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DOMINATING VERTEX COVERS: THE VERTEX-EDGE DOMINATION PROBLEM

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Abstract

The *vertex-edge domination number* of a graph, $\gamma_{ve}(G)$, is defined to be the cardinality of a smallest set D such that there exists a vertex cover C of G such that each vertex in C is dominated by a vertex in D. This is motivated by the problem of determining how many guards are needed in a graph so that a searchlight can be shone down each edge by a guard either incident to that edge or at most distance one from a vertex incident to the edge. Our main result is that for any cubic graph G with n vertices, $\gamma_{ve}(G) \leq 9n/26$. We also show that it is NP-hard to decide if $\gamma_{ve}(G) = \gamma(G)$ for bipartite graph G.

Keywords: cubic graph, dominating set, vertex cover, vertex-edge dominating set.

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1. Introduction

Let $G = (V, E)$ be an undirected graph with n vertices. A *dominating set* of graph G is a set $D \subseteq V$ such that for each $u \in V \setminus D$, there exists an $x \in D$ adjacent to u. A vertex u is said to *dominate* a vertex v if either $u = v$ or u is adjacent to v. The minimum cardinality amongst all dominating sets of G is the *domination number*, denoted $\gamma(G)$. A *vertex cover* of graph G is a set $C \subseteq V$ such that for each edge $uv \in E$, at least one of u, v is an element of C. The minimum cardinality amongst all vertex covers of G is the *vertex cover number*, denoted $\tau'(G)$.

A number of recent papers have studied problems associated with defending or searching a finite, undirected graph $G = (V, E)$. These problems sometimes refer to *protecting* the graph with guards. A variety of graph protection problems and models have been considered in the literature of late, see the survey [5]. In the usual protection model, each attack in a sequence of attacks is defended by a mobile guard that is sent to the attacked vertex from a neighboring vertex or, in the case when edges are attacked, by sending a guard across the attacked edge (as introduced in [4]). A dominating set can then be viewed as a static positioning of guards which protect the vertices of the graph, while a vertex cover can be viewed a static positioning of guards which protect the edges of the graph.

A number of other papers have considered so-called *searchlight* problems which, inspired by the famous art gallery problem, attempt to use searchlights to find an intruder in a graph or a polygon. See for example [2] and [12]. In this paper, we study a variation on the searchlight problem. We shall consider the problem in which the guards, each of whom holds a searchlight, must shine a searchlight down some edge (where they think there might be an intruder). The problem is formally defined below and was initially defined by Peters in [10]. The problem was also studied in [1, 7–9, 11].

We now define what one may informally think of as a vertex-cover-dominating-set, or what is called a *vertex-edge dominating* set, for simplicity. The parameter $\gamma_{ve}(G)$ is called the *vertex-edge domination number* of G (see [10]) and is defined to be equal to the cardinality of a smallest set D such that there exists a vertex cover C of G such that each vertex in C is dominated by a vertex in D. Alternatively, a set D is a vertex-edge dominating set if and only if the set of vertices not dominated by D form an independent set.

We shall say that an edge uv is *protected* if there is a guard on u, v, or any neighbor of u, v. As examples, observe that $\gamma_{ve}(P_4) = 1$ and $\gamma_{ve}(C_5) = 2$. It is clear that $\tau'(G) \geq \gamma(G) \geq \gamma_{ve}(G)$ for any graph G without isolated vertices.

Informally, we wish to place guards on the vertices of a graph so that any edge is "close" to any guard; that is, each edge is incident to a vertex with a guard or incident with a vertex adjacent to a vertex with a guard. Following the art gallery metaphor, one may suppose that an alarm is triggered on edge uv. A guard must be able to quickly view uv to determine whether there is an intruder on the edge or a false alarm. Thus, if guards occupy the vertices of a vertex-edge dominating set and an alarm is triggered on edge uv, there is a guard nearby: on an endpoint of u, v or on a vertex adjacent to u or v. Such a guard can shine a flashlight down incident edge uv to check for an intruder or move to one of u, v and shine a flashlight down incident edge uv. As a simple example, consider the graph G shown in Figure 1 with a guard located on vertex y. Suppose an alarm is triggered on some edge e of G. If e is incident with y, the guard simply shines a flashlight down edge e. Otherwise, the guard moves to x or z and shines a flashlight down edge e.

With respect to the formal definition of the vertex-edge domination number, observe that $C = \{x, z\}$ is a vertex cover of graph G shown in Figure 1. It is clear that $D = \{y\}$ is a set of minimum cardinality such that each vertex of C is dominated by D. Thus, $\gamma_{ve}(G) = 1$.

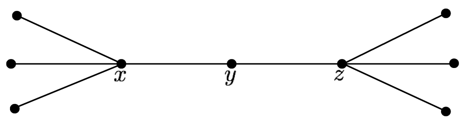

Figure 1. A graph G with $\gamma_{ve}(G) = 1$.

Upper bounds on the vertex-edge domination number of graphs of order n were presented in [1] for non-trivial connected graphs (upper bound of $\gamma_{ve}(G) \leq n/2$) and connected C_5-free graphs (upper bound of $\gamma_{ve}(G) \leq n/3$).

In this paper, we present results on the vertex-edge domination number of some graphs. Our main result is shown in Section 2: $\gamma_{ve}(G) \leq 9n/26$ for any cubic graph G with n vertices. In Section 3, we show that it is NP-hard to determine whether a bipartite graph, B, satisfies $\gamma_{ve}(B) = \gamma(B)$. We start with a simple result.

Proposition 1. *Let G be a connected graph of order at least 2. Then $\gamma_{ve}(G) = \tau'(G)$ if and only if $\tau'(G) = 1$.*

Proof. As G is a connected graph of order at least 2 we have $\tau'(G) \geq 1$. If $\tau'(G) = 1$, then the proposition follows, as $\tau'(G) \geq \gamma_{ve}(G)$ for all G.

Now suppose $\tau'(G) > 1$. Let C be a minimum vertex cover of G. We construct a vertex-edge dominating set D with fewer vertices than C. Initially, let $D = C$. If any two vertices in C are adjacent, then one of them can be removed from D. So suppose no two vertices in C are adjacent. If there exist two vertices in C that are distance two apart, then these two vertices can replaced in D by the vertex that lies on the path of length two between them. If there are no such vertices of distance two apart in C, then it follows that the closest pair of vertices in C are distance at least three apart and thus C cannot be a vertex cover, as there is an edge on the shortest path between any two vertices in C that is not covered by any vertex in C. ■

2. Cubic Graphs

Kostochka and Stocker proved that the domination number of a cubic graph with n vertices is at most $5n/14$, see [6]. There exists a cubic graph on 14 vertices where the domination number is 5, so the bound is tight. Thus, trivially, for any cubic graph G, $\gamma_{ve}(G) \leq \gamma(G) \leq 5n/14 \approx 0.35714n$. In this section, we prove our main result, that for any cubic graph G, $\gamma_{ve}(G) \leq 9n/26 \approx 0.34615n$.

In Section 2.1, we define a useful class of hypergraphs and state two useful hypergraph results. In Section 2.2, we state and prove our main result, Theorem 4.

2.1. Main result on hypergraphs from [3]

For the hypergraph H, let $n(H)$ denote the number of vertices in H, $m(H)$ denote the number of edges in H and $e_i(H)$ denote the number of edges in H of size i. For hypergraph H with the vertex set V, a smallest subset of V that contains vertices from every edge is called a *transversal* and its cardinality is denoted by $\tau(H)$.

In order to state the main result from [3], we need to define a particular class of hypergraphs $\mathcal{B}$. Let $\mathcal{B}$ be the class of *bad hypergraphs* defined as exactly those that can be generated using the operations (A)–(D) below.

(A) Let H_2 be the hypergraph with two vertices $\{x, y\}$ and one edge $\{x, y\}$ and let H_2 belong to $\mathcal{B}$.

(B) Given any $B' \in \mathcal{B}$ containing a 2-edge $\{u, v\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let $E(B) = E(B') \cup \{\{u, v, x\}, \{u, v, y\}, \{x, y\}\} \setminus \{u, v\}$. Now add B to $\mathcal{B}$.

(C) Given any $B' \in \mathcal{B}$ containing a 3-edge $\{u, v, w\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let

$$E(B) = E(B') \cup \{\{u, v, w, x\}, \{u, v, w, y\}, \{x, y\}\} \setminus \{u, v, w\}.$$

Now add B to $\mathcal{B}$.

(D) Given any $B_1, B_2 \in \mathcal{B}$, such that B_i contains a 2-edge $\{u_i, v_i\}$, for $i = 1, 2$, define B as follows.

Let $V(B) = V(B_1) \cup V(B_2) \cup \{x\}$ and let $E(B) = E(B_1) \cup E(B_2) \cup \{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \setminus \{\{u_1, v_1\}, \{u_2, v_2\}\}$. Now add B to $\mathcal{B}$.

Definition 1. For any hypergraph H, let $b(H)$ denote the number of connected components in H that belong to $\mathcal{B}$. Further, let $b^1(H)$ denote the maximum number of vertex disjoint subhypergraphs in H which are isomorphic to hypergraphs in $\mathcal{B}$ and which are intersected by exactly one other edge in H.

Theorem 2 [3]**.** *If H is a hypergraph whose all edges have size 2, 3, or 4, and $\Delta(H) \leq 3$, then*

$$24\tau(H) \leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H).$$

Using Theorem 2, we can prove the following result, which is implicit in [3]; therefore we include a short proof for completeness.

Theorem 3. *Let H be a hypergraph whose all edges have size 3 or 4, and $\Delta(H) \leq 3$ and every 4-edge contains a vertex that does not belong to any 3-edge. Then $12\tau(H) \leq 3n(H) + 2e_4(H) + 3e_3(H)$.*

Proof. Assume that $R \in \mathcal{B}$ and that R contains no 2-edge. In this case we note that the last operation carried out in the construction of R is operation (D) (see Subsection 2.1), as operations (A)–(C) all create 2-edges. Therefore there exist five vertices $\{u_1, v_1, u_2, v_2, x\}$ in R where $\{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \subseteq E(R)$. However then R is not a subgraph of H as the edge $\{u_1, v_1, u_2, v_2\}$ contains no vertex that does not belong to a 3-edge. Therefore $b(H) = b^1(H) = 0$ and by Theorem 2 we have the following, which completes the proof of the theorem.

$$\begin{aligned} 24\tau(H) &\leq 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H) \\ &= 6n(H) + 4e_4(H) + 6e_3(H). \end{aligned}$$ ■

2.2. Upper bound for cubic graphs

The bound that we shall present in Theorem 4 cannot be improved to anything below $n/3$, due to the graph in Figure 2. We leave it as an open problem to either find larger connected cubic graphs with $\gamma_{ve}(G) = n/3$ or show that the graph in Figure 2 is the only one; for instance, it does not appear easy to combine copies of the graph in Figure 2 in some way to arrive at another such example.

The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$.

Theorem 4. *If G is a cubic graph, then $\gamma_{ve}(G) \leq 9n/26$.*

Proof. Let S be a maximal independent set in G and assume that $|S| = (5/14 - \varepsilon_1)n$, where $n = |V(G)|$ (ε_1 may be positive or negative). Let T be the set of all vertices in $\overline{S} = V(G) \setminus S$ that have exactly one neighbor in S and let $\varepsilon_2 = (|S| - |T|)/n$. We will now prove the following two claims.

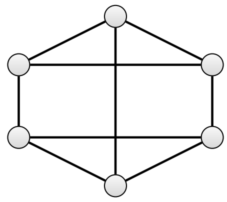

Figure 2. A 6-vertex cubic graph with vertex-edge domination number equal to $n/3$.

Claim A. $\gamma_{ve}(G) \le |S| - \varepsilon_2 n/4 = \left(\frac{5}{14} - \varepsilon_1 - \frac{\varepsilon_2}{4}\right) n.$

Proof. Let U be a maximal subset of S such that $S \setminus U$ dominates $\overline{S}$. As $\overline{S}$ is a vertex cover of G, we note that $\gamma_{ve}(G) \le |S \setminus U|$.

We will now show that $|U| \ge \varepsilon_2 n/4$, which will complete the proof of Claim A. Clearly this is true if $\varepsilon_2 \le 0$, so assume that $\varepsilon_2 > 0$. For the sake of contradiction assume that $|U| < \varepsilon_2 n/4$ and let T' be the set of all vertices not in T that have a unique neighbor in $S \setminus U$; note that $T' \subseteq N(U)$. As G is cubic, we must have $|T'| \le 3|U|$, which implies the following inequality.

$$|S \setminus U| = |S| - |U| \ge (|T| + \varepsilon_2 n) - |U| > (|T| + 4|U|) - |U| = |T| + 3|U| \ge |T| + |T'|.$$

As $|T| + |T'| < |S \setminus U|$, we note that some vertex in $s \in S \setminus U$ is not adjacent to a vertex in $T \cup T'$ (as each vertex in $T \cup T'$ is adjacent to at most one vertex in $S \setminus U$). This is a contradiction to the maximality of U, as s could have been added to U. This completes the proof of Claim A. $\square$

Claim B. $\frac{12}{14}\gamma_{ve}(G) \le \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right) n.$

Proof. We will first construct a 4-uniform hypergraph H as follows. Let $V(H) = V(G)$ and for every vertex $s \in \overline{S}$ add $N_G[s]$ as a hyperedge in H. This completes the definition of H. As G is cubic, we note that H is 4-uniform with $n = |V(G)|$ vertices and $m_H = |\overline{S}|$ edges.

Note that $\Delta(H) \le 3$ as for all $x \in V(G)$ at least one vertex in $N[x]$ belongs to S and therefore at most three vertices from $N[x]$ belongs to $\overline{S}$ (which are the vertices that give rise to edges containing x). Furthermore, no 4-edge in H has all its vertices in S.

Let $Q_1 \subseteq V(H)$ be all degree one vertices in H. Note that every vertex in Q_1 belongs to $\overline{S}$ and it has all its neighbors in S. Let H' be the hypergraph obtained from H by deleting all vertices in Q_1 (by deleting a vertex v, we mean deleting v and shrinking every edge, e, containing v such that it contains the vertex set $V(e) \setminus \{v\}$ instead of $V(e)$). Note that all edges in H' have size three or four and if e is a 3-edge, then all vertices in e belong to S. As no 4-edge is completely contained in S we note that every 4-edge contains a vertex (in $\overline{S}$) which does not belong to any 3-edge. Therefore the following holds by Theorem 3.

$$12\tau(H') \leq 3n(H') + 2e_4(H') + 3e_3(H') \leq 3(n - |Q_1|) + 2(m_H - |Q_1|) + 3|Q_1|.$$

Next, as $m_H = n - |S|$, this implies the following

$$12\tau(H') \leq 5n - 2|S| - 2|Q_1|.$$

We will first show that $\gamma_{ve}(G) \leq \tau(H')$ and then evaluate $5n - 2|S| - 2|Q_1|$. Let R be a transversal in H' with $|R| = \tau(H')$. As R contains a vertex from $N[y]$ for all $y \in \overline{S}$, we note that R dominates all vertices in $\overline{S}$. As $\overline{S}$ is a vertex cover of G, we get that $\gamma_{ve}(G) \leq |R| = \tau(H')$ as desired.

We will now evaluate $5n - 2|S| - 2|Q_1|$. Let Q_2 be the vertices in $\overline{S}$ of degree 2 in H and let Q_3 be the vertices in $\overline{S}$ of degree 3 in H. In G the vertices in Q_1 have 3 neighbors in S, the vertices in Q_2 have 2 neighbors in S, and the vertices in Q_3 have 1 neighbor in S. By double counting the number of edges between S and $\overline{S}$ we get the following

$$3|S| = 3|Q_1| + 2|Q_2| + 1|Q_3|.$$

Recall that $Q_3 = T$ and $|S| - |T| = \varepsilon_2 n$ (and therefore $|S| - \varepsilon_2 n = |T|$), and thus we obtain the following

$$3|S| = 3|Q_1| + 2|Q_2| + (|S| - \varepsilon_2 n).$$

As $Q_1 \cup Q_2 = \overline{S} \setminus T$ we also note that the following holds

$$|Q_1| + |Q_2| = |\overline{S}| - |T| \leq (n - |S|) - (|S| - \varepsilon_2 n).$$

Next, the above two equations can be rewritten as follows

$$3|Q_1| + 2|Q_2| = 2|S| + \varepsilon_2 n \qquad 2|Q_1| + 2|Q_2| = 2n - 4|S| + 2\varepsilon_2 n.$$

Subtracting the second equation from the first, one obtains the following

$$|Q_1| = 6|S| - 2n - \varepsilon_2 n.$$

Now since $|S| = (5n/14 - \varepsilon_1)$, we get the following equality

$$\begin{aligned} 5n - 2|S| - 2|Q_1| &= 5n - 2|S| - 2(6|S| - 2n - \varepsilon_2 n) = 9n - 14|S| + 2\varepsilon_2 n \\ &= 9n - 14(5/14 - \varepsilon_1)n + 2\varepsilon_2 n = n\,(4 + 14\varepsilon_1 + 2\varepsilon_2)\,. \end{aligned}$$

Therefore $12\tau(H') \leq n\,(4 + 14\varepsilon_1 + 2\varepsilon_2)$, which completes the proof of Claim B (by dividing both sides by 14). $\square$

Adding the results in Claim A and Claim B, we get the following inequality

$$\gamma_{ve}(G) + \frac{12}{14}\gamma_{ve}(G) \leq \left(\frac{5}{14} - \varepsilon_1 - \frac{\varepsilon_2}{4}\right) n + \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right) n$$

which implies

$$\frac{26}{14}\gamma_{ve}(G) \leq \left(\frac{9}{14} - \frac{7\varepsilon_2 - 4\varepsilon_2}{28}\right) n.$$

Therefore if $\varepsilon_2 \geq 0$, then we have $\gamma_{ve}(G) \leq 9n/26$, as desired. If $\varepsilon_2 < 0$, then we note that S is a dominating set in G and therefore $\gamma_{ve}(G) \leq |S| = (5/14 - \varepsilon_1)n$. Combining this with Claim B results in the following inequality

$$\gamma_{ve}(G) + \frac{12}{14}\gamma_{ve}(G) \leq \left(\frac{5}{14} - \varepsilon_1\right) n + \left(\frac{4}{14} + \varepsilon_1 + \frac{2\varepsilon_2}{14}\right) n.$$

Analogously to above this implies the following

$$\frac{26}{14}\gamma_{ve}(G) \leq \left(\frac{9}{14} + \frac{2\varepsilon_2}{28}\right) n.$$

This again implies $\gamma_{ve}(G) \leq 9n/26$, as desired. ■

Following the example shown in Figure 2, we leave open the following question.

Question 1. *Is it true that for any cubic graph G of order n, $\gamma_{ve}(G) \leq n/3$?*

In fact, a stronger open problem was stated in [1]. Namely, is it true that $\gamma_{ve}(G) \leq n/3$ for all connected graphs of order $n \geq 6$?

3. NP-Hardness

Recall that a support vertex in a tree is a vertex that is adjacent to a leaf in the tree. The trees, T, satisfying $\gamma_{ve}(T) = \gamma(T)$ were characterized by Theorem 32 of [9]. This result states that $\gamma_{ve}(T) = \gamma(T)$ if and only if T has an efficient dominating set S such that each vertex of S is a support vertex of T. A simple corollary of the result in [9] is the following.

Corollary 5. *We can decide if $\gamma_{ve}(T) = \gamma(T)$ in polynomial time for all trees T.*

We now consider the case when we want to decide whether $\gamma_{ve}(G) = \gamma(G)$ for bipartite graphs G.

Theorem 6. *It is NP-hard to decide whether $\gamma_{ve}(G) = \gamma(G)$ for a bipartite graph G.*

Proof. Recall that if $H = (V, E)$ is a hypergraph, then we denote the cardinality of a smallest subset of V that contains vertices from every edge (called a *transversal*) by $\tau(H)$.

We will reduce from the NP-hard problem of deciding whether a 3-uniform hypergraph, H, has a transversal of size at most k. That is, the hypergraph $H = (V, E)$, where V is the vertex set of H and each edge $e \in E$ is a set containing three vertices. We then want to decide whether there is a subset, $X \subseteq V$, of size at most k that contains at least one vertex of every edge of H.

The idea is to construct a graph, G, such that $\gamma_{ve}(G) < \gamma(G)$ if and only if $\tau(H) \leq k$. Start the construction of graph G with vertex set V. To this, for each edge $e \in E$, we add the vertex set $V_e = \{v_i^e \mid i = 1, 2, \ldots, k\}$ and the edges from each vertex in V_e to the three vertices in V that belong to e. Then we add the vertices $W = \{w_1, w_2, \ldots, w_k\}$ and for all $i = 1, 2, \ldots, k$ add all edges from w_i to v_i^e for all $e \in E$. Finally, we add the two new vertices x and y and all edges from x to $V \cup \{y\}$. This completes the construction of G.

We will show that $\tau(H) \leq k$ if and only if $\gamma_{ve}(G) < \gamma(G)$. Let $S_i = w_i \cup \{v_i^e \mid e \in E\}$. Note that $\gamma(G) = k+1$, as any dominating set in G must contain at least one vertex from each S_i (in order to dominate w_i) and a vertex from $\{x, y\}$ (in order to dominate y) and $W \cup \{x\}$ is a dominating set in G.

If $\tau(H) \leq k$, then let T be a transversal of H of size $\tau(H)$. Note that $T \subseteq V$ and T is a vertex-edge-dominating set in G (as the only vertices not dominated by T in G are $\{w_1, w_2, \ldots, w_k, y\}$ which form an independent set). Therefore $\gamma_{ve}(G) \leq |T| \leq k < k+1 = \gamma(G)$.

Now assume that $\gamma_{ve}(G) < \gamma(G)$. For the sake of contradiction assume that $\tau(H) > k$. Let Q be a vertex-edge dominating set in G of size $\gamma_{ve}(G)$. As $|Q| = \gamma_{ve}(G) < \gamma(G) = k+1$ and $\tau(H) > k$ we note that $Q \cap V$ is not a transversal in H. Therefore some edge $e \in E$ is not covered by $Q \cap V$. Due to the edge $w_i v_i^e$, we note that Q must contain at least one vertex from each S_i, $i = 1, 2, \ldots, k$. As $|Q| \leq k$, we therefore note that the edge xy is not covered by Q, a contradiction. Therefore $\tau(H) > k$, which completes the proof. ∎

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