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LARGE CONTRACTIBLE SUBGRAPHS OF A 3-CONNECTED GRAPH

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Abstract

Let $m \geq 5$ be a positive integer and let G be a 3-connected graph on at least $2m+1$ vertices. We prove that G has a contractible set W such that $m \leq |W| \leq 2m-4$. (Recall that a set $W \subset V(G)$ of a 3-connected graph G is contractible if the graph $G(W)$ is connected and the graph $G-W$ is 2-connected.) A particular case for $m=4$ is that any 3-connected graph on at least 11 vertices has a contractible set of 5 or 6 vertices.

Keywords: connectivity, 3-connected graph, contractible subgraph.

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Basic Definitions

Before introducing results of our paper let us recall main definitions that we need. We consider undirected graphs without loops and multiple edges and use standard notation.

For a graph G, we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We use notation $v(G)$ for the number of vertices of G. For disjoint sets $X, Y \subset V(G)$, we denote by $E_G(X,Y)$ the set of all edges of the graph G joining X and Y. A notation $xy \in E_G(X,Y)$ means that $x \in X$ and $y \in Y$.

We denote the *degree* of a vertex x in the graph G by $d_G(x)$.

Let $\mathrm{N}_G(w)$ denote the *neighborhood* of a vertex $w \in V(G)$ (i.e., the set of all vertices of the graph G adjacent to w). For a subset W of $V(G)$, let $\mathrm{N}_G(W)$

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denote the *neighborhood* of W (i.e., the set of all vertices of the graph G which are adjacent to W and do not belong to W).

For a set of vertices $U \subset V(G)$, we denote by $G(U)$ the *induced subgraph* of the graph G on the set U.

Let $u \in V(G)$, and let $W, U \subset V(G)$. We say that a vertex $u \in V(G)$ is *adjacent* to a set $W \subset V(G)$ if $u \notin W$ and u is adjacent to a vertex of W. Further, U is *adjacent* to W if a vertex of U is adjacent to W.

An *xy-path* is a path between vertices x and y. If P is a path containing x and y then xPy denote the part of P between x and y.

A *component* of a graph G is a maximal up to inclusion connected subgraph of G.

Definition. (1) Let $R \subset V(G)$. We denote by $G - R$ the graph obtained from G by deleting all vertices of the set R and all edges incident to vertices of R. The set R is a *cutset* if the graph $G - R$ is disconnected.

(2) If H is a subgraph of G then $G - H = G - V(H)$.

(3) A graph G is *k-connected* if $|V(G)| > k$ and G has no cutset of size less than k.

Definition. (1) A subset W of $V(G)$ is *connected* if $G(W)$ is connected.

(2) Let G be a 3-connected graph. A subset W of $V(G)$ is *contractible* if W is connected and $G - W$ is 2-connected.

1. Introduction and Main Results

Consider a 2-connected graph G on n vertices, and let n_1 and n_2 be positive integers with $n_1 + n_2 = n$. Clearly, $V(G)$ can be partitioned into two connected sets V_1 and V_2 such that $|V_1| = n_1$ and $|V_2| = n_2$.

In 1994, McCuaig and Ota [4] have formulated the following conjecture for 3-connected graphs. This conjecture was mentioned in Mader's survey on connectivity [3].

Conjecture 1. *Let $m \in \mathrm{N}$. Then there exists an integer n such that every 3-connected graph G on at least n vertices has a contractible set of m vertices.*

For $m = 1$, this statement is clear. For $m = 2$, it is rather easy and well-known (it was proved by Tutte). The case $m = 3$ was proved by the authors of this conjecture [4], the case $m = 4$ was proved by Kriesell [5]. For any $m \geq 5$, Conjecture 1 is open now. It is only known [6] that in case $m = 5$ Conjecture 1 is true for cubic graphs and graphs of average degree close to 3.

We suggest a new result on existence of large contractible sets in 3-connected graphs.

Theorem 2. *Let $m \geq 5$ be a positive integer and G be a 3-connected graph on at least $2m+1$ vertices. Then G has a contractible set W such that $m \leq |W| \leq 2m-4$.*

A particular case of this theorem for $m = 5$ is the following.

Corollary 3. *A 3-connected graph on $n \geq 11$ vertices has a contractible set of 5 or 6 vertices.*

In what follows, we formulate several facts on the structure of 2-connected graphs and after that, with the help of them, we prove Theorem 2.

2. Necessary Tools

We start with well known definitions of block and cutpoint.

2.1. Blocks and cutpoints of a connected graph

We have a classic instrument to study the structure of a connected graph — blocks and cutpoints. First we recall the definitions.

Definition. Let G be a connected graph.

A vertex $a \in V(G)$ is a *cutpoint* of G if the graph $G - a$ is disconnected.

A *block* of the graph G is a subgraph having no cutpoints which is maximal up to inclusion with this property.

The *interior*, denoted by $\mathrm{Int}(B)$, of a block B is the set of all its vertices which are not cutpoints of G.

The structure of mutual disposition of blocks and cutpoints of a connected graph can be described by the *tree of blocks and cutpoints* (see [7]). Recall that the tree of blocks and cutpoints of a graph G is a bipartite graph with bipartition $(\mathcal{B}, \mathcal{S})$, where $\mathcal{B}$ is the set of blocks and $\mathcal{S}$ is the set of cutpoints of G. A cutpoint a and a block B are adjacent if and only if $a \in V(B)$. It is easy to prove that this graph is a tree, all leaves of which correspond to blocks (which are called *pendant blocks*).

We need the following simple lemma.

Lemma 4. *Let G be a 2-connected graph and let $U, W \subset V(G)$. Assume that $U \cap W = \emptyset$ and U is not adjacent to W. If $G - U - W$ is 2-connected, then $G - U$ is 2-connected.*

Proof. Since $G - U - W$ is 2-connected, there exists a block B of $G - U$ which contains $G - U - W$. Suppose $G - U$ has a cutpoint, say a. Then a separates B from another block B'. Clearly, $V(B') \subset W$ and, therefore, $V(B')$ is not adjacent to U. Then a is a cutpoint of G, a contradiction. ■

2.2. The decomposition of a graph by a set of cutsets

We need to describe the structure of decomposition of a 2-connected graph by its 2-vertex cutsets. We define the *block tree* of a 2-connected graph as in [12]. In general, this structure is similar to Tutte's one [1]. We start with the *decomposition of a graph by a set of cutsets*, defined in [10].

Definition. Let $R \subset V(G)$ be a cutset.

(1) Let $X, Y \subset V(G)$, $X \not\subset R$, $Y \not\subset R$. We say that R *separates* X from Y if no two vertices $v_x \in X$ and $v_y \in Y$ belong to the same connected component of the graph $G - R$.

(2) We say that R *splits* a set $X \subset V(G)$ if the set $X \setminus R$ is not contained in one connected component of the graph $G - R$.

In this section, $k \geq 2$ and G is a k-connected graph. Denote by $\mathfrak{R}_k(G)$ the set of all k-vertex cutsets of G.

Definition. Let $\mathfrak{S} \subset \mathfrak{R}_k(G)$.

(1) A set $A \subset V(G)$ is a *part of decomposition* of G by $\mathfrak{S}$ if no cutset of $\mathfrak{S}$ splits A and A is maximal up to inclusion set with this property. By $\mathrm{Part}(G; \mathfrak{S})$ we denote the set of all parts of decomposition of G by $\mathfrak{S}$.

(2) Let $A \in \mathrm{Part}(G; \mathfrak{S})$. A vertex of A is *inner* if it does not belong to any cutset of $\mathfrak{S}$. The set of all inner vertices of the part A is called the *interior* of A, which is denoted by $\mathrm{Int}(A)$.
The *boundary* of A is the set $\mathrm{Bound}(A) = A \setminus \mathrm{Int}(A)$.

(3) For a set $S \in \mathfrak{R}_k(G)$, we will write simply $\mathrm{Part}(G; S)$ instead of $\mathrm{Part}(G; \{S\})$.

It is clear that if two parts of $\mathrm{Part}(G; \mathfrak{S})$ have nonempty intersection then their intersection is a subset of a certain cutset of $\mathfrak{S}$.

It is easy to prove [11] that $\mathrm{Bound}(A)$ consists of all vertices of the part A which are adjacent to $V(G) \setminus A$. If $\mathrm{Int}(A) \neq \emptyset$ then $\mathrm{Bound}(A)$ separates $\mathrm{Int}(A)$ from $V(G) \setminus A$.

Definition. Two cutsets $S, T \in \mathfrak{R}_k(G)$ are *independent* if S does not split T and T does not split S. Otherwise, these cutsets are *dependent*.

Remark 5. Let G be a k-connected graph and let $S, T \in \mathfrak{R}_k(G)$.

(1) Then either S and T are independent or each of them splits the other. For the detail of proof see [2, 8].

(2) Let S and T be independent. By the definition, there exist a part $A \in \mathrm{Part}(G; S)$ such that $T \subset A$ and a part $B \in \mathrm{Part}(G; T)$ such that $S \subset B$. If $A' \in \mathrm{Part}(G; S)$ and $A' \neq A$ then $A' \subset B$. For the detail of proof see [8].

(3) Let S and T be independent. Let $A \in \mathrm{Part}(G; S)$ and $B \in \mathrm{Part}(G; T)$. Clearly, if $A \subset B$ then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.

2.3. The block tree of a 2-connected graph

In this section, the graph G is 2-connected.

Definition. (1) A cutset $S \in \mathfrak{R}_2(G)$ is *single* if S is independent with all other cutsets of $\mathfrak{R}_2(G)$. Denote by $\mathfrak{O}(G)$ the set of all single cutsets of the graph G.

(2) We will write $\mathrm{Part}(G)$ instead of $\mathrm{Part}(G; \mathfrak{O}(G))$. Parts of this decomposition will be called simply *parts of* G.

Definition. The *block tree* $\mathrm{BT}(G)$ of a 2-connected graph G is a bipartite graph with bipartition $(\mathfrak{O}(G), \mathrm{Part}(G))$, where a single cutset S and a part A are adjacent if and only if $S \subset A$.

In what follows we list several properties of $\mathrm{BT}(G)$. Most of them are similar to properties of the classic tree of blocks and cutpoints of a connected graph.

Lemma 6 [13, Lemma 1]**.** *For a 2-connected graph G, the following statements hold.*

(1) $\mathrm{BT}(G)$ *is a tree. Every leaf of* $\mathrm{BT}(G)$ *corresponds to a part of* $\mathrm{Part}(G)$.

(2) *Let $B, B' \in \mathrm{Part}(G)$. Then a cutset $S \in \mathfrak{O}(G)$ separates B from B' in G if and only if S separates B from B' in* $\mathrm{BT}(G)$.

Definition. Let $A \in \mathrm{Part}(G)$. A part A is *pendant* if it corresponds to a leaf of $\mathrm{BT}(G)$.

Remark 7. If $A \in \mathrm{Part}(G)$ is a pendant part then $\mathrm{Bound}(A)$ is a single cutset of the graph G.

Definition. (1) For a 2-connected graph G, we denote by G' the graph obtained from G upon adding all edges of type ab where $\{a, b\} \in \mathfrak{O}(G)$.

(2) A part $A \in \mathrm{Part}(G)$ is called a *cycle* if the graph $G'(A)$ is a cycle. A is called a 3-*block* if $G'(A)$ is a 3-connected graph. If A is a cycle then $|A|$ is the *length* of A.

Lemma 8 [13, Lemma 2]**.** *For a 2-connected graph G, the following statements hold.*

(1) *Every part of* $\mathrm{Part}(G)$ *is either a cycle or a* 3*-block.*

(2) *If $A \in \mathrm{Part}(G)$ is a cycle, then all vertices of* $\mathrm{Int}(A)$ *have degree* 2 *in the graph G.*

(3) *Let $A \in \mathrm{Part}(G)$ be a cycle of length at least* 4*. Then any pair of its non-neighboring vertices form a non-single cutset of the graph G. All non-single cutsets of G are of such type.*

(4) *Let $S \in \mathfrak{R}_2(G)$ be a non-single cutset. Then $|\mathrm{Part}(G; S)| = 2$.*

Lemma 9 [12, Lemma 6]**.** *Assume that G is a 2-connected graph, $S \in \mathfrak{R}_2(G)$ and $B \in \mathrm{Part}(G;S)$. If $G(B)$ is 2-connected then $S \in \mathfrak{O}(G)$.*

Lemma 10. *Assume that G is a 2-connected graph, $S = \{a,b\} \in \mathfrak{R}_2(G)$ and $D \in \mathrm{Part}(G;S)$. Then one of the two following statements holds.*

1° *$G(D)$ is an ab-path.*

2° *There exists a pendant part $A \in \mathrm{Part}(G)$ such that $\mathrm{Int}(A) \subset \mathrm{Int}(D)$.*

Proof. Assume that there exists $T \in \mathfrak{O}(G)$ such that $T \subset D$. Since T is single, T is independent with S or T coincides with S. Hence, there is a part $D' \in \mathrm{Part}(G;T)$ such that $\mathrm{Int}(D') \subset \mathrm{Int}(D)$. By item (2) of Lemma 6, D' is a union of parts of $\mathrm{Part}(G)$ which lie in one component of $\mathrm{BT}(G) - S$. Clearly, among these parts, there is a pendant part $A \in \mathrm{Part}(G)$. Then $\mathrm{Int}(A) \subset \mathrm{Int}(D)$ and statement 2° holds.

Now we may assume that no single cutset is contained in D. In particular, $S \notin \mathfrak{O}(G)$. Then, by item (3) of Lemma 8, there exists a part $C \in \mathrm{Part}(G)$ such that $S \subset C$ and C is a cycle. Since D contains no single cutset, $\mathrm{Int}(D) \subset \mathrm{Int}(C)$. Therefore, $G(D)$ is a simple ab-path. ∎

Lemma 11 [12, Theorem 2]**.** *Let G be a 2-connected graph without single cutsets. Then either G is 3-connected or G is a cycle.*

3. Proof of Theorem 2

In what follows, the graph G will be 3-connected.

Definition. A contractible set $W \subset V(G)$ of a 3-connected graph G is *maximal* if there exists no vertex $x \in V(G) \setminus W$ such that the set $W \cup \{x\}$ is contractible.

Remark 12. Let $W \subset V(G)$ be a maximal contractible set and $x \in V(G) \setminus W$ be a vertex adjacent to W. Then the graph $G - W - x$ is not 2-connected.

Lemma 13. *Let G be a 3-connected graph, and $W \subset V(G)$ be a maximal contractible set such that the graph $H = G - W$ is not a cycle. Then the following statements hold.*

(1) *Let $A \in \mathrm{Part}(H)$ be a cycle. Then each inner vertex of A is adjacent to W.*

(2) *There are at least two pendant parts in $\mathrm{Part}(H)$, all these parts are cycles of length at least 4. The boundary of every pendant part is a single cutset of H.*

(3) *Let $A \in \mathrm{Part}(H)$ be a pendant part. Then $H - \mathrm{Int}(A)$ is 2-connected.*

Proof. (1) Let $A \in \mathrm{Part}(H)$ be a cycle and $x \in \mathrm{Int}(A)$. Then $d_H(x) = 2$. Since G is 3-connected, the vertex x must be adjacent in G to the set W.

(2) Since W is maximal, the graph H is not 3-connected. Since H is not a cycle, by Lemma 11 this graph has single cutsets. Hence, the tree $\mathrm{BT}(H)$ has at least two leaves which correspond to pendant parts of $\mathrm{Part}(H)$. The boundary of a pendant part is a single cutset of the graph H.

Consider a pendant part $A \in \mathrm{Part}(H)$. Let $\mathrm{Bound}(A) = S$. If W is not adjacent to $\mathrm{Int}(A)$ then a 2-vertex cutset S separates $\mathrm{Int}(A)$ in a 3-connected graph G. Since this is impossible, there exists a vertex $x \in \mathrm{Int}(A)$ adjacent to W in G. However, by maximality of W, the graph $H - x$ cannot be 2-connected. Hence, there exists a cutset $R \in \mathfrak{R}_2(H)$ which contains x. Since $x \in \mathrm{Int}(A)$, the cutset R is not single. Then, by item (3) of Lemma 8, the part A is a cycle of length at least 4.

(3) Let $\mathrm{Bound}(A) = \{x, x'\}$ and $H' = H - \mathrm{Int}(A)$. Suppose that H' is not 2-connected. Then H' has a cutpoint w. If both x and x' belong to the same block of H' then w is a cutpoint of H, a contradiction. Therefore, in H', w separates x from x'.

By item (2), vertices of the set $\mathrm{Int}(A)$ form a simple xx'-path in H. Since $\mathrm{Bound}(A) = \{x, x'\}$ is a single cutset in H, no cutset of $\mathfrak{R}_2(H)$ separates x from x'. Then, by Menger's theorem, there exist three independent xx'-paths in H. Clearly, at most one of these paths intersects $\mathrm{Int}(A)$. Therefore, in H', there are two independent xx'-paths. Thus, w cannot separate x from x' in H', a contradiction. ■

Theorem 2 is a consequence of the following lemma.

Lemma 14. *Let $m \geq 4$, $n \geq 2m + 3$ and let G be a 3-connected graph on n vertices. If G has a contractible set of $m \geq 4$ vertices, then G has a contractible set of m' vertices, where $m + 1 \leq m' \leq 2m - 2$.*

The proof of Lemma 14 is rather complicated. We divide this proof into several claims. In all these claims, let G satisfy the condition of Lemma 14, i.e., let G be a 3-connected graph with $v(G) \geq 2m + 3$. We assume that G has a contractible set of $m \geq 4$ vertices. Each such set is maximal, otherwise, Lemma 14 is proved. We try to find in the graph G a *suitable* vertex set W', i.e., a contractible set of size $m + 1 \leq |W'| \leq 2m - 2$.

For a maximal contractible set W of m vertices, we will use the notation $H = G - W$ and $F = G(W)$. Then H is 2-connected and F is connected.

Claim 15. *Let W be a maximal contractible set of m vertices. Assume that the graph $G - W$ is not a cycle and has a pendant part D with $|\mathrm{Int}(D)| \leq m - 2$. Then the assertion of Lemma 14 holds.*

Proof. Consider the set $W' = W \cup \mathrm{Int}(D)$. By item (1) of Lemma 13, the graph $G(W')$ is connected. By item (3) of Lemma 13, the graph

$$G - W' = (G - W) - \mathrm{Int}(D)$$

is 2-connected. Since $m = |W| < |W'| \le 2m - 2$, the set W' is suitable. ■

Claim 16. *Let M be a maximal contractible set of at most m vertices with $|\mathrm{N}_G(M)| = p \le m + 2$. Then the graph $G - M$ is not a cycle and has pendant parts $D_1, \dots, D_k$ such that*

$$\sum_{i=1}^{k} |\mathrm{Int}(D_i)| \le p.$$

Proof. Let $G' = G - M$. If G' is a cycle then all vertices of this cycle are adjacent to M in G. Therefore, $V(G) \subset M \cup \mathrm{N}_G(M)$, whence it follows $v(G) \le |M| + |\mathrm{N}_G(M)| \le 2m + 2$, a contradiction.

Thus, G' is not a cycle. Then the graph G and the set M satisfy the condition of Lemma 13. Therefore, G' has at least two pendant parts $D_1, \dots, D_k$ which interiors are disjoint. By item (1) of Lemma 13, $\bigcup_{i=1}^{k} \mathrm{Int}(D_i) \subset \mathrm{N}_G(M)$, whence our claim follows. ■

Claim 17. *Let M and W be two maximal contractible sets such that $|M| = m$, $|W| \le m$ and $|\mathrm{N}_G(M) \setminus W| \le 2$. Then the assertion of Lemma 14 holds.*

Proof. The contractible set M satisfies the condition of Claim 16. Let $G' = G - M$, let $D_1, \dots, D_k$ be pendant parts of the graph G' and $D = \bigcup_{i=1}^{k} \mathrm{Int}(D_i)$. If $W \not\subset D$ then

$$|D| \le |W| - 1 + 2 = m + 1,$$

whence by $k \ge 2$ the graph G' has a pendant part which interior contains at most $\frac{m+1}{2} < m - 1$ vertices. In this case, by Claim 15, the assertion of Lemma 14 holds.

Now let $W \subset D$. Clearly, $\mathrm{Int}(D_1), \dots, \mathrm{Int}(D_k)$ are vertex sets of components of $G(D)$. Since the graph $G(W)$ is connected, we have $W \subset \mathrm{Int}(D_i)$ for a certain i. Hence, the union of all other interiors consists of at most 2 vertices. Therefore, G' has a pendant part which interior has at most $2 \le m - 2$ vertices and, by Claim 15, the assertion of Lemma 14 holds. ■

Claim 18. *Let W be a maximal contractible set of at most m vertices and let the graph $H = G - W$ be a cycle. Then the assertion of Lemma 14 holds.*

Proof. Let $H = h_1 h_2 \cdots h_k$ be a cycle. It follows that $k \ge m + 3$, since $n \ge 2m + 3$ and $|W| \le m$. In the rest of the proof, subscripts are taken modulo k. Since G is a 3-connected graph, every vertex of H has degree at least 3 in G. Hence, each vertex of H has at least one neighbor in W. Recall that the graph $F = G(W)$ is connected.

Subclaim 18.1. *For $i \in \{1, 2, \ldots, k\}$, if $|\mathrm{N}_G(\{h_i, h_{i+m+1}\}) \cap W| \geq 2$, then Lemma 14 holds.*

Proof. Let x and y be two distinct vertices of W which are adjacent to h_i and h_{i+m+1}, respectively. Let $L = \{h_{i+1}, h_{i+2}, \ldots, h_{i+m}\}$ and let P be a xy-path in F. Then, in the graph $G' = G - L$, all vertices of the path P and the set $V(H) \setminus L$ lie on a cycle (see Figure 1a). Hence, these vertices lie in the same block B of the graph G'.

Let U be the set of all vertices of G' which do not belong to B. Then $U \subset W \setminus \{x, y\}$. Assume that $U \neq \emptyset$. Then every connected component of $G(U)$ is separated in the graph G' from B by a cutpoint and, therefore, is adjacent to L (since G is 3-connected). Let $W' = L \cup U$. It follows that $G(W')$ is connected. Further, W' is a contractible set, since $G - W' = B$ is 2-connected. Moreover,

$$m + 1 \leq |W'| = |L| + |U| \leq |L| + |W \setminus \{x, y\}| \leq 2m - 2$$

and Lemma 14 holds.

Hence, we may assume $U = \emptyset$. Then L is a contractible set of G. Further, we may assume L is a maximal contractible set, otherwise, Lemma 14 holds (since $|L| = m$). Note that $\mathrm{N}_G(L) \subset \{h_i, h_{i+m+1}\} \cup W$. By applying Claim 17 on the set L, the assertion of Lemma 14 holds. □

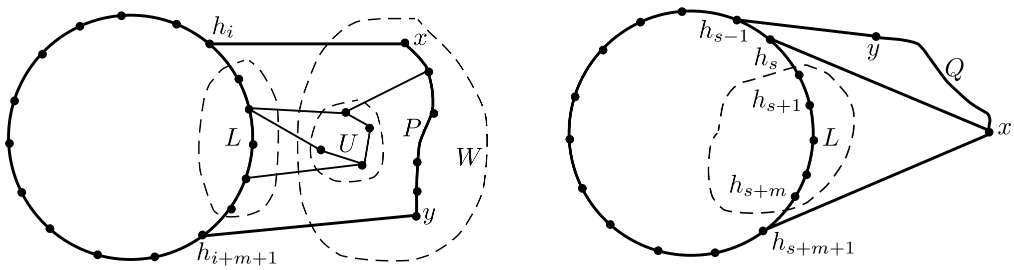


Figure 1. H is a cycle.

By Subclaim 18.1, we assume that $|\mathrm{N}_G(\{h_i, h_{i+m+1}\}) \cap W| \leq 1$, for $i \in \{1, 2, \ldots, k\}$. It follows that $|\mathrm{N}_G(h_i) \cap W| = 1$, for $i \in \{1, 2, \ldots, k\}$.

Subclaim 18.2. *If there exist $i, j \in \{1, 2, \ldots, k\}$ such that $\mathrm{N}_G(h_i) \cap W \neq \mathrm{N}_G(h_j) \cap W$, then Lemma 14 holds.*

Proof. We can pick s such that $\mathrm{N}_G(h_s) \cap W \neq \mathrm{N}_G(h_{s-1}) \cap W$. Let $\mathrm{N}_G(h_s) \cap W = \{x\}$ and $\mathrm{N}_G(h_{s-1}) \cap W = \{y\}$. Further, $L = \{h_{s+1}, h_{s+2}, \ldots, h_{s+m}\}$. Clearly, $h_{s-1} \notin L$.

Let Q be a xy-path in F and let $G' = G - L$. By Subclaim 18.1, x is adjacent to h_{s+m+1}. Therefore, in the graph G', all vertices of the path Q and of the set $V(H) \setminus L$ lie on a cycle (see Figure 1b). Hence, these vertices lie in the same block B of the graph G'. Now, by the same argument as in Subclaim 18.1, Subclaim 18.2 holds. □

By Subclaim 18.2, we may assume that all vertices of H are adjacent to exactly one vertex of W, say, x. Therefore, x is a cutpoint of a 2-connected graph G, a contradiction. Hence, Claim 18 holds. ■

If H is a cycle then Claim 18 shows that Lemma 14 holds. In what follows, we may assume that H is not a cycle.

Claim 19. *Let W be a maximal contractible set. Assume that $|W| \le m$. Let $A \in \mathrm{Part}(H)$ be a pendant part such that $|\mathrm{Int}(A)| \ge m$. Then the assertion of Lemma 14 holds.*

Proof. Recall that $F = G(W)$ is connected. By item (2) of Lemma 13, A is a cycle and $\mathrm{Bound}(A) = \{s, t\}$ is a single cutset of H. Let vertices of $\mathrm{Int}(A)$ follow $a_1, \dots, a_k$ from s to t, where $k \ge m$. Let $L = \{a_1, \dots, a_m\}$ and $G' = G - L$. If $k = m$ then let $t' = t$. If $k > m$ then let $t' = a_{m+1}$. By Lemma 13, $H' = H - \mathrm{Int}(A)$ is 2-connected.

Subclaim 19.1. *If G' is 2-connected, then Lemma 14 holds.*

Proof. Now L is a contractible set of m vertices. We assume that L is maximal, since otherwise Lemma 14 is proved. Since $\mathrm{N}_G(L) \subset W \cup \{s, t'\}$, by applying Claim 17 on the set L, Lemma 14 holds. □

Subclaim 19.2. *Let P be a connected subgraph of $G(\mathrm{Int}(A))$ and let v be a vertex of W which is adjacent to $V(P)$.*

(1) *Let B' be a block of $G-P$ which contains H'. Then $M = G-B'$ is connected.*

(2) *Let B' be a block of $G - P - v$ which contains H'. Then $M = G - B'$ is connected.*

Proof. (1) Suppose M is disconnected. Let M_1 be a component of M which does not contain P (see Figure 2a). Then $V(M_1) \subset W$ and no vertex of M_1 is adjacent to $V(P)$. By the definition of B', we find that $G - P$ has a cutpoint $w \notin M_1$ which separates M_1 from B'. It follows that w is a cutpoint of G, a contradiction. Hence, M is connected.

(2) The proof is similar to that of item (1). □

In what follows, we may assume that G' is not 2-connected. Let B be a block of G' which contains H'. By Lemma 13, there exists a pendant part $A' \in \mathrm{Part}(H)$ which is different from A and every inner vertex of A' is adjacent to W.

Subclaim 19.3. *If $k > m$, then $a_{m+1}, \dots, a_k \in V(B)$.*

Proof. Let $a' \in \mathrm{Int}(A')$. Both a_{m+1} and a' have neighbors in W, say y and y', respectively. There is a yy'-path in F and a $a't$-path in H' (see Figure 2b). Hence, there is a cycle which contains $a_{m+1}, \dots, a_k$, t and a'. Since $a', t \in V(B)$ and $a' \neq t$, all vertices $a_{m+1}, \dots, a_k$ are also contained in $V(B)$. □

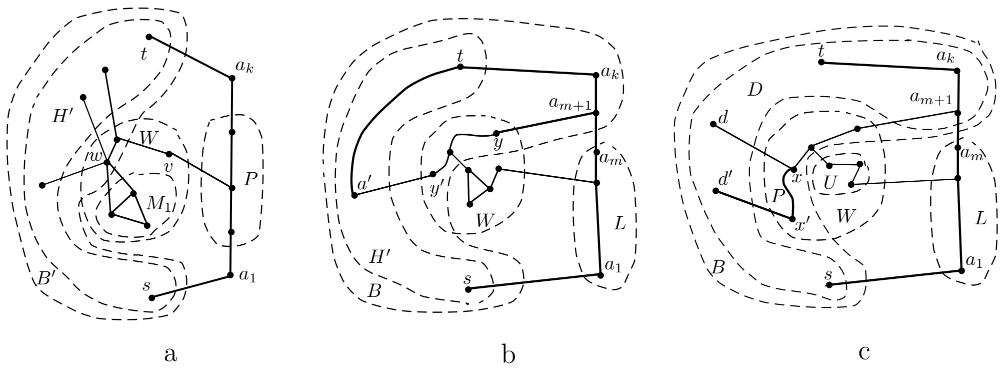


Figure 2. $|\mathrm{Int}(A)| \geq m$. Subclaims 19.2–19.4.

Let $D = V(G) \setminus (L \cup W) = V(H') \cup \{a_{m+1}, \dots, a_k\}$. Clearly, $\mathrm{Int}(A') \subset D$. By Subclaim 19.3, $D \subset V(B)$.

Subclaim 19.4. *If D has two distinct vertices d and d' such that $\mathrm{N}_G(d) \cap W \neq \emptyset$, $\mathrm{N}_G(d') \cap W \neq \emptyset$ and $\mathrm{N}_G(d) \cap W \neq \mathrm{N}_G(d') \cap W$, then Lemma 14 holds.*

Proof. Let $\mathrm{N}_G(d) \cap W = \{x\}$ and $\mathrm{N}_G(d') \cap W = \{x'\}$. There exists a xx'-path P in F (see Figure 2c). Clearly, P is contained in B. Let $U = W \setminus V(B)$ and $W' = L \cup U$. It follows that $U \subset W \setminus V(P)$. Since G' is not 2-connected, $U \neq \emptyset$. By Subclaim 19.2, $G(W')$ is connected. Since $G - W' = B$ is 2-connected, W' is a contractible set. Moreover,

$$m + 1 \leq |W'| = |L| + |U| \leq |L| + |W \setminus V(P)| \leq m + (m - 2) = 2m - 2,$$

and Lemma 14 holds. □

It was mentioned above that $\mathrm{N}_G(D) \cap W \supset \mathrm{N}_G(\mathrm{Int}(A')) \cap W \neq \emptyset$. By Subclaim 19.4, we may assume that $\mathrm{N}_G(D) \cap W = \{x\}$. Since $|\mathrm{Int}(A')| \geq 2$ and every vertex of $\mathrm{Int}(A')$ is adjacent to x, we have $x \in V(B)$.

Subclaim 19.5. *If $|\mathrm{N}_G(\{a_1, a_m\}) \cap W| \geq 2$, then Lemma 14 holds.*

Proof. By Lemma 13, $\mathrm{N}_G(a_1)\cap W \neq \emptyset$ and $\mathrm{N}_G(a_m)\cap W \neq \emptyset$. Hence, we can find in W two distinct vertices u and v such that $a_1u \in E(G)$ and $a_mv \in E(G)$. In F, there exist a xu-path P_u and a xv-path P_v. By symmetry, we may assume that P_u does not contain v. (The vertex u can coincide with x and the vertex v cannot coincide with x.) Let $L' = \{a_2, \dots, a_m, v\}$ and let $G_1 = G - L'$ (see Figure 3a).

Suppose G_1 is 2-connected. We may assume that L' is maximal (otherwise, Lemma 14 is proved). Recall that v has no neighbor in D (since $v \neq x$). Therefore, $\mathrm{N}_G(L') \subset W \cup \{a_1, t'\}$. Then Lemma 14 follows from Claim 17.

Now we may assume G_1 is not 2-connected. Let B' be the block of G_1 which contains H'. Let $a' \in \mathrm{Int}(A')$. Then a' is adjacent to x. Clearly, there is an sa'-path in H' (see Figure 3b). Therefore, the path a_1uP_ux is contained in B'.

Let U' be the set of all vertices of G_1 which do not belong to B' and $W'' = L' \cup U'$. By Subclaim 19.2, W'' is connected. Since $G - W'' = B'$ is 2-connected, W'' is contractible. Further,

$$m \leq |L'| < |W''| = |L'| + |U'| \leq |L'| + |W \setminus \{x, v\}| \leq m + (m-2) = 2m-2,$$

and Lemma 14 holds. $\square$

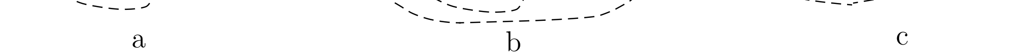


Figure 3. $|\mathrm{Int}(A)| \geq m$. Subclaim 19.5.

By Subclaim 19.5, we may assume that $|\mathrm{N}_G(\{a_1, a_m\}) \cap W| = 1$. Let $\mathrm{N}_G(\{a_1, a_m\}) \cap W = \{v\}$, where v can coincide with x. Let

$$M = \{a_2, \dots, a_{m-1}\} \cup W \setminus \{x, v\}$$

(see Figure 3c). Since $\mathrm{N}_G(D) \cap W = \{x\}$ and $\mathrm{N}_G(\{a_1, a_m\}) \cap W = \{v\}$, $G - M$ is a block of $G - \{a_2, \dots, a_{m-1}\}$. By Subclaim 19.2, M is a connected set. Thus, M is contractible. Further,

$$\begin{aligned} 2m - 4 = (m-2) + (m-2) &\leq |M| = |\{a_2, \dots, a_{m-1}\}| + |W \setminus \{x, v\}| \\ &\leq (m-2) + (m-1) = 2m-3. \end{aligned}$$

If $m \geq 5$ then M is a contractible set such that $m + 1 \leq |M| \leq 2m - 3$ and Lemma 14 holds. So, we may assume $m = 4$. Then $|M| = 4$. Therefore, M is a maximal contractible set (otherwise, Lemma 14 is proved). Since $W \setminus \{x, v\}$ is not adjacent to D, we have $\mathrm{N}_G(M) \subseteq W \cup \{a_1, a_m\}$. Hence, by Claim 17, Lemma 14 holds. ■

By Lemma 13, H has at least two pendant parts, say A and A'. Further, by Claims 15 and 19, assume that the interior of any pendant part of H consists of exactly $m - 1$ vertices. Let

$$\mathrm{Bound}(A) = \{s, t\}, \quad L = \mathrm{Int}(A) = \{a_1, \ldots, a_{m-1}\},$$

$$\mathrm{Bound}(A') = \{s', t'\}, \quad L' = \mathrm{Int}(A') = \{a'_1, \ldots, a'_{m-1}\},$$

where vertices of L are enumerated from s to t and vertices of L' are enumerated from s' to t'. Set the notation $N = V(H) \setminus (L \cup L')$. Recall that both graphs $H - L$ and $H - L'$ are 2-connected by Lemma 13.

Claim 20. *Assume that, for any vertex $w \in W$ and for any part $B \in \mathrm{Part}(H)$, there is at most one edge from w to $\mathrm{Int}(B)$. Then Lemma 14 holds.*

Proof. For each vertex $a \in L \cup L'$, we choose one edge from a to W. The chosen edges are called *good*. By the condition of the claim, any two good edges incident to vertices of L have distinct ends in W. Then, since $|L| = m - 1$, exactly one vertex in W (say, z) is not an end of a good edge incident to L. Similarly, exactly one vertex of W (say, z') is not an end of a good edge incident to L'.

Subclaim 20.1. *Assume that there exist two adjacent vertices $x, y \in W \setminus \{z'\}$. Then Lemma 14 holds.*

Proof. Consider the set $W' = L \cup W \setminus \{x, y\}$ (see Figure 4a). Then $|W'| = 2m - 3$. The graph $G - W'$ is 2-connected since it can be obtained from a 2-connected graph $H - L$ upon adding adjacent vertices x, y which have different neighbors in the set $L' \subset V(H - L)$. If the graph $G(W')$ is connected, the set W' is suitable and Lemma 14 is proved.

Assume that the graph $G(W')$ is disconnected. Then the only vertex of the set W which can be not adjacent to L (the vertex z) is separated in F by the set $\{x, y\}$ from all other vertices. Since F is connected, z is adjacent to at least one of x and y, say, to y. Since G is 3-connected, $d_G(z) \geq 3$. Thus, z is adjacent to $L' \cup N$. If z is adjacent to N (see Figure 4b) then $G(N \cup L' \cup \{z, y\})$ is 2-connected.

In the remaining case, z is not adjacent to N. Then z is adjacent to exactly one vertex of the set $N \cup L'$, say, to $a'_i \in L'$. Therefore, $zy, zx \in E(G)$. One of the vertices x and y (say, y) is adjacent to a vertex of the set $L' \setminus \{a'_i\}$ (see Figure 4c). Then $G(N \cup L' \cup \{z, y\})$ is 2-connected again. In both cases, the set

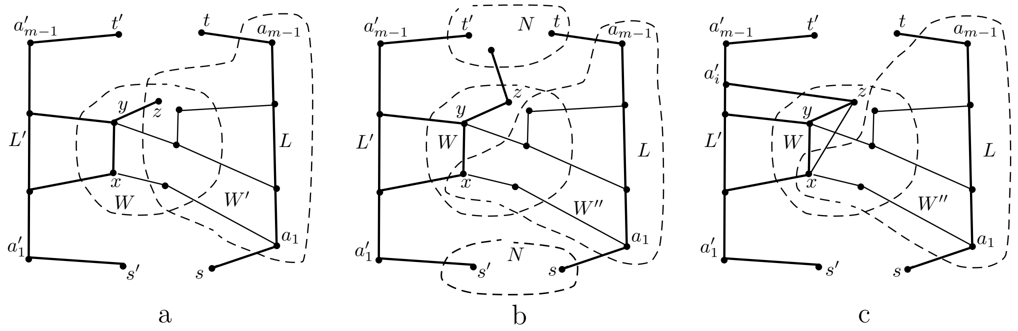


Figure 4. Subclaim 20.1.

$W'' = L \cup (W \setminus \{z, y\})$ is suitable: the graph $G - W'' = G(N \cup L' \cup \{z, y\})$ is 2-connected, the graph $G(W'')$ is connected (all vertices of the set $W \setminus \{z, y\}$ are adjacent to L) and $|W''| = 2m - 3$. Thus, Lemma 14 holds. □

Now we return to the proof of Claim 20. We may assume that all edges of the graph F are incident to the vertex z' (otherwise, by Subclaim 20.1, Lemma 14 holds). By symmetry, all edges of F are incident to z. Thus, $z = z'$ and F is a star with the center z (see Figure 5a). In this case, consider a vertex $y \in W$, adjacent to a'_2 and the set $M = L \cup \{y\}$. We will prove that the graph $G_1 = G - M$ is 2-connected. Since $H - L$ is 2-connected, vertices of the set $N \cup L' = V(H - L)$ lie in one block of G_1, say, B. All leaves of the star $F - y$ are incident to good edges, and other ends of these edges are distinct vertices of the set $L' \subset V(B)$. Therefore, we have $W \setminus \{y\} \subset V(B)$. Hence, $G_1 = B$ is a 2-connected graph.

Note that M is connected, $|M| = m$ and $\mathrm{N}_G(M) \subseteq (W \setminus \{y\}) \cup \{a'_2, s, t\}$. Thus, M is a maximal contractible set. By Claim 16, the graph $G - M$ is not a cycle and has pendant parts $D_1, \ldots, D_k$ (where $k \geq 2$) such that $\sum_{i=1}^{k} |\mathrm{Int}(D_i)| \leq m + 2$. Then $|\mathrm{Int}(D_i)| = m - 1$ for all $i \in \{1, \ldots, k\}$ (otherwise, by Claims 15 and 19, Lemma 14 holds). This is possible only if $m = 4$ and $k = 2$ (in this case, $m + 2 = 2(m - 1)$). Hence, $\mathrm{N}_G(M) = (W \setminus \{y\}) \cup \{a'_2, s, t\}$ and the graph $G(\mathrm{N}_G(M))$ has two connected components $\mathrm{Int}(D_1)$ and $\mathrm{Int}(D_2)$ such that $|\mathrm{Int}(D_1)| = |\mathrm{Int}(D_2)| = m - 1$. Since $G(W \setminus \{y\}) = F - y$ is connected and have exactly $m - 1$ vertices, $W \setminus \{y\}$ and $\{a'_2, s, t\}$ must be components of $G(\mathrm{N}_G(M))$. However, a'_2 can be adjacent only to a'_1, a'_3 and vertices of W. Hence, a'_2 is not adjacent to $\{s, t\}$, a contradiction. ■

Claim 21. *Let $y \in W$ be adjacent to two vertices of L. If $F - y$ is disconnected, then Lemma 14 holds.*

Proof. Let $U_1, \ldots, U_p$ be all connected components of $F - y$. Assume that U_1 is not adjacent to L' (see Figure 5b) and consider a block B' of U_1. Recall that

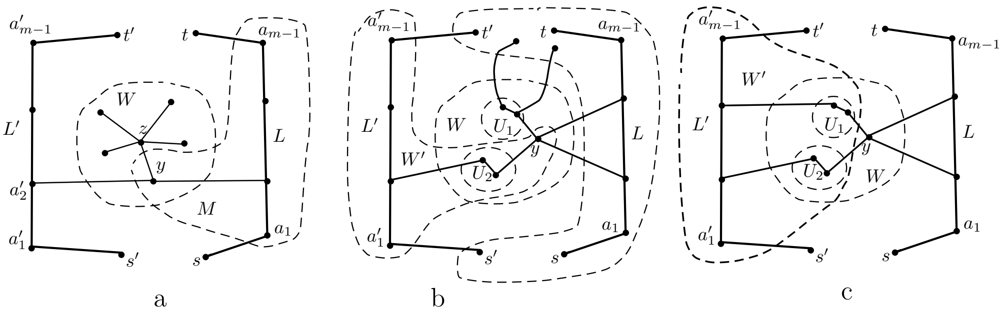


Figure 5. Claims 20 and 21.

$G - y$ is 2-connected and U_1 is not adjacent to $U_2, \dots, U_p$. Hence, in $G - y$, there exist two disjoint paths from B' to $L \cup N$ which inner vertices belong to U_1. Therefore, the graph $G' = G(N \cup L \cup U_1)$ is 2-connected.

Let $W' = L' \cup W \setminus U_1$. The graph $G - W' = G'$ is 2-connected, the graph $G(W')$ is connected (all components $U_2, \dots, U_k$ are adjacent to $y \in W \setminus U_1$ and $W \setminus U_1$ is adjacent to L') and

$$m + 1 \le |L'| + |U_2 \cup \{y\}| \le |W'| \le |L'| + |W| - |U_1| \le 2m - 2.$$

Hence, the set W' is suitable and Lemma 14 is proved.

Now we may assume that all components $U_1, \dots, U_p$ are adjacent to L' (see Figure 5c). In this case, $W' = L' \cup W \setminus \{y\}$. The graph $G - W' = G(N \cup L \cup \{y\})$ is 2-connected, the graph $G(W')$ is connected and $|W'| = 2m - 2$. Hence, the set W' is suitable and Lemma 14 is proved. ∎

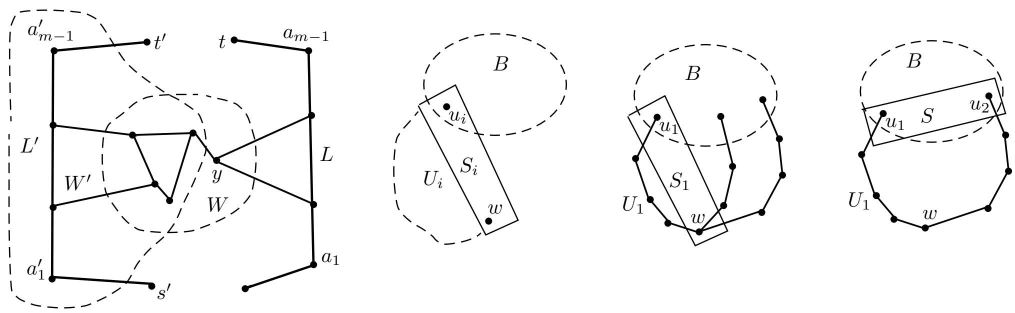
Next two claims will study properties of G under the assumption that Lemma 14 does not hold. In the proofs, we use the same notation as above.

Claim 22. *Assume that Lemma* 14 *does not hold. Let W be a contractible set of m vertices. Then there exists a vertex $y \in W$ such that, for any pendant part $D \in \mathrm{Part}(H)$, all vertices of* $\mathrm{Int}(D)$ *are adjacent to y and are not adjacent to $W \setminus \{y\}$.*

Proof. There exist a vertex $y \in W$ and a pendant part $A \in \mathrm{Part}(H)$ such that y has two neighbors in $L = \mathrm{Int}(A)$ (otherwise, Lemma 14 is proved by Claim 20). Moreover, $F - y$ is connected (otherwise, Lemma 14 is proved by Claim 21).

First, we prove the claim for a pendant part $A' \in \mathrm{Part}(H)$ which is different from A. Let $L' = \mathrm{Int}(A')$. We know that $|L| = |L'| = m - 1$ (otherwise, Lemma 14 is proved). Assume that $W \setminus \{y\}$ and L' are adjacent (see Figure 6a). Let $W' = L' \cup (W \setminus \{y\})$. The graph $G - W' = G(N \cup L \cup \{y\})$ is 2-connected, the graph $G(W')$ is connected and $|W'| = 2m-2$. Hence, the set W' is suitable and Lemma 14 holds, a contradiction.

Hence, L' and $W \setminus \{y\}$ are not adjacent. Then every vertex of L' is adjacent to y. Since $|L'| \geq 2$, we may exchange L and L' and, by symmetry, assure that each vertex of L is adjacent to y and is not adjacent to $W \setminus \{y\}$. Thus, we have proved the claim for every pendant part of $\mathrm{Part}(H)$. ■



a b c d

Figure 6. Claim 22 and Subclaim 23.1.

Claim 23. *Assume that Lemma* 14 *does not hold. Let W be a contractible set of m vertices. Then there exists a vertex $y \in W$ which is adjacent to all vertices of $W \setminus \{y\}$. Then there exists a vertex $y \in W$ which is adjacent to all vertices of $W \setminus \{y\}$. Moreover, for any pendant part $D \in \mathrm{Part}(H)$, all vertices of* $\mathrm{Int}(D)$ *are adjacent to y and are not adjacent to $W \setminus \{y\}$.*

Proof. By Claim 22, all inner vertices of pendant parts of $\mathrm{Part}(H)$ are adjacent to a certain vertex $y \in W$ and are not adjacent to $W \setminus \{y\}$.

Consider pendant parts A and A' of H and there interiors L and L', respectively. Let $M = L \cup \{y\}$, $G' = G - M$ and $W' = W \setminus \{y\}$. Clearly, $G - y$ is 2-connected. By Lemma 13, $H' = (G - y) - W' - L$ is 2-connected. Since L is not adjacent to W', by Lemma 4, the graph $G' = (G - y) - L$ is 2-connected. Since $|M| = m$ and the graph $G(M)$ is connected, M is a maximal contractible set.

Subclaim 23.1. *W' is a pendant part of the graph G'.*

Proof. We will prove that there exists a pendant part $D \in \mathrm{Part}(G')$ such that $\mathrm{Int}(D) \subset W'$. Then, by Claims 15 and 19, $|\mathrm{Int}(D)| = m - 1$ whence it follows $W' = \mathrm{Int}(D)$ and the subclaim holds.

Since F is connected, there exists a vertex $w \in W'$ which is adjacent to y. Since M is a maximal contractible set and $M \cup \{w\}$ is connected, $G' - w$ is not 2-connected. Since $G' - W' = H'$ is 2-connected, there is a block B of $G' - w$ which contains H'. Let $u_1, \dots, u_k$ be all cutpoints of $G'-w$ which belong to $V(B)$. Then, for all $i \in \{1, \dots, k\}$, $S_i = \{w, u_i\} \in \mathfrak{R}_2(G')$ and there is a part $U_i \in \mathrm{Part}(G'; S_i)$ such that $\mathrm{Int}(U_i) \subset W'$ (see Figure 6b). If S_i is single then, by Lemma 10, there exists a pendant part $D \in \mathrm{Part}(G')$ such that $\mathrm{Int}(D) \subset \mathrm{Int}(U_i)$, and we are done. In what follows, assume that S_i is not single. Then, by Lemma 6, $|\mathrm{Part}(G', S_i)| = 2$. Therefore, $G' - \mathrm{Int}(U_i) \in \mathrm{Part}(G'; S_i)$.

If $G'(U_i)$ is not a u_iw-path then, by Lemma 10, there exists a pendant part $D \in \mathrm{Part}(G')$ such that $\mathrm{Int}(D) \subset \mathrm{Int}(U_i)$, and we are done. Thus, we may assume that, for all $i \in \{1, \dots, k\}$, $G'(U_i)$ is a simple wu_i-path.

If $k = 1$ then $G' - \mathrm{Int}(U_1) = B$ and B is 2-connected. If $k \geq 3$ then $G' - \mathrm{Int}(U_1)$ is also 2-connected (see Figure 6c). In both cases, by Lemma 9, S_1 is single, a contradiction.

If $k = 2$ then $S = \{u_1, u_2\} \in \mathfrak{R}_2(G')$ and $\mathrm{Part}(G'; S) = \{V(B), U_1 \cup U_2\}$ (see Figure 6d). Since B is 2-connected, by Lemma 9, $S \in \mathfrak{O}(G)$. In this case, $U_1 \cup U_2$ is a pendant part of G'. Clearly, $\mathrm{Int}(U_1 \cup U_2) \subset W$ and the subclaim is proved. □

Now we finish the proof of Claim 23. By Subclaim 23.1, W' is a pendant part of the graph G'. By Claim 22, there exists a vertex $y' \in M$ which is adjacent to all vertices of W'. Since L is not adjacent to W', $y' = y$ (see Figure 7a). ■

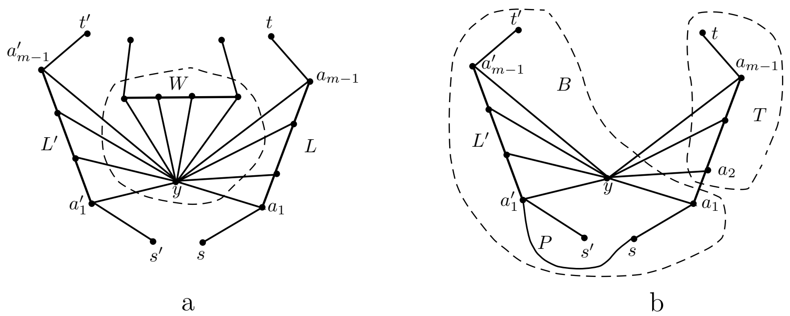


Figure 7. The vertex y.

Claim 24. *The set $T = \{a_2, \dots, a_{m-1}, t\}$ is contractible.*

Proof. First, let us prove that $G - T$ is 2-connected. Indeed, this graph is obtained from a 2-connected graph $G - t$ upon deleting vertices of the set $T' = T \setminus \{t\}$ which are adjacent in $G - t$ only to y and a_1 (see Figure 7b). In a 2-connected graph $H' = H - L$, there are two disjoint a'_1s-paths and at most one of them contains t. Thus, in $H' - t$, there is an a'_1s-path P which forms a cycle together with the path $sa_1ya'_1$. Thus, in $G - T$, there is a block B which contains a_1 and y. If $G - T$ is not 2-connected then it has a cutpoint x which separates B from another block B'. Since vertices of the set T' are adjacent in

$G - T = G - t - T'$ only to vertices of the block B, the vertex x also separates B from B' in $G - t$, a contradiction.

Thus, $G - T$ is 2-connected. Clearly, $G(T)$ is connected. Therefore, T is contractible. ■

The end of the proof of Lemma 14. Assume the statement of Lemma 14 does not hold. By Claim 24, the set $T = \{a_2, \dots, a_{m-1}, t\}$ is contractible. Consider two cases.

1. *The set T is not maximal.*

Then there exists a vertex $u \in \mathrm{N}_G(T)$ such that $G - T - u$ is 2-connected. Note that $u \neq y$, since $d_{G-T-y}(a_1) = 1$. However, any vertex $u \in V(G - T) \setminus \{y\}$ is not adjacent to $\{a_3, \dots, a_{m-1}\}$. Since $a_2t \notin E(G)$, the graph $G(T \cup \{u\})$ has no vertex adjacent to all others, a contradiction with Claim 23.

2. *The set T is maximal.*

The graph $H_0 = G - T$ is 2-connected. If H_0 is a cycle then Lemma 14 follows from Claim 18. Let H_0 be not a cycle. Consider a pendant part $D \in \mathrm{Part}(H)$. If $|\mathrm{Int}(D)| \geq m$ then Lemma 14 follows from Claim 19. Now assume that $|\mathrm{Int}(D)| \leq m - 1$ and consider a set $W' = T \cup \mathrm{Int}(D)$. By Lemma 13, the graph $G - W' = H_0 - \mathrm{Int}(D)$ is 2-connected and the graph $G(W')$ is connected. Thus, W' is contractible. Since $2 \leq |\mathrm{Int}(D)| \leq m - 1$, we have $m + 1 \leq |W'| \leq 2m - 2$, i.e., Lemma 14 is proved. ■

***Proof of Theorem* 2.** Consider the maximal $s \leq m$ such that the graph G has a contractible set U of s vertices. If $s = m$ we are done. Assume that $s \leq m - 1$. By Lemma 14, there exists another contractible set U' such that $s+1 \leq |U'| \leq 2s-2 \leq 2m-4$. By the maximality of s, we have $|U'| > m$. Thus, the set U' is suitable for Theorem 2. ■

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