

## INTERSECTION DIMENSION AND GRAPH INVARIANTS

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### Abstract

We show that the intersection dimension of graphs with respect to several hereditary properties can be bounded as a function of the maximum degree. As an interesting special case, we show that the circular dimension of a graph with maximum degree  $\Delta$  is at most  $O\left(\Delta \frac{\log \Delta}{\log \log \Delta}\right)$ . It is also shown that permutation dimension of any graph is at most  $\Delta(\log \Delta)^{1+o(1)}$ . We also obtain bounds on intersection dimension in terms of treewidth.

**Keywords:** circular dimension, dimensional properties, forbidden-subgraph colorings.

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### 1. INTRODUCTION

A graph property is a class of labeled finite graphs closed under isomorphism. A graph property  $\mathcal{P}$  is said to be *hereditary* if, for every  $G \in \mathcal{P}$ , every vertex induced subgraph of  $G$  is also in  $\mathcal{P}$ . We often refer to a graph property simply as a property or a class. In [10], Cozzens and Roberts introduced the notion of dimensional properties of graphs. They termed a graph property  $\mathcal{P}$  as *dimensional* if any graph can be written as the intersection of graphs from  $\mathcal{P}$ , i.e., for any graph  $G = (V, E)$ , there are  $k$  graphs  $\{G_i = (V, E_i) \in \mathcal{P} : 1 \leq i \leq k\}$  (for some

$k$ ) such that  $E = \bigcap_i E_i$ . Throughout the paper, we focus only on dimensional properties of graphs. Also, we use the terms “set” and “family” interchangeably.

For a dimensional property  $\mathcal{P}$  and a graph  $G$ , the minimum number  $k$  such that  $G$  can be written as the intersection of  $k$  graphs from  $\mathcal{P}$  is defined as the *intersection dimension* of  $G$  with respect to  $\mathcal{P}$  and is denoted by  $\dim_{\mathcal{P}}(G)$ .

In [17], Kratochvil and Tuza showed that a property  $\mathcal{P}$  is dimensional if and only if all complete graphs and all complete graphs minus an edge are in  $\mathcal{P}$ . They also proved that for any dimensional hereditary property  $\mathcal{P}$ , either  $\dim_{\mathcal{P}}(G) = 1$  for every  $G$  or it can take arbitrarily large values. However, it may still be possible to express  $\dim_{\mathcal{P}}(G)$  in terms of other invariants of  $G$ . In this paper, we bound  $\dim_{\mathcal{P}}(G)$  in terms of the maximum degree  $\Delta(G)$ .

Some interesting specializations of intersection dimension include the boxicity  $\text{box}(G)$  of a graph (with respect to the class of interval graphs), cubicity  $\text{cub}(G)$  (with respect to the class of unit interval graphs), circular dimension  $\dim_{\text{CA}}(G)$  (with respect to the class of circular arc graphs), overlap dimension  $\dim_o(G)$  (with respect to the class of overlap graphs), permutation dimension  $\dim_{\text{perm}}(G)$  (with respect to the class of permutation graphs), split dimension  $\dim_{\text{split}}(G)$  (with respect to the class of split graphs), chordal dimension  $\dim_{\text{chord}}(G)$  (with respect to the class of chordal graphs), perfect dimension  $\dim_{\text{perf}}(G)$  (with respect to the class of perfect graphs). Of these, boxicity is the most studied notion and various results on boxicity for special graph classes are known. For example, in [23], it was shown that every planar graph has boxicity at most 3. It is also known that every bipartite planar graph has boxicity at most 2 [15] and also that every outerplanar graph has boxicity at most 2 [22]. Upper bounds on boxicity have also been obtained in terms of treewidth [9] ( $\text{box}(G) \leq \text{tw}(G) + 2$  for any  $G$ ) and maximum degree [1, 8, 11] ( $\text{box}(G) \leq c\Delta(\log \Delta)^2$  for any  $G$ ,  $c$  is a constant, due to [1]).

Circular dimension was first studied by Feinberg in [12], where, for every  $n$ , the maximum value of circular dimension was determined exactly for the class of complete multi-partite graphs on  $n$  vertices. Since the class of interval graphs is contained in the class CA of circular arc graphs, the boxicity of a graph is an upper bound on its circular dimension. However, the circular dimension can be much smaller than boxicity. Moreover, boxicity cannot be bounded by any function of circular dimension, that is, there is no  $f(\cdot)$  such that  $\text{box}(G) \leq f(\dim_{\text{CA}}(G))$  for every  $G$ . This assertion follows from the infinite family of graphs  $\{P_n^c, C_n^c \mid n \geq 4\}$ . It was established by Cozzens and Roberts [10], for each  $n \geq 4$ , that each of  $P_n^c$  and  $C_n^c$  has boxicity at least  $(n-1)/3$ , whereas each of them has circular dimension at most 2. Here,  $P_n^c$  and  $C_n^c$  denote respectively the complement of a path  $P_n$  and a cycle  $C_n$  on  $n$  vertices. In [17], it is established that  $\frac{\dim_{\mathcal{A}_1}(G)}{\dim_{\mathcal{A}_2}(G)}$  can become arbitrarily large, for various specific pairs of hereditary graph classes  $(\mathcal{A}_1, \mathcal{A}_2)$ .

In particular, it is established for the case of  $\mathcal{A}_1 = CA$  and  $\mathcal{A}_2$  denoting interval graphs.

In this paper, we present upper bounds for the intersection dimension  $\dim_{\mathcal{P}}(G)$  of an arbitrary graph  $G$  with respect to any member  $\mathcal{P}$  of a set  $\mathcal{C}$  of hereditary and dimensional properties, in terms of its invariants like maximum degree  $\Delta(G)$ , treewidth  $tw(G)$ , star chromatic number  $\chi_s(G)$  and its generalizations to  $(2, \mathcal{F})$ -chromatic numbers  $\chi_{2, \mathcal{F}}(G)$ . As a consequence, it follows that for every such property  $\mathcal{P}$ ,  $\dim_{\mathcal{P}}(G)$  is bounded for graphs  $G$  from any proper minor closed class and in particular, for graphs of bounded treewidth.

Moreover, for some specific hereditary classes  $\mathcal{P}$  such as the class of circular-arc graphs, the class of overlap graphs and the class of permutation graphs, we also upper-bound  $\dim_{\mathcal{P}}(G)$  (for any  $G$ ) by a “nearly” linear function of either  $\Delta(G)$  or  $\chi(G)$ . The proofs of these bounds are based on relating the intersection dimension with  $(2, \mathcal{F})$ -subgraph colorings, in particular, frugal colorings (these notions are defined in Section 2). No bound (applicable to all graphs) was known before for permutation dimension and overlap dimension. For the cases of planar and planar bipartite graphs  $G$ , [17] establishes that permutation dimension is at most 12 and 4, respectively.

This paper is organized as follows. In Section 2, we present some definitions of graph operations and graph coloring notions like forbidden subgraph colorings and frugal colorings. In Section 3, we obtain the basic results of this paper relating intersection dimension (with respect to certain hereditary classes) and forbidden subgraph colorings. Section 4 contains improved bounds on intersection dimension in terms of maximum degree obtained by using frugal colorings. In Section 5, we obtain an improved bound on the circular dimension and conclude with some open problems in Section 6.

## 2. DEFINITIONS AND FACTS

We first need a few preliminaries. For a simple undirected graph  $G = (V, E)$ , we shall denote by  $G^c$  the complement of  $G$ , defined as  $G^c = \left(V, \binom{V}{2} \setminus E\right)$  and for a subset  $S \subseteq V$ , we shall denote by  $G[S]$  the subgraph induced by  $G$  on  $S$ , that is,  $G_S = (S, E_S)$ , where  $E_S = \{\{u, v\} \in E : u, v \in S\}$ . A graph  $H$  is said to be a vertex induced (shortly induced) subgraph of  $G$  if  $H = G[S]$  for some  $S \subseteq V$ . A graph  $H$  is said to be a *minor* of  $G$  (denoted by  $H \triangleleft G$ ) if  $H$  can be obtained from  $G$  by applying a sequence of edge contractions and deletions of vertices or edges. We say that  $G$  is  *$H$ -minor free* if  $G$  does not have a minor which is isomorphic to  $H$ . A class  $\mathcal{C}$  of graphs is said to be *proper minor-closed* if  $\mathcal{C}$  is closed under minors (that is,  $G \in \mathcal{C} \wedge H \triangleleft G \Rightarrow H \in \mathcal{C}$ ) and is not the class of all graphs. A family of graphs is *non-trivial* if it contains at least one graph

and is not the class of all graphs.

We also recall some special classes of graphs. A *split* graph is a graph  $G = (V, E)$  which admits a partition  $V = S \cup T$  such that  $G[S]$  and  $G[T]$  induce respectively a complete graph and an empty graph. A graph is a *permutation* graph if it is isomorphic to a graph  $G = (V, E)$  where  $V = \{1, 2, \dots, n\}$ ,  $E = \{\{i, j\} : i < j, \pi_i > \pi_j\}$ , for some permutation  $\pi : V \rightarrow V$ . A graph is an *interval* graph if it is isomorphic to a graph  $G = (V, E)$  where  $V = \{u_i = [a_i, b_i] : 1 \leq i \leq n, a_i, b_i \in \mathbb{R}, a_i \leq b_i \forall i\}$  and  $E = \{\{u_i, u_j\} : u_i \cap u_j \neq \emptyset\}$ . A graph is an *overlap* graph (also known as a *circle* graph) if it is isomorphic to a graph  $G = (V, E)$  where  $V = \{u_i = [a_i, b_i] : 1 \leq i \leq n, a_i, b_i \in \mathbb{R}, a_i \leq b_i \forall i\}$  and  $E = \{\{u_i, u_j\} : u_i \cap u_j \neq \emptyset, u_i \not\subseteq u_j, u_j \not\subseteq u_i\}$ . A graph is a *circular-arc* graph if it is isomorphic to a graph  $G = (V, E)$  where  $V = \{c_i : 1 \leq i \leq n\}$ ,  $E = \{\{c_i, c_j\} : c_i \cap c_j \neq \emptyset\}$  and each  $c_i$  is an arc on a planar circle (without loss of generality assumed to be of radius 1 and centered at  $(0, 0)$ ). A graph is a *chordal* graph if there is no induced cycle on four or more edges. A graph  $G = (V, E)$  is a *perfect* graph if, for every induced subgraph  $G_S = G[S]$  ( $S \subseteq V$ ), we have  $\chi(G_S) = \omega(G_S)$  where  $\chi()$  and  $\omega()$  denote respectively the chromatic number and maximum size of a clique in a graph. For more details on these graph classes, the reader is referred to [10, 14, 7].

**Definition 1.** We say that a class  $\mathcal{A}$  of graphs is additive if, for every two vertex disjoint members  $G = (U, E)$  and  $H = (V, F)$  of  $\mathcal{A}$ , their disjoint union  $G \cup H$  defined to be  $(U \cup V, E \cup F)$  is also a member of  $\mathcal{A}$ .

Some examples of additive and hereditary classes are: the class of perfect graphs, the class of chordal graphs, the class of interval graphs, and the class of permutation graphs. The class of circular-arc graphs and the class of split graphs are examples of classes which are not additive.

**Definition 2.** Following [17], we say that a class  $\mathcal{A}$  of graphs has the Full Degree Completion (FDC) property if for any graph  $G = (V, E)$  in  $\mathcal{A}$ , the graph  $H = (V \cup \{u\}, E \cup \{\{u, v\} : v \in V\})$  ( $u \notin V$ ) obtained by adding a universal vertex (i.e., a vertex adjacent to all of  $V$ ) also belongs to  $\mathcal{A}$ .

Some examples of hereditary classes which are additive and which also satisfy FDC property are: the class of perfect graphs, the class of chordal graphs, the class of interval graphs, the class of permutation graphs. The class of overlap graphs is an example of a graph class which does not satisfy the FDC property. Each of the two classes of split graphs and circular-arc graphs satisfies the FDC property, but is not additive.

**Definition 3.** The Zykov sum of two graphs  $G = (U, E)$  and  $H = (V, F)$  with disjoint vertex sets is formed by taking the union of the two graphs and adding

all edges between the graphs, that is, the graph  $G' = (U \cup V, E \cup F \cup \{\{u, v\} : u \in U, v \in V\})$ . We say that a class  $\mathcal{A}$  of graphs has the Zykov sum property if the Zykov sum of any two vertex disjoint graphs in  $\mathcal{A}$  is also in  $\mathcal{A}$ .

It follows from definitions that if a graph class satisfies the Zykov sum property, then it also satisfies the FDC property. Some examples of additive and hereditary classes which satisfy the Zykov sum property are: the class of perfect graphs and the class of permutation graphs. Examples of graph classes which do not satisfy the Zykov sum property include the class of interval graphs and the class of chordal graphs.

**Definition 4.** Given two graphs  $G$  and  $H$ , we say that  $G$  is  $H$ -free if  $G$  has no isomorphic copy of  $H$  as a subgraph (not necessarily induced). Given a family  $\mathcal{F}$  of graphs, we say that  $G$  is  $\mathcal{F}$ -free if  $G$  is  $H$ -free for each  $H \in \mathcal{F}$ .

For a family  $\mathcal{F}$  of graphs, we use  $\text{Forb}(\mathcal{F})$  to denote the class of all graphs which are  $\mathcal{F}$ -free. Thus, for example, if  $\mathcal{F}$  is the set of all cycles,  $\text{Forb}(\mathcal{F})$  is the class of all forests. An acyclic coloring of a graph is a proper vertex coloring in which the subgraph induced by the union of any two color classes is a forest. The following definition from [5] generalizes the notion of acyclic coloring, and is related to intersection dimension, as we shall prove later.

**Definition 5.** Let  $\mathcal{F}$  be a family of connected bipartite graphs on at least 3 vertices each. We define a  $(2, \mathcal{F})$ -subgraph coloring (or just a  $(2, \mathcal{F})$ -coloring) to be a proper coloring of the vertices of a graph  $G$  so that the subgraph of  $G$  induced by the union of any 2 color classes is  $\mathcal{F}$ -free. We denote by  $\chi_{2, \mathcal{F}}(G)$  the minimum number of colors sufficient to guarantee a  $(2, \mathcal{F})$ -subgraph coloring of  $G$ .

In recent works [5, 4, 6], the present authors defined a generalization of the above notion for any fixed  $j \geq 2$  (by considering the union of any  $j$  color classes) and obtained upper bounds on  $(j, \mathcal{F})$ -chromatic numbers and their edge analogues of an arbitrary  $G$  in terms of  $\Delta(G)$ . Tightness (up to polylogarithmic factors in  $\Delta$ ) were also established for the case  $j = 2$ .

Four special cases of this notion, which are of interest to the present work are the following.

- *Acyclic coloring* is a  $(2, \mathcal{F})$ -coloring, where  $\mathcal{F}$  is the set of all cycles. The minimum number of colors used in any acyclic coloring of  $G$  is known as its *acyclic chromatic number* and is denoted by  $\chi_a(G)$ . [3] presents an upper bound of  $O(\Delta^{4/3})$  on graphs of maximum degree  $\Delta$ .
- *Star coloring* is a  $(2, \{P_4\})$ -coloring — in such a coloring, the union of any two color classes induces a forest of vertex disjoint stars. The minimum number of colors used in any star coloring of a graph is called its *star chromatic number*

and is denoted by  $\chi_s(G)$ . [13] presents an upper bound of  $O(\Delta^{3/2})$  on graphs of maximum degree  $\Delta$ .

- $\beta$ -frugal coloring is a  $(2, \{K_{1, \beta+1}\})$ -coloring — it is a coloring in which each vertex has at most  $\beta$  neighbors in any other color class. The corresponding chromatic number is referred to as its  $\beta$ -frugal chromatic number and is denoted by  $\chi_\beta^{\text{frugal}}(G)$ . A simple probabilistic argument (see [16]) establishes an upper bound of  $O(\Delta^{(\beta+1)/\beta})$  on graphs of maximum degree  $\Delta$ .
- $(2, \text{planar})$ -coloring is a proper coloring in which the union of any two color classes induces a planar subgraph. The corresponding chromatic number is denoted by  $\chi_2^{\text{planar}}(G)$ . [5] presents an upper bound of  $O(\Delta^{8/7})$  on graphs of maximum degree  $\Delta$ .

The following fact is easy to verify and will be often used later in the proof.

**Fact 6.** Suppose  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are two properties such that  $\mathcal{P}_1 \subseteq \mathcal{P}_2$ . If  $\mathcal{P}_1$  is dimensional, then  $\mathcal{P}_2$  is also dimensional and also  $\dim_{\mathcal{P}_2}(G) \leq \dim_{\mathcal{P}_1}(G)$  for any  $G$ .

### 3. INTERSECTION DIMENSION AND FORBIDDEN SUBGRAPH COLORINGS

In their paper [17], Kratochvil and Tuza proved the following lemmas which we shall need.

**Lemma 7** [17]. *Let  $\mathcal{A}$  be a dimensional class of graphs satisfying the FDC requirement. Suppose  $G = (V, E)$  is a graph and  $G_i = (V_i, E_i)$ ,  $i = 1, 2, \dots, k$  are induced subgraphs of  $G$  such that each non-edge of  $G$  is present as a non-edge in some  $G_i$ . Then,  $\dim_{\mathcal{A}}(G) \leq \sum_{i=1}^k \dim_{\mathcal{A}}(G_i)$ .*

**Lemma 8** [17]. *Let  $\mathcal{A}$  be a dimensional class of graphs satisfying the Zykov sum property. If  $G = (V, E)$  is a graph and  $G_{ij} = (V_{ij}, E_{ij})$ ,  $i = 1, 2, \dots, k$ ,  $j = 1, \dots, l_i$ , are induced subgraphs of  $G$  such that (i) each non-edge of  $G$  is present as a non-edge in some  $G_{ij}$  and (ii) for every  $i$ , the vertex sets  $V_{ij}$ ,  $j = 1, 2, \dots, l_i$  form a partition of  $V$ . Then  $\dim_{\mathcal{A}}(G) \leq \sum_{i=1}^k \max_{1 \leq j \leq l_i} \dim_{\mathcal{A}} G_{ij}$ .*

Using Lemmas 7 and 8, we now obtain a result which connects intersection dimension and  $(2, \mathcal{F})$ -subgraph colorings. This result generalizes the bounds obtained (in Section 3) of [17] for some specific hereditary and dimensional classes like split graphs and chordal graphs to arbitrary hereditary and dimensional graph classes. Below, we use  $BIP$  to refer to the class of all bipartite graphs.

**Theorem 9.** *Let  $\mathcal{A}$  be a hereditary and additive class of graphs which satisfies the FDC property. Let  $\mathcal{F}$  be a family of connected graphs and suppose there exists a constant  $t = t(\mathcal{F})$  such that for all graphs  $H \in \text{Forb}(\mathcal{F}) \cap BIP$ ,*

the intersection dimension of  $H$  w.r.t  $\mathcal{A}$  is at most  $t$ . Then for any graph  $G$ ,  $\dim_{\mathcal{A}}(G) \leq t(\chi_{2,\mathcal{F}}^{(G)})$ . Further, if  $\mathcal{A}$  has the Zykov sum property, then  $\dim_{\mathcal{A}}(G) \leq t\chi_{2,\mathcal{F}}(G)$ .

In particular, if  $\text{Forb}(\mathcal{F}) \cap BIP \subseteq \mathcal{A}$  for some  $\mathcal{F}$ , then  $\dim_{\mathcal{A}}(G) \leq (\chi_{2,\mathcal{F}}^{(G)})$ . If  $\mathcal{A}$  also satisfies Zykov sum property, then  $\dim_{\mathcal{A}}(G) \leq \chi_{2,\mathcal{F}}(G)$ .

**Proof.** Since  $\mathcal{A}$  is hereditary and is additive, it contains all empty graphs. Let  $G = (V, E)$  be any graph and let  $C_1, \dots, C_k$  be the color classes in a  $(2, \mathcal{F})$ -subgraph coloring of  $G$  where  $k = \chi_{2,\mathcal{F}}(G)$ .

For all  $i \neq j$ , let  $G_{i,j}$  be the subgraph of  $G$  induced by the union of the color classes  $C_i$  and  $C_j$ . We have  $G_{i,j} \in \text{Forb}(\mathcal{F}) \cap BIP$  and hence  $\dim_{\mathcal{A}}(G_{i,j}) \leq t$ . Also, each non-edge of  $G$  is present as a non-edge in some  $G_{i,j}$ . Hence, by Lemma 7,  $\dim_{\mathcal{A}}(G) \leq \sum_{1 \leq i < j \leq k} \dim_{\mathcal{A}}(G_{i,j}) \leq t(\chi_{2,\mathcal{F}}^{(G)})$ .

Suppose that  $\mathcal{A}$  also satisfies the Zykov sum property. Consider a  $(2, \mathcal{F})$ -subgraph coloring of  $G$  with the color classes  $C_1, \dots, C_k$ , where  $k = \chi_{2,\mathcal{F}}(G)$ . Now consider a proper edge coloring of  $K_k$  (the complete graph on  $[k] = \{1, 2, \dots, k\}$ ) using  $k$  colors. Let  $M_1, \dots, M_k$  be the matchings forming the  $k$  color classes in this edge coloring.

For each  $i \in [k]$ , let  $\mathcal{H}_i = \{G_{i,j}\}_j$  be a collection of induced subgraphs of  $G$  obtained as follows. For each matching edge  $(l, m)$  in  $M_i$ , include in  $\mathcal{H}_i$  the induced subgraph  $G_{i,(l,m)} = G[C_l \cup C_m]$ . For each  $l \in [k]$  such that vertex  $l$  is unmatched in  $M_i$ , include the subgraph  $G_{i,l} = G[C_l]$  in  $\mathcal{H}_i$ . Clearly, the vertex sets of  $G_{i,j}$  form a partition of  $V$  for each  $i$ . Also, each non-edge of  $G$  is present as a non-edge in some  $G_{i,j}$ . Further, for all  $i, j$ ,  $G_{i,j} \in \text{Forb}(\mathcal{F})$ . Applying Lemma 8, we get  $\dim_{\mathcal{A}}(G) \leq kt = t\chi_{2,\mathcal{F}}(G)$ . This proves Theorem 9. ■

The following two corollaries are consequences of the above theorem.

**Corollary 10.** For any  $G$ , the following are true:

- (a)  $\dim_{\text{perf}}(G) \leq \chi(G)$ ;
- (b)  $\dim_o(G) \leq \dim_{\text{perm}}(G) \leq 4\chi_2^{\text{planar}}(G)$ .

**Proof.** By setting  $\mathcal{F} = \{C_5, C_7, \dots\}$  and  $\mathcal{A}$  to be the class of perfect graphs, we note that  $\text{Forb}(\mathcal{F}) \cap BIP \subseteq \mathcal{A}$  and also that  $\chi_{2,\mathcal{F}}(G) = \chi(G)$ . Also,  $\mathcal{A}$  satisfies the Zykov sum property. Hence (a) follows. It should be noted that (a) is an immediate corollary of the fact ( $\dim_{\text{split}}(G) \leq \chi(G)$  for any  $G$ ) established in [17] and also Fact 6. Here, this derivation is presented (as an alternate proof) of this fact to illustrate the applicability of Theorem 9. As mentioned before, the class of split graphs is not even additive.

For (b), note that permutation graphs form an additive and hereditary class satisfying the Zykov sum property. Applying the bound of  $\dim_{\text{perm}}(G) \leq 4$  obtained in [17] for planar bipartite graphs, we deduce the stated bound. Since

every permutation graph is an overlap graph (see Section 4.7 of [7]), an application of Fact 6 implies that  $\dim_o(G) \leq \dim_{\text{perm}}(G)$ . This completes the proof of Corollary 10. ■

**Corollary 11.** *For any  $G$  and for any additive and hereditary class  $\mathcal{A}$ , the following are true:*

- (a) *if  $\mathcal{A}$  satisfies the FDC property, then  $\dim_{\mathcal{A}}(G) \leq \binom{\chi_s(G)}{2}$ ;*
- (b) *if  $\mathcal{A}$  satisfies the Zykov sum property, then  $\dim_{\mathcal{A}}(G) \leq \chi_s(G)$ .*

**Proof.** For (a) and (b), we set  $\mathcal{F} = \{P_4\}$ . Then,  $\text{Forb}(\mathcal{F})$  is the collection of all graphs each of whose connected components is either a star or a triangle (that is, a complete graph on 3 vertices). Also, any hereditary class of graphs which is additive and which satisfies the FDC property must contain all stars and also all complete graphs. Thus,  $\text{Forb}(\mathcal{F}) \subseteq \mathcal{A}$ .

Statements (a) and (b) now follow from an application of Theorem 9. ■

We now apply some results of Fertin *et al.* [13], Albertson *et al.* [2], Nešetřil and Ossona de Mendez [20], and Mohar and Špacapan [18] on the star chromatic number in conjunction with Corollaries 10 and 11 to obtain the following corollary.

**Corollary 12.** *Let  $\mathcal{A}$  be a hereditary class of graphs which is additive. Then, there exists positive constants  $c_1, c_2, c_3, c_4, c_5, c_6$  and  $c_H$  (for every fixed graph  $H$ ) such that: for any graph  $G$  with maximum degree  $\Delta$ , treewidth  $t$  and genus  $g > 0$ , we have the following:*

- (a) *If  $\mathcal{A}$  satisfies the FDC property, then*

$$\dim_{\mathcal{A}}(G) \leq c_1 \Delta^3; \quad \dim_{\mathcal{A}}(G) \leq c_2 t^4; \quad \dim_{\mathcal{A}}(G) \leq c_3 g^{6/5}.$$

*If  $G$  is  $H$ -minor free, then  $\dim_{\mathcal{A}}(G) \leq c_H$ .*

- (b) *If  $\mathcal{A}$  satisfies the Zykov sum property, then*

$$\dim_{\mathcal{A}}(G) \leq c_4 \Delta^{3/2}; \quad \dim_{\mathcal{A}}(G) \leq \frac{(t+2)(t+1)}{2}; \quad \dim_{\mathcal{A}}(G) \leq c_5 g^{3/5}.$$

- (c)  $\dim_o(G) \leq \dim_{\text{perm}}(G) \leq 100 \cdot \Delta^{8/7}$ .

**Proof.** The stated bounds follow from an application of Corollaries 10 and 11 combined with the following upper bounds on star chromatic numbers.

- $\chi_s(\Delta) = O(\Delta^{3/2})$  ([13]).
- If graph  $G$  has treewidth at most  $t$ , then  $\chi_s(G) \leq (t+2)(t+1)/2$  ([2, 13]).
- For any fixed graph  $H$ , there is a constant  $d_H$  such that for any  $H$ -minor free graph  $G$ ,  $\chi_s(G) \leq d_H$  ([20]).



- For a graph  $G$  of genus  $g > 0$ ,  $\chi_s(G) \leq c_6 g^{3/5}$ , where  $c_6$  is some absolute constant ([18]).
- $\chi_2^{\text{planar}}(G) \leq 25 \cdot \Delta^{8/7}$  ([5]).

This completes the proof of Corollary 12. ■

**Remark.** It follows that for every proper minor-closed class  $\mathcal{C}$ , there is a  $d = d_{\mathcal{C}}$  such that  $\dim_{\mathcal{A}}(G) \leq d$  for every  $G \in \mathcal{C}$  and for every hereditary and additive class  $\mathcal{A}$  satisfying FDC. In particular, it is true for graphs of bounded treewidth.

#### 4. IMPROVED BOUNDS

In this section, we significantly improve the bounds of Corollary 12 (stated in terms of  $\Delta(G)$ ) by combining Theorem 9 with the following result of Molloy and Reed [19] on frugal colorings. Throughout this section, all logarithms are with respect to base 2.

**Theorem 13** [19]. *There exists a positive constant  $\Delta_0$  such that every graph  $G$  of maximum degree  $\Delta \geq \Delta_0$  can be properly colored using  $\Delta + 1$  colors so that any vertex has at most  $\beta$  neighbors in any color class, where  $\beta = \lfloor a(\log \Delta) / (\log \log \Delta) \rfloor$  and  $a (\geq 2$  without loss of generality) is some absolute positive constant.*

**Notation.** Let  $\mathcal{A}$  be a hereditary, additive and dimensional class of graphs satisfying the FDC property. For such classes, and for any positive real number  $t$ , we define  $\dim_{\mathcal{A}}(t) = \max\{\dim_{\mathcal{A}}(G) : \Delta(G) \leq t\}$ . By Corollary 12,  $\dim_{\mathcal{A}}(t)$  is well-defined.

By combining Theorem 9 with Theorem 13, we obtain the following result. In what follows, for  $x \geq 1$ ,  $\log^* x$  denotes  $\min\{k \geq 0 : \log^{(k)} x < 2\}$  where  $\log^{(0)} x = x$  and for  $i \geq 1$ , define  $\log^{(i)} x = \log_2(\log^{(i-1)} x)$ . We note that  $\log^* x = o(\log^{(i)} x)$  for every fixed  $i \geq 0$ . In particular, we have  $\log^* x = o(\log \log x)$ .

**Theorem 14.** *Let  $\mathcal{A}$  be an additive, hereditary class of graphs satisfying the FDC property. Then for all sufficiently large  $\Delta$  and for some positive constant  $B$ , the following holds.*

- $\dim_{\mathcal{A}}(\Delta) \leq \Delta^2(\log \Delta)^2 \cdot B^{\log^* \Delta}$ ;
- If  $\mathcal{A}$  satisfies the Zykov sum property as well, then  $\dim_{\mathcal{A}}(\Delta) \leq \Delta(\log \Delta) \cdot B^{\log^* \Delta}$ ;
- In particular,  $\dim_o(\Delta) \leq \dim_{\text{perm}}(\Delta) \leq \Delta(\log \Delta) \cdot B^{\log^* \Delta}$ .

**Proof.** Let  $G$  be a graph of maximum degree  $\Delta \geq \Delta_0$ , as in Theorem 13. We apply Theorem 9 with  $\mathcal{F} = \{K_{1,\beta+1}\}$  where  $\beta = \lfloor a(\log \Delta)/(\log \log \Delta) \rfloor$ ,  $a$  being the constant in Theorem 13. By Theorem 13,  $\chi_{2,\mathcal{F}}(G) \leq \Delta + 1$ . Applying Theorem 9, we get  $\dim_{\mathcal{A}}(G) \leq \binom{\Delta+1}{2} \dim_{\mathcal{A}}(\beta)$ . Thus, we get for all  $\Delta \geq \Delta_0$ ,

$$\dim_{\mathcal{A}}(\Delta) \leq \binom{\Delta+1}{2} \dim_{\mathcal{A}}\left(\left\lfloor \frac{a \log \Delta}{\log \log \Delta} \right\rfloor\right) \leq \Delta^2 \dim_{\mathcal{A}}\left(\left\lfloor \frac{a \log \Delta}{\log \log \Delta} \right\rfloor\right).$$

For  $x > 2$ , we define

$$f^0(x) = x \quad \text{and} \quad f(x) = \left\lfloor \frac{a \log x}{\log \log x} \right\rfloor,$$

and for  $i \geq 1$ ,

$$f^{i+1}(x) = \left\lfloor \frac{a \log f^i(x)}{\log \log f^i(x)} \right\rfloor.$$

Let  $k = \max \{i : f^i(\Delta) \geq 2^{2^a}\}$ . Note that  $f^{i+1}(\Delta) \leq \lfloor \log f^i(\Delta) \rfloor$  for  $i \leq k$ .

Hence  $k \leq \log^* \Delta$ . Applying statement (a) of Corollary 12, we obtain that  $\dim_{\mathcal{A}}(\Delta_0) \leq c_1 \Delta_0^3$ , where  $c_1$  is the constant mentioned in Corollary 12.

Assume without loss of generality that  $\Delta_0 \geq 2^{2^a}$ . As a result, we have

$$\begin{aligned} \dim_{\mathcal{A}}(\Delta) &\leq \Delta^2 \dim_{\mathcal{A}}(f(\Delta)) \leq \Delta^2 (f(\Delta))^2 \dim_{\mathcal{A}}(f^2(\Delta)) \\ &\leq \dots \leq \Delta^2 \left( \prod_{1 \leq i \leq k} (f^i(\Delta))^2 \right) \dim_{\mathcal{A}}(\lfloor e^{e^a} \rfloor) \\ &\leq c_1 \Delta_0^3 \cdot \Delta^2 \left( \prod_{1 \leq i \leq k} (f^i(\Delta))^2 \right). \end{aligned}$$

We now bound the product

$$S = \prod_{1 \leq i \leq k} f^i(\Delta).$$

Using the fact that  $f^{i+1}(\Delta) \leq \log f^i(\Delta)$  for  $i \leq k$ , we get

$$S \leq \left( \frac{a \log \Delta}{\log \log \Delta} \right) \left( \frac{a \log \log \Delta}{\log \log f(\Delta)} \right) \left( \frac{a \log \log f(\Delta)}{\log \log f^2(\Delta)} \right) \cdots \left( \frac{a \log \log f^{k-2}(\Delta)}{\log \log f^{k-1}(\Delta)} \right).$$

Thus,

$$S \leq a^k \log \Delta.$$

Hence, we get

$$\dim_{\mathcal{A}}(\Delta) \leq c \Delta^2 (\log \Delta)^2 \cdot a^{2(\log^* \Delta)}, \quad \text{where } c = c_1 \Delta_0^3.$$

By suitably choosing  $B$ , we can infer that  $c \cdot a^{2(\log^* \Delta)} \leq B^{\log^* \Delta}$ .

If  $\mathcal{A}$  satisfies the Zykov sum property, applying Theorem 9 yields

$$\dim_{\mathcal{A}}(\Delta) \leq (\Delta + 1) \dim_{\mathcal{A}} \left( \left\lfloor \frac{a \log \Delta}{\log \log \Delta} \right\rfloor \right) \leq 2\Delta \dim_{\mathcal{A}} \left( \left\lfloor \frac{a \log \Delta}{\log \log \Delta} \right\rfloor \right).$$

By a similar analysis carried out as before, one can deduce that  $\dim_{\mathcal{A}}(\Delta) \leq \Delta(\log \Delta)B^{\log^* \Delta}$ . This completes the proof of Theorem 14.  $\blacksquare$

**Remark.** As noted before, we have  $\log^* \Delta = o(\log^{(i)} \Delta)$  for every fixed  $i$ . In fact,  $\log^* \Delta \leq 5$  for any  $1 \leq \Delta \leq 2^{65536}$ . Thus, one sees that the above theorem replaces the  $O(\Delta^3)$  bound (for classes satisfying FDC) by a bound which is essentially  $\Delta^2(\log \Delta)^2$  ignoring the multiplicative factor which grows very slowly compared to other factors. Similar significant improvements can be noticed for other other types of classes also.

The assumption of  $\mathcal{A}$  being additive used in Theorems 9 and 14 is essential, as otherwise the dimension number need not always be expressed as a function of the maximum degree as the following examples illustrate.

**Unbounded dimension with only the FDC assumption.** Consider the class of graphs consisting of complete graph ( $K_n$ ) and complete graphs minus an edge ( $K_n - e$ ). This is the intersection of all dimensional classes satisfying the FDC property. The intersection dimension of a graph  $G$  with respect to this class is  $|E(G^c)|$ , which is not bounded by any function of the maximum degree.

**Unbounded dimension with only the Zykov sum assumption.** The Zykov sum property carries over intersection and thus we can consider the smallest dimensional class of graphs with Zykov sum property. This class is in fact the class of all complete graphs and complete graphs minus a matching (of any size). It is easy to see that the intersection dimension of a graph  $G$  with respect to this class is in fact  $\chi'(G^c)$ .  $\chi'(G^c)$  is the chromatic index of the complement  $G^c$ . This shows that for hereditary classes satisfying only the Zykov sum property, the intersection dimension need not always be bounded by a function of the maximum degree.

## 5. CIRCULAR DIMENSION — A SPECIAL CASE

Circular arc graphs (shortly, CA graphs) are defined as the intersection graphs of closed arcs of a circle. Despite their similarity to interval graphs (which are a subclass of CA graphs), these need not be perfect graphs while interval graphs are also perfect graphs. The class CA is clearly dimensional and hereditary but

it is not additive. As a result, the results of Sections 3 and 4 cannot be employed to obtain upper bounds for  $\dim_{CA}(G)$ .

Since the class of circular arcs is a superclass of the class of interval graphs, it follows that for any graph  $G$ ,  $\dim_{CA}(G) \leq \text{box}(G)$ . Employing the best known bound ([1]) on  $\text{box}(G)$ , we deduce that  $\dim_{CA}(G) \leq c\Delta(\log \Delta)^2$  for any  $G$ . We present below an asymptotic improvement over this bound on  $\dim_{CA}(G)$  that is asymptotically  $O\left(\Delta \left(\frac{\log \Delta}{\log \log \Delta}\right)\right)$ .

**Lemma 15.** *Let  $G$  be a split graph such that every clique vertex has at most  $t$  neighbors in the independent set. Then  $G$  has circular dimension at most  $t + 1$ .*

**Proof.** Let  $G$  be  $G = (I \cup C, E)$ . Form  $t + 1$  CA graphs  $G_0, G_1, \dots, G_t$  with  $G_i = (I \cup C, E_i)$  and  $E = E_0 \cap E_1 \cap \dots \cap E_t$  as follows. Assume, without loss of generality, that  $I = \{1, \dots, n\}$  is the independent set in  $G$ . Consider  $n + 1$  distinct points on the unit circle and label them consecutively with  $0, 1, \dots, n$ , traversing in the clockwise direction. In each  $G_k$  ( $0 \leq k \leq t$ ), each  $i \in I$  is identified with the closed circular arc consisting of just the point  $i$  on the circle. Define  $i_0 = 0$ . For any clique vertex  $u$  with  $r \geq 1$  neighbors in  $I$ , say  $i_1 < i_2 < \dots < i_r$ , and for any  $s$ ,  $0 \leq s \leq r$ , we identify  $u$  with the closed circular arc (clockwise) joining  $i_{s+1}$  with  $i_s$  (modulo  $r + 1$ ) in the graph  $G_s$ . For  $s > r$ , identify  $u$  in  $G_s$  with the circular arc used in  $G_r$ . If  $u$  has no neighbor in  $I$ , then identify  $u$  with the closed arc consisting of just the point  $i_0$ , in each  $G_s$  ( $0 \leq s \leq t$ ). It can be verified that  $E(G) = E(G_0) \cap E(G_1) \cap \dots \cap E(G_t)$  and that each  $G_i$  is a split graph. This proves Lemma 15. ■

**Theorem 16.** *The circular dimension satisfies  $\dim_{CA}(\Delta) \leq c\Delta \left(\frac{\log \Delta}{\log \log \Delta}\right)$  for some constant  $c$ .*

**Proof.** For  $\Delta \leq \Delta_0$  ( $\Delta_0$  is defined in Theorem 13), employ the  $O(\Delta(\log \Delta)^2)$  bound on  $\text{box}(G)$ . For  $\Delta \geq \Delta_0$ , applying Theorem 13, we obtain a  $\beta \left(\leq \frac{a(\log \Delta)}{\log \log \Delta}\right)$ -frugal coloring of  $V(G)$  using  $k = \Delta + 1$  colors. Let  $V_1, \dots, V_k$  be the color classes. We now form  $k$  split supergraphs  $G_1, \dots, G_k$  where  $G_i$  is obtained from  $G$  by making  $G[V - V_i]$  a complete graph, that is,  $G_i = (V, E \cup \{\{u, v\} : u, v \in V \setminus V_i\})$ . It can be seen that  $E(G) = E(G_1) \cap \dots \cap E(G_k)$ . Now we apply Lemma 15 to each  $G_i$  and deduce that  $\dim_{CA}(G_i) \leq \beta + 1$  and hence  $\dim_{CA}(G) \leq k(\beta + 1) \leq c\Delta \left(\frac{\log \Delta}{\log \log \Delta}\right)$  for a suitably chosen constant  $c > 0$ . This proves Theorem 16. ■

In this context, we recall the following lower bound on the maximum value of circular dimension over  $n$ -vertex graphs, obtained by Shearer [21].

**Theorem 17.** *There exist graphs on  $n$  vertices for which the circular dimension is at least  $\Omega\left(\frac{n}{\log_2 n}\right)$ .*

## 6. CONCLUDING REMARKS

We obtained upper bounds (in terms of some invariants like maximum degree) on intersection dimension of an arbitrary graph, for several dimensional, hereditary properties. It would be interesting to determine how tight these bounds are. Also, studying the computational complexity of determining the intersection dimension will be an interesting problem. In particular, we suggest the following open problems.

- Determine the asymptotically best bound for circular dimension in terms of maximum degree. We conjecture that it is  $O(\Delta)$ .
- It is known [11] that testing whether a graph has boxicity at most 2 is NP-complete. Can similar statements be established for other nontrivial dimensional graph properties?

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