ and there exists $\varepsilon \in \mathbb{R}$ such that when rotating the drawing D_{k-2} by an angle $\alpha \in (-\varepsilon, \varepsilon)$ around x_{k-3}, the Euclidean distance between x_{k-2} and x_k remains smaller than $d_{k-1} + d_k$. The possible positions for x_{k-1} form a circular arc (intersection of the circle with unit diameter centered in x_k and the disk centered in x_{k-3} with radius d_{k-2}). Thus, there are infinitely many possible positions for x_{k-2} and x_{k-1} and now we can again apply the argument as in the previous cases to avoid a degeneracy. □

Now, that Lemma 7 is proved, the proof of Claim 6 is complete. ■

3. Concluding Remarks

Since our result covers only a subset of the set of positive integers, it would be desirable to prove an exponential lower bound for $f(\mathcal{UD}, n)$ for all n.

Note that, in unit-distance drawings of graphs from $\mathcal{U}D$, we do not require the condition $dist(x, y) \neq 1$ for nonadjacent vertices x, y (unit-distance drawings which satisfy this condition are called *faithful*). The recent paper [1] considers labelled unit-distance and faithful unit-distance graphs in general Euclidean spaces $\mathbb{R}^d$; it is proved that, for prescribed d and given number of vertices, there is far more unit-distance graphs than the faithful ones. Concerning the structure of minimal non-faithful unit-distance graphs (for fixed d), the authors address the problem of minimum number of edges of such graphs and provide lower and upper bounds for bipartite case.

Along with unit-distance graphs, there are studied also *odd distance graphs* for which there exist drawings with all edges represented by line segments of odd lengths. It is known (see [3]) that the complete graph K_4 is not an odd distance graph, and recently in [8] it was proved that this is also not the case for the wheel graph W_6 (observe that both these graphs also belong to $\mathcal{M}(\mathcal{UD})$). Hence, the related problem would be to characterize all minimal non-odd-distance graphs; note that, by the result of Piepmeyer (see [7]), these graphs have chromatic number at least 4.

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