

COMPARING ECCENTRICITY-BASED GRAPH INVARIANTS

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Abstract

The first and second Zagreb eccentricity indices (EM_1 and EM_2), the eccentric distance sum (EDS), and the connective eccentricity index (CEI) are all recently conceived eccentricity-based graph invariants, some of which found applications in chemistry. We prove that $EDS \geq EM_1$ for any connected graph, whereas $EDS > EM_2$ for trees. Moreover, in the case of trees, $EM_1 \geq CEI$, whereas $EM_2 > CEI$ for trees with at least three vertices. In addition, we compare EDS with EM_2 , and compare EM_1 , EM_2 with CEI for general connected graphs under some restricted conditions.

Keywords: eccentricity (of vertex), Zagreb eccentricity index, eccentric distance sum, connective eccentricity index.

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1. INTRODUCTION

Throughout this paper we consider only simple connected graphs. For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the *degree* of a

vertex v in G , denoted by $d_G(v)$, is the number of edges incident with v . Denote by $d_G(u, v)$ the distance between the vertices u and v in G . The *eccentricity* of a vertex v in the graph G is defined as $\varepsilon_G(v) = \max\{d_G(u, v) | u \in V(G)\}$. If $d_G(u, v) = \varepsilon_G(v)$ for some vertex u in G , then u is said to be an *eccentric vertex* of v . The *diameter* of a connected graph G is equal to $\max\{\varepsilon_G(v) | v \in V(G)\}$, while the *radius* is equal to $\min\{\varepsilon_G(v) | v \in V(G)\}$.

A connected graph is said to be a *tree* if it contains no cycles. Let P_n , S_n , C_n , and K_n be the path, star, cycle, and complete graph of order n , respectively. For notation and terminology not defined here, the readers are referred to [4].

The first and second Zagreb indices have been introduced more than forty years ago [16, 17] and became one of the best studied degree-based graph invariants [5, 27]. These are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

In an analogy with them, Vukićević and Graovac [30], and Ghorbani and Hosseinzadeh [13], independently, introduces the *first* and *second Zagreb eccentricity indices* as

$$(1) \quad EM_1(G) = \sum_{u \in V(G)} \varepsilon_G(u)^2$$

and

$$(2) \quad EM_2(G) = \sum_{uv \in E(G)} \varepsilon_G(u) \varepsilon_G(v).$$

For recent results on the Zagreb eccentricity indices of graphs, see [6, 7, 25, 28, 29, 31] and the references cited therein.

The *eccentric distance sum* (EDS) of a connected graph G , denoted by $\xi^d(G)$, is defined as

$$(3) \quad \xi^d(G) = \sum_{u \in V(G)} \varepsilon_G(u) D_G(u) = \sum_{\{u, v\} \subseteq V(G)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v).$$

This graph invariant was put forward by Gupta, Singh and Madan [15] as an eccentricity weighted version of the Wiener index [10]. It proved to be a structure descriptor that can be used to predict biological and physical properties of chemical compounds, in particular in structure activity/property relationships studies.

More recently, the mathematical properties of EDS have been extensively investigated. For recent results on the EDS, see [3, 12, 21–24, 26, 33].

Somewhat earlier, in 2000, Gupta *et al.* introduced the *connective eccentricity index* (CEI), denoted by $C^\xi(G)$, which is defined [14] as

$$(4) \quad C^\xi(G) = \sum_{u \in V(G)} \frac{d_G(u)}{\varepsilon_G(u)}.$$

For recent results on the CEI see [1, 32, 34] and the references cited therein. Obviously, by considering the total contribution of each edge, one can rewrite CEI as

$$(5) \quad C^\xi(G) = \sum_{uv \in E(G)} \left[\frac{1}{\varepsilon_G(u)} + \frac{1}{\varepsilon_G(v)} \right].$$

Relationships between various eccentricity-based graph invariants have received much attention over the past few decades, see e.g., [7–9, 18–20, 35]. Some of these researches were motivated by conjectures created by the Grafitti [11] and AutoGraphiX [2] software.

In this paper, we investigate the relationship between EM_1 , EM_2 , and EDS, and the relationship between EM_1 , EM_2 , and CEI. We organize this paper as follows. In Section 2, we first prove that EDS is greater than or equal to EM_1 for any connected graph. Then we prove that EDS is greater than EM_2 for trees. Moreover, we compare EDS with EM_2 for general connected graphs under some restricted conditions. In Section 3, we first show that for trees, EM_1 is greater than or equal to CEI. Then we prove that EM_2 is strictly greater than CEI for trees with at least three vertices. In addition, we compare EM_1 and EM_2 with CEI for general connected graphs under some restricted conditions.

2. COMPARISON OF EM_1 AND EM_2 WITH EDS

We begin our investigation by comparing EM_1 and EM_2 with EDS.

Theorem 1. *Let G be a connected graph of order n . Then*

$$\xi^d(G) \geq EM_1(G)$$

with equality if and only if $G \cong K_2$.

Proof. If $G \cong K_2$, then $\xi^d(G) = EM_1(G)$. Therefore, we assume that $n \geq 3$. If $\varepsilon_G(v) = 1$ for each v in G , then $G \cong K_n$. Elementary calculation yields

$$\xi^d(K_n) = n(n-1) > n = EM_1(K_n).$$

Then the statement of the theorem is true. So, we may assume that there exists at least one vertex, say v , in G such that $\varepsilon_G(v) \geq 2$. If so, then

$$\begin{aligned} D_G(v) &\geq [d_G(v) - 1] + [1 + 2 + \cdots + \varepsilon_G(v)] \\ &\geq \frac{1}{2} \varepsilon_G(v) [\varepsilon_G(v) + 1] > \varepsilon_G(v). \end{aligned}$$

Also, for any vertex u with $\varepsilon_G(u) = 1$, $D_G(u) = n - 1 > 1 = \varepsilon_G(u)$. Thus, by (1) and (3),

$$\xi^d(G) > EM_1(G).$$

This completes the proof. ■

Next, we examine the relationship between EM_2 and EDS. We first compare EM_2 with EDS for trees. Then we compare EM_2 with EDS for connected graphs under given restricted conditions.

Before proceeding, we prove the following result.

Lemma 2. *Let T be a tree. Then*

$$(6) \quad \sum_{u \in V(T)} \varepsilon_T(u)^2 > \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1].$$

Proof. Let n be the order of T . If $n = 2$, then $T \cong P_2$, and

$$\sum_{u \in V(T)} \varepsilon_T(u)^2 = 1 + 1 = 2 > 0 = \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1].$$

Hence, (6) follows readily. Therefore, we assume that $n \geq 3$. Denote by d be the diameter of T . Clearly, $d \geq 2$.

We proceed by induction on the order n .

Note that $n \geq d + 1$. We first prove that the inequality (6) holds for the case when $n = d + 1$.

If $n = d + 1$, then $T \cong P_n$. If n is even, then

$$(7) \quad \sum_{u \in V(T)} \varepsilon_T(u)^2 = 2 \left[\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2} + 1\right)^2 + \cdots + (n - 1)^2 \right]$$

and

$$\begin{aligned} \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1] &= 2 \left[\left(\frac{n}{2} - 1\right) \frac{n}{2} + \frac{n}{2} \left(\frac{n}{2} + 1\right) + \cdots \right. \\ (8) \quad &\quad \left. \cdots + (n - 3)(n - 2) \right] + \left(\frac{n}{2} - 1\right)^2. \end{aligned}$$

Thus, by (7) and (8),

$$\begin{aligned} & \sum_{u \in V(T)} \varepsilon_T(u)^2 - \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1] \\ &= 2 \left[\frac{n}{2} + \left(\frac{n}{2} + 1\right) + \dots + (n - 2) + (n - 1)^2 \right] - \left(\frac{n}{2} - 1\right)^2 \\ &> 2(n - 1)^2 - \left(\frac{n}{2} - 1\right)^2 = \frac{7}{4}n^2 - 3n + 1 > 0. \end{aligned}$$

Consider now the case when n is odd. Then

$$(9) \quad \sum_{u \in V(T)} \varepsilon_T(u)^2 = 2 \left[\left(\frac{n+1}{2}\right)^2 + \left(\frac{n+3}{2}\right)^2 + \dots + (n-1)^2 \right] + \left(\frac{n-1}{2}\right)^2,$$

$$(10) \quad \begin{aligned} \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1] &= 2 \left[\left(\frac{n-1}{2} - 1\right)\left(\frac{n+1}{2} - 1\right) + \left(\frac{n+1}{2} - 1\right) \right. \\ &\quad \left. \cdot \left(\frac{n+3}{2} - 1\right) + \dots + (n-3)(n-2) \right]. \end{aligned}$$

By (9) and (10), we have

$$\sum_{u \in V(T)} \varepsilon_T(u)^2 - \sum_{xy \in E(T)} [\varepsilon_T(x) - 1][\varepsilon_T(y) - 1] > \left(\frac{n-1}{2}\right)^2 > 0.$$

Thus, the inequality (6) holds for the case when $n = d + 1$. Now, let $n \geq d + 2$ and assume that (6) holds for smaller values of n .

Let $P_{d+1} = v_0v_1 \cdots v_{d-1}v_d$ be a diametrical path in T . For each $1 \leq j \leq d - 1$, let T_j be a component containing v_j of $T \setminus \{v_{j-1}, v_{j+1}\}$. Note that $n \geq d + 2$. Then $|T_k| \geq 2$ for some k . Among all such subtrees T_k with $|T_k| \geq 2$, we choose any one, say T_i . Let $u \in V(T_i)$ such that $d_T(u, v_i) = \max\{d_T(w, v_i) | w \in V(T_i)\}$. Then u is a pendent vertex which is farthest from v_i among all vertices in T_i .

We have the following claim.

Claim 3. *The vertex u cannot be the unique eccentric vertex of any vertex in T .*

Proof. Suppose to the contrary that there exists a vertex, say z , in $V(T)$ such that the unique eccentric vertex of z is u . Let T^1 be the component containing v_0 of $T \setminus \{v_i\}$, and let T^2 be the component containing v_d of $T \setminus \{v_i\}$. First, we assume that $z \in V(T_i)$. Without loss of generality, suppose that $i \geq \lfloor d/2 \rfloor$. (If this is not so, then we can relabel all vertices of the diametrical path in a reverse order.) By the assumption that the unique eccentric vertex of z is u , we have $d_T(z, u) > d_T(z, v_0)$. But, at the same time,

$$d_T(z, u) \leq d_T(z, v_i) + d_T(v_i, u) \leq d_T(z, v_i) + d_T(v_i, v_0) = d_T(z, v_0),$$

a contradiction. So $z \notin V(T_i)$.

Now, it must be either $z \in V(T^1)$ or $z \in V(T^2)$. If $z \in V(T^1)$, then by our assumption,

$$d_T(z, v_i) + d_T(v_i, v_d) = d_T(z, v_d) < d_T(z, u) = d_T(z, v_i) + d_T(v_i, u).$$

Thus, $d_T(v_i, v_d) < d_T(v_i, u)$, and then

$$d = d_T(v_0, v_d) = d_T(v_0, v_i) + d_T(v_i, v_d) < d_T(v_0, v_i) + d_T(v_i, u) = d_T(v_0, u),$$

a contradiction. So, $z \notin V(T^1)$. Similarly, we can prove that $z \notin V(T^2)$. This completes the proof. □

Let $T' = T \setminus \{u\}$. By our choice of u , T' is connected. In addition, the diameter of T' is equal to d . By the assumption that $i \geq \lfloor d/2 \rfloor$ and $d \geq 3$, we have $\varepsilon_T(u) \geq d_T(v_0, u) \geq i + 1 \geq \lfloor d/2 \rfloor + 1 \geq 2$. By Claim 3, for each vertex in $w \in V(T) \setminus \{u\}$, $\varepsilon_T(w) = \varepsilon_{T'}(w)$. By the induction hypothesis,

$$(11) \quad \sum_{w \in V(T')} \varepsilon_{T'}(w)^2 > \sum_{xy \in E(T')} [\varepsilon_{T'}(x) - 1] [\varepsilon_{T'}(y) - 1].$$

Therefore,

$$\begin{aligned} \sum_{w \in V(T)} \varepsilon_T(w)^2 &= \sum_{w \in V(T')} \varepsilon_T(w)^2 + \varepsilon_T(u)^2 = \sum_{w \in V(T')} \varepsilon_{T'}(w)^2 + \varepsilon_T(u)^2 \\ &> \sum_{xy \in E(T')} [\varepsilon_{T'}(x) - 1] [\varepsilon_{T'}(y) - 1] + \varepsilon_T(u)^2 \quad (\text{by (11)}) \\ &> \sum_{xy \in E(T')} [\varepsilon_{T'}(x) - 1] [\varepsilon_{T'}(y) - 1] + [\varepsilon_T(u) - 1] [\varepsilon_T(u) - 2] \\ &= \sum_{xy \in E(T) \setminus \{ut\}} [\varepsilon_T(x) - 1] [\varepsilon_T(y) - 1] + [\varepsilon_T(u) - 1] [\varepsilon_T(u) - 2] \\ &= \sum_{xy \in E(T) \setminus \{ut\}} [\varepsilon_T(x) - 1] [\varepsilon_T(y) - 1] + [\varepsilon_T(u) - 1] [\varepsilon_T(t) - 1] \\ &= \sum_{xy \in E(T)} [\varepsilon_T(x) - 1] [\varepsilon_T(y) - 1], \end{aligned}$$

where t is the unique neighbor of u in T .

The proof is complete by mathematical induction. ■

Next, we compare EM_2 with EDS for trees. We first need the following result.

Lemma 4. *Let G be a connected graph with radius $r \geq 2$. Then*

$$\xi^d(G) \geq \sum_{uv \in E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] + \sum_{u \in V(G)} \varepsilon_G(u)^2.$$

Proof. It can be easily seen that

$$\sum_{uv \notin E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G) \setminus N_G(u)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v).$$

Let x be a vertex in G such that $d_G(u, x) = \varepsilon_G(u)$. Then $\varepsilon_G(x) \geq d_G(u, x) = \varepsilon_G(u)$. For the above specified u and x , it holds

$$[\varepsilon_G(u) + \varepsilon_G(x)] d_G(u, x) \geq 2\varepsilon_G(u) \cdot \varepsilon_G(u) = 2\varepsilon_G(u)^2.$$

Since $\varepsilon_G(u) \geq r \geq 2$, $V(G) \setminus N_G(u) \neq \emptyset$. Moreover, if x is an eccentric vertex of u in G , then $x \in V(G) \setminus N_G(u)$. So

$$\sum_{v \in V(G) \setminus N_G(u)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v) \geq 2\varepsilon_G(u)^2$$

and then

$$\sum_{uv \notin E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v) \geq \sum_{u \in V(G)} \varepsilon_G(u)^2.$$

According to (3),

$$\begin{aligned} \xi^d(G) &= \sum_{uv \in E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] + \sum_{uv \notin E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] d_G(u, v) \\ &\geq \sum_{uv \in E(G)} [\varepsilon_G(u) + \varepsilon_G(v)] + \sum_{u \in V(G)} \varepsilon_G(u)^2. \end{aligned}$$

This completes the proof. ■

Theorem 5. *Let T be a tree. Then*

$$\xi^d(T) > EM_2(T).$$

Proof. Let n be the order of T . If the radius of T is 1, then T is a star. Thus, $\xi^d(T) = (n - 1)(4n - 5)$ and $EM_2(T) = 2(n - 1)$. So, the statement of the theorem is true as $n \geq 2$. Now, we may suppose that the radius of T is at least

two. According to Lemmas 2 and 4,

$$\begin{aligned} \xi^d(T) &\geq \sum_{uv \in E(T)} [\varepsilon_T(u) + \varepsilon_T(v)] + \sum_{w \in V(T)} \varepsilon_T(w)^2 \\ &> \sum_{uv \in E(T)} [\varepsilon_T(u) + \varepsilon_T(v)] + \sum_{uv \in E(T)} [\varepsilon_T(u) - 1] [\varepsilon_T(v) - 1] \\ &= \sum_{uv \in E(T)} \varepsilon_T(u) \varepsilon_T(v) + (n - 1) > \sum_{uv \in E(T)} \varepsilon_T(u) \varepsilon_T(v) = EM_2(T). \end{aligned}$$

This completes the proof. \blacksquare

In what follows, we compare EM_2 with EDS for connected graphs under given restricted conditions.

Theorem 6. *Let G be a connected graph with radius r and maximum degree Δ . If $r \geq \Delta$, then*

$$\xi^d(G) \geq EM_2(G).$$

Proof. Let n be the order of G . If $n = 2$, then $G \cong K_2$, and $\xi^d(G) > EM_2(G)$, as claimed. Now, we assume that $n \geq 3$. According to (2), we can rewrite the second Zagreb eccentricity indices as

$$(12) \quad EM_2(G) = \frac{1}{2} \sum_{x \in V(G)} \left[\varepsilon_G(x) \cdot \sum_{y \in N_G(x)} \varepsilon_G(y) \right].$$

For any edge $xy \in E(G)$ and any vertex $u \in V(G) \setminus \{x, y\}$, it holds

$$d_G(y, u) - d_G(x, u) \leq 1,$$

implying that

$$(13) \quad \varepsilon_G(y) \leq \varepsilon_G(x) + 1.$$

Thus, by (12) and (13),

$$(14) \quad EM_2(G) \leq \sum_{x \in V(G)} \varepsilon_G(x) \cdot \frac{1}{2} [d_G(x)(\varepsilon_G(x) + 1)].$$

As proved in Theorem 1, for each $x \in V(G)$, we have

$$D_G(x) \geq \frac{1}{2} \varepsilon_G(x) [\varepsilon_G(x) + 1].$$

Note that $\varepsilon_G(x) \geq r \geq \Delta \geq d_G(x)$. Therefore, for each $x \in V(G)$,

$$(15) \quad D_G(x) \geq \frac{1}{2} d_G(x) [\varepsilon_G(x) + 1].$$

So, by (3), (14) and (15),

$$\xi^d(G) \geq EM_2(G).$$

This completes the proof. ■

3. COMPARISON OF EM_1 AND EM_2 WITH CEI

In order to find the relationship between EM_1 , EM_2 , and CEI, we first consider the following three special graphs.

For the complete graph K_n ,

$$C^\xi(K_n) = n(n - 1), \quad EM_1(K_n) = n, \quad EM_2(K_n) = \frac{n(n - 1)}{2},$$

implying that $C^\xi(K_n) > EM_1(K_n)$ and $C^\xi(K_n) > EM_2(K_n)$ for $n \geq 2$.

For the star S_n ($n \geq 3$),

$$C^\xi(S_n) = \frac{3(n - 1)}{2}, \quad EM_1(S_n) = 4n - 3, \quad EM_2(S_n) = 2(n - 1),$$

implying that $C^\xi(S_n) < EM_1(S_n)$ and $C^\xi(S_n) < EM_2(S_n)$ for $n \geq 3$.

For the cycle C_n ,

$$C^\xi(C_n) = n \frac{2}{\lfloor n/2 \rfloor}, \quad EM_1(C_n) = EM_2(C_n) = n (\lfloor n/2 \rfloor)^2,$$

implying that $C^\xi(C_n) < EM_1(C_n) = EM_2(C_n)$ for $n \geq 4$, and $C^\xi(C_n) > EM_1(C_n) = EM_2(C_n)$ for $n = 3$.

From the above examples, it is seen that in the general case, the graph invariants EM_1 , EM_2 , and CEI are not comparable. Bearing this in mind, in what follows we examine the relationship between EM_1 , EM_2 , and CEI for trees. We first compare EM_1 with CEI.

Theorem 7. *Let T be a tree of order n . Then*

$$EM_1(T) \geq C^\xi(T)$$

with equality if and only if $T \cong P_2$.

Proof. If $n = 2$, then $T \cong P_2$, and $C^\xi(T) = 2 = EM_1(T)$. If $n = 3$, then $T \cong P_3$, and $EM_1(T) = 9 > 3 = C^\xi(T)$. Assume therefore that $n \geq 4$. Let d be the diameter of T . If $d = 2$, then $T \cong S_n$. It can be easily seen that $EM_1(T) = 4n - 3$ and $C^\xi(T) = \frac{3}{2}(n - 1)$. Thus, $EM_1(T) > C^\xi(T)$. Now, we assume that $d \geq 3$. Then $\varepsilon_T(x) \geq 2$ for each vertex x in T . Thus, for each edge uv ,

$$[\varepsilon_T(u) - 1][\varepsilon_T(v) - 1] \geq 1$$

and

$$\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, for each edge uv ,

$$(16) \quad [\varepsilon_T(u) - 1][\varepsilon_T(v) - 1] - \left(\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \right) \geq 0.$$

By (6) and (16),

$$\begin{aligned} EM_1(T) - C^\xi(T) &= \sum_{w \in V(T)} \varepsilon_T(w)^2 - \sum_{uv \in E(T)} \left(\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \right) \\ &> \sum_{uv \in E(T)} \left[[\varepsilon_T(u) - 1][\varepsilon_T(v) - 1] - \left(\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \right) \right] \geq 0. \end{aligned}$$

This completes the proof. ■

Next, we compare EM_1 with CEI for connected graphs under given restricted conditions.

We first state a result due to Ilić, Yu and Feng.

Lemma 8 [23]. *Let G be a connected graph of order n . For each vertex v in G , it holds*

$$(17) \quad \varepsilon_G(v) \leq n - d_G(v).$$

Moreover, all equalities hold together if and only if $G \cong P_4$ or $G \cong K_n \setminus iK_2$, (for $0 \leq i \leq \lfloor n/2 \rfloor$), where for each i , $K_n \setminus iK_2$ is the graph obtained by removing i independent edges from G .

Remark 9. The path P_2 also achieves the equality of (17) in Lemma 8.

Theorem 10. *Let G be a connected graph of order n with radius r . If $r \geq \lceil \sqrt{n/2} \rceil$, then*

$$EM_1(G) \geq C^\xi(G)$$

with equality if and only if $G \cong P_2$.

Proof. Since $r \geq \lceil \sqrt{n/2} \rceil$, $\varepsilon_G(x) \geq \sqrt{n/2}$ for each vertex x in G . By Lemma 8,

$$\begin{aligned}
 (18) \quad \varepsilon_G(x)^2 - \frac{d_G(x)}{\varepsilon_G(x)} &\geq \varepsilon_G(x)^2 - \frac{n - \varepsilon_G(x)}{\varepsilon_G(x)} \\
 &= \frac{\varepsilon_G(x) [\varepsilon_G(x)^2 + 1] - n}{\varepsilon_G(x)} \\
 (19) \quad &\geq \frac{\varepsilon_G(x) \cdot 2\varepsilon_G(x) - n}{\varepsilon_G(x)} \\
 (20) \quad &\geq 0.
 \end{aligned}$$

Therefore, by (1) and (4),

$$EM_1(G) - C^\xi(G) = \sum_{u \in V(G)} \left[\varepsilon_G(u)^2 - \frac{d_G(u)}{\varepsilon_G(u)} \right] \geq 0.$$

We now consider the equality condition. If $EM_1(G) = C^\xi(G)$, then all inequalities (18)–(20) becomes equalities for each x in G . Thus, $\varepsilon_G(x) = n - d_G(x)$, $\varepsilon_G(x) = 1$ and $\varepsilon_G(x) = \sqrt{n/2}$ hold together for each x in G . Therefore, $G \cong P_2$.

Conversely, if $G \cong P_2$, then $EM_1(G) = C^\xi(G)$. ■

Theorem 11. *Let T be a tree of order n . Then*

$$EM_2(T) > C^\xi(T)$$

for $n \geq 3$, and

$$EM_2(T) < C^\xi(T)$$

for $n = 2$.

Proof. If $n = 2$, then $T \cong P_2$, and $C^\xi(T) = 2 > 1 = EM_2(T)$. Now, we assume that $n \geq 3$. According to (2) and (5),

$$EM_2(T) - C^\xi(T) = \sum_{uv \in E(T)} \left[\varepsilon_T(u) \varepsilon_T(v) - \left(\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \right) \right].$$

Let d be the diameter of T . If $d \geq 3$, then $\varepsilon_T(x) \geq 2$ for each vertex x in T . Thus, for each edge uv ,

$$\varepsilon_T(u) \varepsilon_T(v) \geq 4$$

and

$$\frac{1}{\varepsilon_T(u)} + \frac{1}{\varepsilon_T(v)} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore, $EM_2(T) > C^\xi(T)$ for $d \geq 3$.

If $d = 2$, then $T \cong S_n$. Then $EM_2(T) = 2(n - 1)$ and $C^\xi(T) = \frac{3}{2}(n - 1)$. Again, $EM_2(T) > C^\xi(T)$.

This completes the proof. ■

Remark 12. In Theorems 7 and 11, we restricted the consideration to trees. Already for general connected graphs, the statements of Theorems 7 and 11 may be violated. For instance, consider a graph with diameter two, say the complete bipartite graph $K_{2,n-2}$ ($n \geq 5$). It is easy to check that $C^\xi(K_{2,n-2}) = 2n(n-2)$, $EM_1(K_{2,n-2}) = 4n$, and $EM_2(K_{2,n-2}) = 8(n-2)$. Thus, $C^\xi(K_{2,n-2}) > EM_1(K_{2,n-2})$ and $C^\xi(K_{2,n-2}) > EM_2(K_{2,n-2})$, contradicting to Theorems 7 and 11.

By the same reasoning as in Theorem 11, we can prove the following result.

Theorem 13. *Let G be a connected graph with diameter at least three. Then*

$$EM_2(G) > C^\xi(G).$$

4. CONCLUDING REMARKS

In this paper, we have investigated the relationships between the Zagreb eccentricity indices and the eccentric distance sum, and the relationships between the Zagreb eccentricity indices and the connective eccentricity index. We proved that the eccentric distance sum is always greater than or equal to the first Zagreb eccentricity index for any connected graph. However, for the comparison of the second Zagreb eccentricity index with the eccentric distance sum, and the comparison of the first and second Zagreb eccentricity indices with the connective eccentricity index, the considerations had to be restricted to trees; in the case of general connected graphs these eccentricity-based invariants are not comparable.

For trees, we proved that the eccentric distance sum is always greater than the second Zagreb eccentricity index, and that the first Zagreb eccentricity index is always greater than or equal to the connective eccentricity index. Moreover, the second Zagreb eccentricity index is always greater than the connective eccentricity index for trees of order 3 or greater.

In addition, we compared the eccentric distance sum with the second Zagreb eccentricity index, and compared the first and second Zagreb eccentricity indices with the connective eccentricity index for general connected graphs under some restricted conditions.

Comparing these indices for connected graphs under other restricted conditions seem to be interesting, and remains as an open problem.

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REFERENCES

- [1] Y. Alizadeh and S. Klavžar, *Complexity of topological indices: The case of connective eccentric index*, MATCH Commun. Math. Comput. Chem. **76** (2016) 659–667.
- [2] M. Aouchiche and P. Hansen, *Proximity and remoteness in graphs: results and conjectures*, Networks **58** (2011) 95–102.
doi:10.1002/net.20450
- [3] M. Azari and A. Iranmanesh, *Computing the eccentric-distance sum for graph operations*, Discrete Appl. Math. **161** (2013) 2827–2840.
doi:10.1016/j.dam.2013.06.003
- [4] J.A. Bondy and U.S.R. Murty, *Graph Theory* (Springer, New York, 2008).
- [5] B. Borovičanić, K.C. Das, B. Furtula and I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. **78** (2017) 17–100.
- [6] K.C. Das, *Some properties of the Zagreb eccentricity indices*, Ars Math. Contemp. **6** (2013) 117–125.
doi:10.26493/1855-3974.237.48a
- [7] K.C. Das, *Comparison between Zagreb eccentricity indices and the eccentric connectivity index, the second geometric-arithmetic index and the Graovac-Ghorbani index*, Croat. Chem. Acta **89** (2016) 505–510.
doi:10.5562/cca3007
- [8] K.C. Das, I. Gutman and M.J. Nadjafi-Arani, *Relations between distance-based and degree-based topological indices*, Appl. Math. Comput. **270** (2015) 142–147.
doi:10.1016/j.amc.2015.08.061
- [9] K.C. Das and M.J. Nadjafi-Arani, *Comparison between the Szeged index and the eccentric connectivity index*, Discrete Appl. Math. **186** (2015) 74–86.
doi:10.1016/j.dam.2015.01.011
- [10] A. Dobrynin, R. Entringer and I. Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math. **66** (2001) 211–249.
doi:10.1023/A:1010767517079
- [11] S. Fajtlowicz and W.A. Waller, *On two conjectures of GRAFFITI II*, Congr. Numer. **60** (1987) 187–197.
- [12] X. Geng, S. Li and M. Zhang, *Extremal values on the eccentric distance sum of trees*, Discrete Appl. Math. **161** (2013) 2427–2439.
doi:10.1016/j.dam.2013.05.023

- [13] M. Ghorbani and M.A. Hosseinzadeh, *A new version of Zagreb indices*, *Filomat* **26** (2012) 93–100.
doi:10.2298/FIL1201093G
- [14] S. Gupta, M. Singh and A.K. Madan, *Connective eccentricity index: a novel topological descriptor for predicting biological activity*, *J. Mol. Graph. Model.* **18** (2000) 18–25.
doi:10.1016/S1093-3263(00)00027-9
- [15] S. Gupta, M. Singh and A.K. Madan, *Eccentric distance sum: A novel graph invariant for predicting biological and physical properties*, *J. Math. Anal. Appl.* **275** (2002) 386–401.
doi:10.1016/S0022-247X(02)00373-6
- [16] I. Gutman, B. Rušćić, N. Trinajstić and C.F. Wilcox, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, *J. Chem. Phys.* **62** (1975) 3399–3405.
doi:10.1063/1.430994
- [17] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons*, *Chem. Phys. Lett.* **17** (1972) 535–538.
doi:10.1016/0009-2614(72)85099-1
- [18] H. Hua, Y. Chen and K.C. Das, *The difference between remoteness and radius of a graph*, *Discrete Appl. Math.* **187** (2015) 103–110.
doi:10.1016/j.dam.2015.02.007
- [19] H. Hua and K.C. Das, *The relationship between the eccentric connectivity index and Zagreb indices*, *Discrete Appl. Math.* **161** (2013) 2480–2491.
doi:10.1016/j.dam.2013.05.034
- [20] H. Hua and K.C. Das, *Proof of conjectures on remoteness and proximity in graphs*, *Discrete Appl. Math.* **171** (2014) 72–80.
doi:10.1016/j.dam.2014.02.011
- [21] H. Hua, K. Xu and S. Wen, *A short and unified proof of Yu et al.'s two results on the eccentric distance sum*, *J. Math. Anal. Appl.* **382** (2011) 364–366.
doi:10.1016/j.jmaa.2011.04.054
- [22] H. Hua, S. Zhang and K. Xu, *Further results on the eccentric distance sum*, *Discrete Appl. Math.* **160** (2012) 170–180.
doi:10.1016/j.dam.2011.10.002
- [23] A. Ilić, G. Yu and L. Feng, *On the eccentric distance sum of graphs*, *J. Math. Anal. Appl.* **381** (2011) 590–600.
doi:10.1016/j.jmaa.2011.02.086
- [24] S. Li and Y. Wu, *On the extreme eccentric distance sum of graphs with some given parameters*, *Discrete Appl. Math.* **206** (2016) 90–99.
doi:10.1016/j.dam.2016.01.027
- [25] Z. Luo and J. Wu, *Zagreb eccentricity indices of the generalized hierarchical product graphs and their applications*, *J. Appl. Math.* **1** (2014) 1–8.
doi:10.1155/2014/241712

- [26] V. Mukungunugwa and S. Mukwembi, *On eccentric distance sum and minimum degree*, Discrete Appl. Math. **175** (2014) 55–61.
doi:10.1016/j.dam.2014.05.019
- [27] S. Nikolić, G. Kovačević, A. Miličević and N. Trinajstić, *The Zagreb indices 30 years after*, Croat. Chem. Acta **76** (2003) 113–124.
- [28] X. Qi and Z. Du, *On Zagreb eccentricity indices of trees*, MATCH Commun. Math. Comput. Chem. **78** (2017) 241–256.
- [29] X. Qi, B. Zhou and J. Li, *Zagreb eccentricity indices of unicyclic graphs*, Discrete Appl. Math. **233** (2017) 166–174.
doi:10.1016/j.dam.2017.08.001
- [30] D. Vukičević and A. Graovac, *Note on the comparison of the first and second normalized Zagreb eccentricity indices*, Acta Chim. Sloven. **57** (2010) 524–528.
- [31] R. Xing, B. Zhou and N. Trinajstić, *On Zagreb eccentricity indices*, Croat. Chem. Acta **84** (2011) 493–497.
doi:10.5562/cca1801
- [32] G. Yu and L. Feng, *On the connective eccentricity index of graphs*, MATCH Commun. Math. Comput. Chem. **69** (2013) 611–628.
- [33] G. Yu, L. Feng and A. Ilić, *On the eccentric distance sum of trees and unicyclic graphs*, J. Math. Anal. Appl. **375** (2011) 99–107.
doi:10.1016/j.jmaa.2010.08.054
- [34] G. Yu, H. Qu, L. Tang and L. Feng, *On the connective eccentricity index of trees and unicyclic graphs with given diameter*, J. Math. Anal. Appl. **420** (2014) 1776–1786.
doi:10.1016/j.jmaa.2014.06.050
- [35] H. Zhang, S. Li and B. Xu *Extremal graphs of given parameters with respect to the eccentricity distance sum and the eccentric connectivity index*, Discrete Appl. Math. **254** (2019) 204–221.
doi:10.1016/j.dam.2018.07.013

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