

## ON IMPLICIT HEAVY SUBGRAPHS AND HAMILTONICITY OF 2-CONNECTED GRAPHS

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### Abstract

A graph  $G$  of order  $n$  is *implicit claw-heavy* if in every induced copy of  $K_{1,3}$  in  $G$  there are two non-adjacent vertices with sum of their implicit degrees at least  $n$ . We study various implicit degree conditions (including, but not limiting to, Ore- and Fan-type conditions) imposing of which on specific induced subgraphs of a 2-connected implicit claw-heavy graph ensures its Hamiltonicity. In particular, we improve a recent result of [X. Huang, *Implicit degree condition for Hamiltonicity of 2-heavy graphs*, *Discrete Appl. Math.* 219 (2017) 126–131] and complete the characterizations of pairs of o-heavy and f-heavy subgraphs for Hamiltonicity of 2-connected graphs.

**Keywords:** implicit degree, implicit o-heavy, implicit f-heavy, implicit c-heavy, Hamilton cycle.

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### 1. INTRODUCTION

We use [3] for terminology and notation not defined here. In the paper only finite, simple and undirected graphs are considered.

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Let  $G$  be a graph and  $H$  be a subgraph of  $G$ . For a vertex  $u \in V(G)$ , the *neighbourhood* of  $u$  in  $H$  is denoted by  $N_H(u) = \{v \in V(H) : uv \in E(G)\}$  and the *degree* of  $u$  in  $H$  is denoted by  $d_H(u) = |N_H(u)|$ . For two vertices  $u, v \in V(H)$ , the *distance* between  $u$  and  $v$  in  $H$ , denoted by  $d_H(u, v)$ , is the length of a shortest  $(u, v)$ -path in  $H$  (if there are no  $(u, v)$ -paths in  $H$ , then  $d_H(u, v) := +\infty$ ). When there is no danger of ambiguity, we can use  $N(u), d(u)$  and  $d(u, v)$  in place of  $N_G(u), d_G(u)$  and  $d_G(u, v)$ , respectively. We use  $N_2(u)$  to denote the set of vertices which are at distance two from  $u$ , i.e.,  $N_2(u) = \{v \in V(G) : d(u, v) = 2\}$ .

Let  $S$  be a graph. If there are no induced copies of  $S$  in  $G$ , then  $G$  is said to be  $S$ -free. Similarly, for a family  $\mathcal{S}$  of graphs,  $G$  is  $\mathcal{S}$ -free if it is  $S$ -free for every  $S \in \mathcal{S}$ . If one demands  $G$  being  $\mathcal{S}$ -free, then the family  $\mathcal{S}$  is *forbidden* in  $G$ . A cycle in a graph  $G$  is called its *Hamilton cycle* (or *Hamiltonian cycle*), if it contains all vertices of  $G$ , and  $G$  is called *Hamiltonian* if it contains a Hamilton cycle. Forbidden subgraph conditions and degree conditions are two important types of sufficient conditions for the existence of Hamilton cycles in graphs.

The only connected graph of order at least three forbidding of which in a 2-connected graph  $G$  implies Hamiltonicity of  $G$ , is the path  $P_3$  (we use  $P_i$  for a path with  $i$  vertices). When disconnected subgraphs are also considered, forbidding of  $3K_1$  also ensures Hamiltonicity. The former fact can be deduced from [17] and the latter from Chvátal-Erdős theorem [13]. Actually, the graphs  $P_3$  and  $3K_1$  are the only graphs of order at least three having this property. In [26], Li and Vrána proved the necessity part of the following theorem.

**Theorem 1** (Li and Vrána [26]). *Let  $G$  be a 2-connected graph and  $S$  be a graph of order at least three. Then  $G$  being  $S$ -free implies that  $G$  is Hamiltonian if and only if  $S$  is  $P_3$  or  $3K_1$ .*

The case with pairs of forbidden subgraphs other than  $P_3$  and  $3K_1$  is much more interesting. The complete characterization of forbidden pairs of connected subgraphs for Hamiltonicity, based partially on results from [5, 14, 18] and [19], was obtained by Bedrossian in [1]. The ‘only if’ part of the following theorem is due to Faudree and Gould.

**Theorem 2** (Bedrossian [1], Faudree and Gould [17]). *Let  $R$  and  $S$  be connected graphs with  $R, S \neq P_3$  and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -free implies  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$  or  $W$  (see Figure 1).*

In [26], Li and Vrána considered pairs of forbidden subgraphs that are not necessarily connected.

**Theorem 3** (Li and Vrána [26]). *Let  $R$  and  $S$  be graphs of order at least three other than  $P_3$  and  $3K_1$  and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -free implies  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S$  is an induced subgraph of  $P_6, W, N$  or  $K_2 \cup P_4$ .*

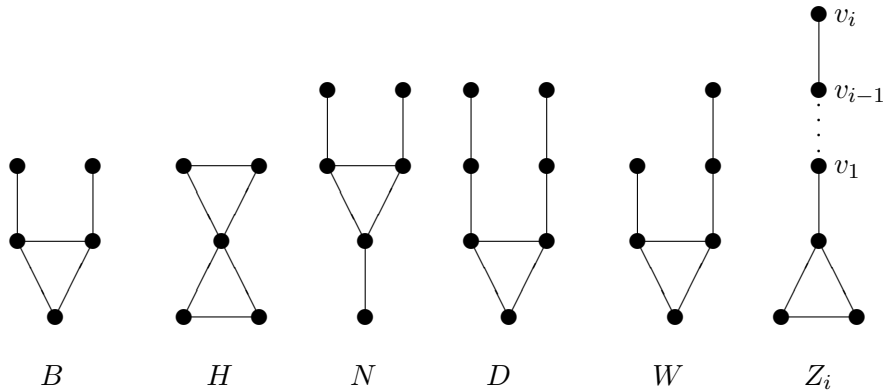


Figure 1. Graphs  $B$  (bull),  $H$  (hourglass),  $N$  (net),  $D$  (deer),  $W$  (wounded) and  $Z_i$ .

A widely studied way of relaxing the forbidden subgraph conditions for Hamiltonicity is allowing the subgraphs in the graph, but with some requirements regarding degrees of their vertices imposed on them. Some of these extensions exploit the concept of implicit degree, introduced by Zhu *et al.* in [32].

**Definition 1** (Zhu, Li and Deng [32]). Let  $v$  be a vertex of a graph  $G$  and  $d(v) = l + 1$ . Set  $M_2 = \max\{d(u) : u \in N_2(v)\}$ . If  $N_2(v) \neq \emptyset$  and  $d(v) \geq 2$ , then let  $d_1 \leq d_2 \leq d_3 \leq \dots \leq d_l \leq d_{l+1} \leq \dots$  be the degree sequence of vertices of  $N(v) \cup N_2(v)$ . Define

$$d^*(v) = \begin{cases} d_{l+1}, & \text{if } d_{l+1} > M_2, \\ d_l, & \text{otherwise.} \end{cases}$$

Then the *implicit degree* of  $v$  in  $G$  is defined as  $id(v) = \max\{d(v), d^*(v)\}$ . If  $N_2(v) = \emptyset$  or  $d(v) \leq 1$ , then define  $id(v) = d(v)$ .

Observe that, by the above definition, for every  $v \in V(G)$  the inequality  $id(v) \geq d(v)$  holds.

Some of the (implicit) degree conditions suitable for relaxing the forbidden subgraph conditions originate from the following classical results.

**Theorem 4** (Fan [15]). *Let  $G$  be a 2-connected graph of order  $n \geq 3$ . If*

$$d(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq n/2$$

*for every pair of vertices  $u$  and  $v$  in  $G$ , then  $G$  is Hamiltonian.*

**Theorem 5** (Ore [31]). *Let  $G$  be a graph of order  $n$ . If for every pair of its non-adjacent vertices the sum of their degrees is not less than  $n$ , then  $G$  is Hamiltonian.*

The authors of [32] prove a counterpart of Ore's Theorem 5, where the degree sum condition is replaced with the implicit degree sum condition. Theorems 4 and 5, and their extensions, gave rise to notions of *f-heavy* [30], *o-heavy* [7, 30], *implicit f-heavy* [9] and *implicit o-heavy* graphs. Here, we cite the definitions of *o-heavy* and *f-heavy* from [30] which are given as follows. Let  $G$  be a graph of order  $n$ . A vertex  $v$  of  $G$  is called *heavy* (or *implicit heavy*) if  $d(v) \geq n/2$  (or  $id(v) \geq n/2$ ). If  $v$  is not heavy (or not implicit heavy), we call it *light* (*implicit light*, respectively). For a given graph  $H$  we say that  $G$  is  *$H$ -o-heavy* (or *implicit  $H$ -o-heavy*) if in every induced subgraph of  $G$  isomorphic to  $H$  there are two non-adjacent vertices with the sum of their degrees (implicit degrees, respectively) in  $G$  at least  $n$ . And  $G$  is said to be  *$H$ -f-heavy* (or *implicit  $H$ -f-heavy*), if for every subgraph  $S$  of  $G$  isomorphic to  $H$ , and every two vertices  $u, v \in V(S)$  holds

$$d_S(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq n/2$$

( $\max\{id(u), id(v)\} \geq n/2$ , respectively).

For a family of graphs  $\mathcal{H}$ ,  $G$  is said to be (*implicit*)  *$\mathcal{H}$ -o-heavy*, if  $G$  is (implicit)  *$H$ -o-heavy* for every  $H \in \mathcal{H}$ . Classes of  *$\mathcal{H}$ -f-heavy* and *implicit  $\mathcal{H}$ -f-heavy* graphs are defined similarly. We note that the above definitions of  *$H$ -f-heavy*,  *$\mathcal{H}$ -o-heavy*, and  *$\mathcal{H}$ -f-heavy* are also all from [30]. When a graph is implicit  $K_{1,3}$ -o-heavy we will call it *implicit claw-heavy*.

Observe that every  $H$ -free graph is trivially  *$H$ -o-heavy* and  *$H$ -f-heavy*. Furthermore, every  *$H$ -o-heavy* (or  *$H$ -f-heavy*) graph is implicit  *$H$ -o-heavy* (implicit  *$H$ -f-heavy*, respectively). Replacing forbidden subgraph conditions with conditions expressed in terms of heavy subgraphs yielded the following extensions of Theorem 2.

**Theorem 6** (Li, Ryjáček, Wang and Zhang [25]). *Let  $R$  and  $S$  be connected graphs with  $R \neq P_3$ ,  $S \neq P_3$  and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -o-heavy implies  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = C_3, P_4, P_5, Z_1, Z_2, B, N$  or  $W$ .*

**Theorem 7.** *Let  $R$  and  $S$  be connected graphs with  $R \neq P_3$ ,  $S \neq P_3$  and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -f-heavy implies that  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S$  is one of the following:*

- $P_4, P_5, P_6$  (Chen, Wei and X. Zhang [11]),
- $Z_1$  (Bedrossian, Chen and Schelp [2]),
- $B$  (G. Li, Wei and Gao [27]),
- $N$  (Chen, Wei and X. Zhang [10]),
- $Z_2, W$  (Ning and S. Zhang [30]).

Recently, motivated by the main result of [20], Li and Ning [23] introduced another type of heavy subgraphs. We say that an induced subgraph  $H$  of  $G$  is

$c$ -heavy in  $G$ , if for every maximal clique  $C$  of  $H$  every non-trivial component of  $H - C$  contains a vertex that is heavy in  $G$ . Graph  $G$  is said to be  $H$ - $c$ -heavy if every induced subgraph of  $G$  isomorphic to  $H$  is  $c$ -heavy. For a family  $\mathcal{H}$  of graphs,  $G$  is called  $\mathcal{H}$ - $c$ -heavy if  $G$  is  $H$ - $c$ -heavy for every  $H \in \mathcal{H}$ .

Observe that every graph is trivially  $\{K_{1,3}, C_3, P_3\}$ - $c$ -heavy, since removal of a maximal clique from any of the three subgraphs results in a graph consisting of trivial components (or an empty graph). With that remark in mind, the authors of [23] extended Theorem 2 in the following way.

**Theorem 8** (Li and Ning [23]). *Let  $S$  be a connected graph of order at least three and let  $G$  be a 2-connected claw- $o$ -heavy graph. Then  $G$  being  $S$ - $c$ -heavy implies that  $G$  is Hamiltonian if and only if  $S = P_4, P_5, P_6, Z_1, Z_2, B, N$  or  $W$ .*

Similarly to implicit  $o$ -heavy and implicit  $f$ -heavy graphs, we can define *implicit  $H$ - $c$ -heavy* and *implicit  $\mathcal{H}$ - $c$ -heavy* graphs by replacing the degree condition in the definition of  $c$ -heavy graphs with implicit degree condition. In the light of the results presented so far, and noting that every implicit claw- $f$ -heavy graph is implicit claw-heavy, it seems worthwhile to tackle the following problems.

**Problem 1.** Characterize all graphs  $S$  such that every 2-connected implicit claw-heavy and implicit  $S$ - $o$ -heavy graph is Hamiltonian.

**Problem 2.** Characterize all graphs  $S$  such that every 2-connected implicit claw-heavy and implicit  $S$ - $f$ -heavy graph is Hamiltonian.

**Problem 3.** Characterize all graphs  $S$  such that every 2-connected implicit claw-heavy and implicit  $S$ - $c$ -heavy graph is Hamiltonian.

As byproducts of the proof of our main result, we obtained the following partial answers to Problems 1–3.

**Theorem 9.** *Let  $G$  be a 2-connected implicit claw-heavy graph. If  $G$  is implicit  $S$ - $o$ -heavy for  $S$  being a subgraph of  $K_2 \cup P_4$ , then  $G$  is Hamiltonian.*

**Theorem 10.** *Let  $G$  be a 2-connected implicit claw-heavy graph. If  $G$  is implicit  $S$ - $f$ -heavy, with  $S$  being one of the graphs  $K_1 \cup P_3, K_2 \cup P_3, K_1 \cup P_4, K_2 \cup P_4, P_4, Z_1$  and  $Z_2$ , then  $G$  is Hamiltonian.*

Theorem 10 implies in particular that every 2-connected implicit  $\{K_{1,3}, Z_1\}$ - $f$ -heavy graph is Hamiltonian. This fact has been proved before in [12].

**Theorem 11.** *Let  $G$  be a 2-connected implicit claw-heavy graph. If  $G$  is implicit  $S$ - $c$ -heavy, with  $S$  being one of the graphs  $K_1 \cup K_2, 2K_1 \cup K_2, K_1 \cup 2K_2, K_2 \cup K_2, K_1 \cup P_3, K_2 \cup P_3, K_1 \cup P_4, K_2 \cup P_4, P_4, P_5$  and  $P_6$ , then  $G$  is Hamiltonian.*

Clearly, for  $S$  being any of the graphs  $K_1 \cup K_2$ ,  $2K_1 \cup K_2$ ,  $K_2 \cup K_2$  and  $K_1 \cup 2K_2$ , every graph is  $S$ -f-heavy. Observe also that each of the remaining subgraphs of  $K_2 \cup P_4$  appear in each of Theorems 9–11. Hence, as corollaries from these theorems and Theorems 6–8, we get the following complete characterizations of heavy pairs of (not necessarily connected) subgraphs for Hamiltonicity.

**Corollary 12.** *Let  $R$  and  $S$  be graphs other than  $P_3$  and  $3K_1$ , and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -o-heavy implies  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S$  is an induced subgraph of  $P_5, W, N$  or  $K_2 \cup P_4$ .*

**Corollary 13.** *Let  $R$  and  $S$  be graphs other than  $P_3$  and  $3K_1$ , and let  $G$  be a 2-connected graph. Then  $G$  being  $\{R, S\}$ -f-heavy implies  $G$  is Hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S$  is one of  $P_4, P_5, P_6, Z_1, Z_2, B, N, W, K_1 \cup P_3, K_2 \cup P_3, K_1 \cup P_4$  and  $K_2 \cup P_4$ .*

**Corollary 14.** *Let  $S$  be a graph of order at least three other than  $P_3$  and  $3K_1$ , and let  $G$  be a 2-connected graph, claw-o-heavy graph. Then  $G$  being  $S$ -c-heavy implies  $G$  is Hamiltonian if and only if  $S$  is one of  $P_4, P_5, P_6, Z_1, Z_2, B, N, W, K_1 \cup K_2, 2K_1 \cup K_2, K_1 \cup 2K_2, K_2 \cup K_2, K_1 \cup P_3, K_2 \cup P_3, K_1 \cup P_4$  and  $K_2 \cup P_4$ .*

We note that the assumption of the graph  $S$  being of order at least three in Corollary 14 is necessary, since every graph is trivially  $\{K_1, 2K_1, K_2\}$ -c-heavy.

For triples of forbidden subgraphs there are also many results. The following are two well-known results of this type (graphs  $D$  and  $H$ , called *deer* and *hourglass*, respectively, are represented in Figure 1).

**Theorem 15** (Broersma and Veldman [5], Brousek [6]). *Let  $G$  be a 2-connected graph. If  $G$  is  $\{K_{1,3}, P_7, D\}$ -free, then  $G$  is Hamiltonian.*

**Theorem 16** (Faudree, Ryjáček and Schiermeyer [16], Brousek [6]). *Let  $G$  be a 2-connected graph. If  $G$  is  $\{K_{1,3}, P_7, H\}$ -free, then  $G$  is Hamiltonian.*

Note that the pair  $\{K_{1,3}, P_6\}$  that is present in Theorem 2 is missing in Theorem 6. A construction of a 2-connected, claw-free and  $P_6$ -o-heavy graph that is not Hamiltonian can be found in [25]<sup>2</sup>. Since every  $P_6$ -o-heavy graph is also implicit  $\{P_7, D\}$ -o-heavy, it is clear that Theorems 15 and 16 cannot be improved by imposing the condition of implicit o-heaviness on all of their forbidden subgraphs. However, a slightly stronger implicit degree sum conditions are sufficient to ensure Hamiltonicity. Our main result is the following.

<sup>2</sup>Nevertheless, the condition of  $P_6$ -o-heaviness can be replaced with other degree conditions on paths  $P_6$  to ensure Hamiltonicity of 2-connected claw-o-heavy graphs. We refer an interested reader to [24] for details.

**Theorem 17.** *Let  $G$  be a 2-connected, implicit claw-heavy graph of order  $n$  such that in every path  $v_1v_2v_3v_4v_5v_6v_7$  induced in  $G$  at least one of the following conditions is satisfied:*

- (1)  $id(v_4) \geq n/2$ , or
- (2)  $id(v_i) + id(v_j) \geq n$  for some  $i \in \{1, 2\}$ ,  $j \in \{6, 7\}$ .

*If*

- (i) *in every induced  $D$  of  $G$  with the set of vertices  $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$  and the set of edges  $\{u_1u_2, u_2u_3, u_3u_4, u_3u_5, u_4u_5, u_5u_6, u_6u_7\}$  at least one of the following conditions is satisfied:*
  - (a)  $id(u_4) \geq n/2$ , or
  - (b)  $id(u_i) + id(u_j) \geq n$  for some  $i \in \{1, 2, 4\}$ ,  $j \in \{6, 7\}$ , or
- (ii) *in every induced  $H$  of  $G$  with the set of vertices  $\{u_1, u_2, u_3, u_4, u_5\}$  and the set of edges  $\{u_1u_2, u_2u_3, u_1u_3, u_3u_4, u_3u_5, u_4u_5\}$  at least one of the following conditions is satisfied:*
  - (a) both  $u_1$  and  $u_2$  are implicit heavy, or
  - (b)  $id(u_i) + id(u_j) \geq n$  for some  $i \in \{1, 2\}$ ,  $j \in \{4, 5\}$ ,

*then  $G$  is Hamiltonian.*

Note that the conditions imposed on paths of order seven in Theorem 17 are satisfied in particular by implicit  $P_7$ -f-heavy and implicit  $P_7$ -c-heavy graphs. Similarly, the conditions imposed on induced deers are satisfied by implicit  $D$ -f-heavy graphs and implicit  $D$ -c-heavy graphs, and the conditions imposed on hourglasses are satisfied by implicit  $H$ -c-heavy graphs, implicit  $H$ -f-heavy graphs and implicit  $H$ -o-heavy graphs. Hence, Theorem 17 implies the following new results.

**Corollary 18.** *Let  $G$  be a 2-connected, implicit claw-heavy graph. If  $G$  is*

- *implicit  $\{P_7, D\}$ -c-heavy or implicit  $\{P_7, H\}$ -c-heavy, or*
- *implicit  $P_7$ -f-heavy and implicit  $D$ -c-heavy, or*
- *implicit  $P_7$ -f-heavy and implicit  $H$ -c-heavy, or*
- *implicit  $P_7$ -f-heavy and implicit  $H$ -o-heavy, or*
- *implicit  $P_7$ -c-heavy and implicit  $H$ -o-heavy, or*
- *implicit  $P_7$ -c-heavy and implicit  $H$ -f-heavy,*

*then  $G$  is Hamiltonian.*

Some previously known results, including recent extensions of Theorem 15 and Theorem 16, can also be deduced from Theorem 17.

**Corollary 19** (Huang [21]). *Let  $G$  be a 2-connected, implicit claw-heavy graph. If  $G$  is  $P_6$ -free, then  $G$  is Hamiltonian.*

**Corollary 20** (Broersma, Ryjáček and Schiermeyer [4]). *Let  $G$  be a 2-connected, claw- $f$ -heavy graph. If  $G$  is  $\{P_7, D\}$ -free or  $\{P_7, H\}$ -free, then  $G$  is Hamiltonian.*

**Corollary 21** (Cai and Li [8]). *Let  $G$  be a 2-connected, implicit claw- $f$ -heavy graph. If  $G$  is  $\{P_7, D\}$ -free or  $\{P_7, H\}$ -free, then  $G$  is Hamiltonian.*

**Corollary 22** (Ning [29]). *Let  $G$  be a 2-connected, claw- $f$ -heavy graph. If  $G$  is  $\{P_7, D\}$ - $f$ -heavy or  $\{P_7, H\}$ - $f$ -heavy, then  $G$  is Hamiltonian.*

**Corollary 23** (Huang [22]). *Let  $G$  be a 2-connected, claw- $f$ -heavy graph. If  $G$  is implicit  $\{P_7, D\}$ - $f$ -heavy or implicit  $\{P_7, H\}$ - $f$ -heavy, then  $G$  is Hamiltonian.*

**Corollary 24** (Cai and Zhang [9]). *Let  $G$  be a 2-connected, implicit claw-heavy graph. If  $G$  is implicit  $\{P_7, D\}$ - $f$ -heavy or implicit  $\{P_7, H\}$ - $f$ -heavy, then  $G$  is Hamiltonian.*

The rest of the paper is organized as follows. In Section 2 we define some auxiliary notions and present lemmas used throughout the proof. The proof of Theorems 9, 10, 11 and 17 is presented in Section 3.

## 2. PRELIMINARIES

In this section, we present two lemmas that will be used throughout the proofs of our main results. They make use of the notion of an *implicit heavy cycle*, which is a cycle that contains all implicit heavy vertices of a graph. For a vertex  $v \in V(G)$  lying on a cycle  $C$  with a given orientation, we denote by  $v^+$  its successor on  $C$  and by  $v^-$  its predecessor. For a set  $A \subset V(C)$  the sets  $A^+$  and  $A^-$  are defined analogously, i.e.,  $A^+ = \{v^+ : v \in A\}$  and  $A^- = \{v^- : v \in A\}$ . We write  $xCy$  for the path from  $x \in V(C)$  to  $y \in V(C)$  following the orientation of  $C$ , and  $x\bar{C}y$  denotes the path from  $x$  to  $y$  opposite to the direction of  $C$ . Similar notation is used for paths.

The next lemma is implicit in [28].

**Lemma 25** (Li, Ning and Cai [28]). *Every 2-connected graph contains an implicit heavy cycle.*

A cycle  $C$  is called *nonextendable* if there is no cycle longer than  $C$  in  $G$  containing all vertices of  $C$ . We use  $E^*(G)$  to denote the set  $\{xy : xy \in E(G) \text{ or } id(x) + id(y) \geq n\}$ .

**Lemma 26** (Huang [21]). *Let  $G$  be a 2-connected graph on  $n \geq 3$  vertices and  $C$  be a nonextendable cycle of  $G$  of length at most  $n - 1$ . If  $P$  is an  $xy$ -path in  $G$  such that  $V(C) \subset V(P)$ , then  $xy \notin E^*(G)$ .*



3. PROOFS OF THEOREMS 9–11 AND 17

For a proof by contradiction suppose that a graph  $G$  satisfying the assumptions of any of the Theorems 9, 10, 11 or 17 is not Hamiltonian. Then  $G$  is a 2-connected implicit claw-heavy graph. By Lemma 25, there is an implicit heavy cycle in  $G$ . Let  $C$  be a longest implicit heavy cycle in  $G$  and give  $C$  an orientation. From the assumption of 2-connectivity of  $G$  it follows that there is a path  $P$  connecting two vertices  $x_1, x_2 \in V(C)$  internally disjoint with  $C$  such that  $|V(P)| \geq 3$ . Let  $P = x_1 u_1 u_2 \cdots u_r x_2$  be such a path of minimum length. Note that this implies that  $P$  is induced unless  $x_1 x_2 \in E(G)$ . The following four claims, as readers can see, also appeared in [9, 21, 22], since they are basic properties of a longest implicit heavy cycle. We also use them to start our proof.

**Claim 27.**  $u_k x_i^+ \notin E^*(G)$  and  $u_k x_i^- \notin E^*(G)$  for every  $k \in \{1, 2, \dots, r\}$  and  $i \in \{1, 2\}$ .

*Proof.* Since  $P_1 = x_1^+ C x_1 P u_k$  and  $P_2 = x_1^- \bar{C} x_1 P u_k$  are paths such that  $V(C) \subset V(P_1)$  and  $V(C) \subset V(P_2)$ ,  $u_k x_1^+ \notin E^*(G)$  and  $u_k x_1^- \notin E^*(G)$  by Lemma 26. Similarly,  $u_k x_2^+ \notin E^*(G)$  and  $u_k x_2^- \notin E^*(G)$ . ■

**Claim 28.**  $x_1^- x_1^+ \in E^*(G)$  and  $x_2^- x_2^+ \in E^*(G)$ .

*Proof.* If  $x_1^- x_1^+ \notin E(G)$ , then the set  $\{x_1, x_1^-, x_1^+, u_1\}$  induces a claw. By Claim 27, we have  $id(u_1) + id(x_1^-) < n$  and  $id(u_1) + id(x_1^+) < n$ . Since  $G$  is implicit claw-heavy, this implies that  $id(x_1^-) + id(x_1^+) \geq n$ . Thus,  $x_1^- x_1^+ \in E^*(G)$ . Similarly,  $x_2^- x_2^+ \in E^*(G)$ . ■

**Claim 29.**  $x_1^- x_2^- \notin E^*(G)$  and  $x_1^+ x_2^+ \notin E^*(G)$ .

*Proof.* Observe that the paths  $P_1 = x_1^- \bar{C} x_2 \bar{P} x_1 C x_2^-$  and  $P_2 = x_1^+ C x_2 \bar{P} x_1 \bar{C} x_2^+$  are paths such that  $V(C) \subset V(P_1)$  and  $V(C) \subset V(P_2)$ . Thus, the Claim follows from Lemma 26. ■

**Claim 30.**  $x_1^- x_1^+ \in E(G)$  or  $x_2^- x_2^+ \in E(G)$ .

*Proof.* Suppose to the contrary that  $x_1^- x_1^+ \notin E(G)$  and  $x_2^- x_2^+ \notin E(G)$ . Then  $id(x_1^-) + id(x_1^+) \geq n$  and  $id(x_2^-) + id(x_2^+) \geq n$  by Claim 28. Thus,  $id(x_1^-) + id(x_2^-) \geq n$  or  $id(x_1^+) + id(x_2^+) \geq n$ , contradicting Claim 29. ■

By Claim 30, without loss of generality, we assume that  $x_1^- x_1^+ \in E(G)$ . The following two claims were proved in [9], here we omit their proofs.

**Claim 31** (Cai and Zhang [9]).  $x_i x_{3-i}^- \notin E^*(G)$  and  $x_i x_{3-i}^+ \notin E^*(G)$  for  $i \in \{1, 2\}$ .

By Claim 31, there is a vertex in  $x_i^+ C x_{3-i}^-$  not adjacent to  $x_i$  in  $G$  for  $i = 1, 2$ . Let  $y_i$  be the first vertex in  $x_i^+ C x_{3-i}^-$  not adjacent to  $x_i$  in  $G$  for  $i = 1, 2$ . Let  $u$  be any vertex of  $P$  other than  $x_1$  and  $x_2$  and let  $z_i$  be an arbitrary vertex in  $x_i^+ C y_i$  for  $i = 1, 2$ .

**Claim 32** (Cai and Zhang [9]).  $uz_1, uz_2, z_1x_2, z_2x_1, z_1z_2 \notin E^*(G)$ .

The proof splits now into subcases, depending on the conditions satisfied by  $G$ .

*Case 1.*  $G$  is implicit  $K_2 \cup P_4$ -o-heavy or implicit  $K_2 \cup P_4$ -f-heavy. By Claim 32, we have that both sets  $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$  and  $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$  induce a graph isomorphic to  $K_2 \cup P_4$  in  $G$ .

Assume that  $G$  is implicit  $K_2 \cup P_4$ -f-heavy. Since none of the vertices  $u_1$  and  $u_r$  belongs to  $C$ , both these vertices are implicit light. This implies that both  $y_2^-$  and  $y_1^-$  are implicit heavy, contradicting Claim 32. This contradiction proves the part of Theorem 10 regarding implicit  $K_2 \cup P_4$ -f-heavy graphs. By taking induced subgraphs from  $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$  and  $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$  corresponding to  $K_1 \cup P_4, P_4, K_1 \cup P_3$  and  $K_2 \cup P_3$ , we get the same contradiction which can also prove the part of Theorem 10 regarding implicit  $K_1 \cup P_4$ -f-heavy graphs, implicit  $P_4$ -f-heavy graphs, implicit  $K_1 \cup P_3$ -f-heavy graphs and implicit  $K_2 \cup P_3$ -f-heavy graphs, respectively.

Consider now the case when  $G$  is implicit  $K_2 \cup P_4$ -o-heavy. Then there is a pair of nonadjacent vertices in both  $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$  and  $\{y_2^-, y_2, u_1, x_1, y_1^-, y_1\}$  which have implicit degree sum not less than  $n$ . Let us focus on the set  $\{y_1^-, y_1, u_r, x_2, y_2^-, y_2\}$ . Since  $uz_1, z_1x_2, z_1z_2 \notin E^*(G)$  by Claim 32, it follows that the pair of nonadjacent vertices with implicit degree sum at least  $n$  belongs to the set  $\{u_r, x_2, y_2^-, y_2\}$ . Since  $uz_2 \notin E^*(G)$  by Claim 32, we have  $id(x_2) + id(y_2) \geq n$ . Now by  $id(x_1) + id(y_1) + id(x_2) + id(y_2) \geq 2n$ , we have  $id(x_1) + id(y_2) \geq n$  or  $id(x_2) + id(y_1) \geq n$ , which contradicts Claim 32. This contradiction proves the part of Theorem 9 regarding implicit  $K_2 \cup P_4$ -o-heavy graphs, and the left part regarding implicit  $S$ -o-heavy graphs for any proper subgraph  $S$  of  $K_2 \cup P_4$  is implied by the validity of theorem for  $K_2 \cup P_4$ . Thus, the proof of Theorem 9 is completed.

*Case 2.*  $G$  is implicit  $S$ -f-heavy for  $S$  being one of  $Z_1$  and  $Z_2$ . Suppose first that  $G$  is implicit  $Z_1$ -f-heavy. Then, since the vertex  $u_1$  is implicit light by the choice of  $C$  and the set  $\{x_1^-, x_1^+, x_1, u_1\}$  induces  $Z_1$ , both vertices  $x_1^-$  and  $x_1^+$  are implicit heavy. Now it follows from Claim 29 that both  $x_2^-$  and  $x_2^+$  are implicit light. Then  $x_2^- x_2^+ \in E(G)$ , by Claim 28. But now the set  $\{x_2^-, x_2^+, x_2, u_r\}$  induces  $Z_1$ . A contradiction. Thus,  $G$  is  $Z_2$ -f-heavy.

Suppose that  $r \geq 2$  or  $r = 1$  and  $x_1x_2 \notin E(G)$ . Then one of the sets  $\{x_1^-, x_1^+, x_1, u_1, u_2\}$  or  $\{x_1^-, x_1^+, x_1, u_1, x_2\}$  induces  $Z_2$ . Similarly to the previous

paragraph, this implies that both  $x_1^-$  and  $x_1^+$  are implicit heavy, and in consequence  $x_2^-$  and  $x_2^+$  are implicit light vertices forming an edge in  $G$ . But then either  $\{x_2^-, x_2^+, x_2, u_r, u_{r-1}\}$  or  $\{x_2^-, x_2^+, x_2, u_r, x_1\}$  also induces a  $Z_2$ , a contradiction.

Thus,  $r = 1$  and  $x_1x_2 \in E(G)$ . But now both sets  $\{u_1, x_2, x_1, y_1^-, y_1\}$  and  $\{u_1, x_1, x_2, y_2^-, y_2\}$  induce  $Z_2$ , implying that both  $y_1^-$  and  $y_2^-$  are implicit heavy. This contradicts Claim 32. Together with Case 1, this contradiction completes the proof of Theorem 10.

*Case 3.*  $G$  is implicit  $K_1 \cup P_3$ -c-heavy.

**Claim 33.**  $x_1$  and  $x_2$  are implicit heavy.

**Proof.** By Claim 32, we have that both sets  $\{x_1^+, x_2, y_2^-, y_2\}$  and  $\{x_2^+, x_1, y_1^-, y_1\}$  induce a graph isomorphic to  $K_1 \cup P_3$  in  $G$ . Since  $G$  is implicit  $K_1 \cup P_3$ -c-heavy and the independent vertex of  $K_1 \cup P_3$  is a maximal clique, there is an implicit heavy vertex in both sets  $\{x_2, y_2^-, y_2\}$  and  $\{x_1, y_1^-, y_1\}$ . If  $y_1$  or  $y_1^-$  is implicit heavy, then none of the vertices of  $\{x_2, y_2^-, y_2\}$  can be implicit heavy by Claim 32, a contradiction. Hence,  $x_1$  is implicit heavy. Similarly,  $x_2$  is also implicit heavy. ■

**Claim 34.**  $x_2^-x_2^+ \in E(G)$ .

**Proof.** By Claim 31 and Claim 33, we have that  $x_2^-$  and  $x_2^+$  are implicit light. Since  $G$  is implicit claw-heavy,  $x_2^-x_2^+ \in E(G)$ . ■

By Claim 29, there is a vertex in  $x_i^+Cx_{3-i}^-$  not adjacent to  $x_i^-$  in  $G$  for  $i = 1, 2$ . Let  $w_i$  be the first vertex in  $x_i^+Cx_{3-i}^-$  not adjacent to  $x_i^-$  in  $G$  for  $i = 1, 2$ . Note that  $w_i \neq x_i^+$ .

**Claim 35.**  $uw_i^- \notin E(G)$  and  $uw_i \notin E(G)$ .

**Proof.** Suppose that  $uw_1^- \in E(G)$ . By Claim 27, we have that  $w_1^- \neq x_1^+$ . Then  $C' = x_1Puw_1^-Cx_1^-w_1^-Cx_1$  is a cycle such that  $V(C) \subset V(C')$ , a contradiction. Hence,  $uw_1^- \notin E(G)$ . We also have that  $uw_1 \notin E(G)$ ; otherwise,  $C'' = x_1Puw_1Cx_1^-w_1^-Cx_1$  is a cycle such that  $V(C) \subset V(C'')$ , a contradiction. By symmetry, we have that  $uw_2^- \notin E(G)$  and  $uw_2 \notin E(G)$ . ■

From Claim 27 and Claim 35 we have that  $\{u, x_1^-, w_1^-, w_1\}$  induces a graph isomorphic to  $K_1 \cup P_3$  in  $G$ . Since  $G$  is implicit  $K_1 \cup P_3$ -c-heavy, there is an implicit heavy vertex in the set  $\{x_1^-, w_1^-, w_1\}$ . By Claim 31 and Claim 33 we have that  $x_1^-$  is implicit light. If  $w_1^-$  is implicit heavy, then  $w_1^- \neq x_1^+$  by Claim 31 and Claim 33. Thus  $P_1 = w_1^-Cx_2^-x_2^+Cx_1^-w_1^-Cx_1Px_2$  is a path such that  $V(C) \subset V(P_1)$  and  $w_1^-x_2 \in E^*(G)$ , contradicting Lemma 26. If  $w_1$  is implicit heavy, then  $P_2 = w_1Cx_2^-x_2^+Cx_1^-w_1^-Cx_1Px_2$  is a path such that  $V(C) \subset V(P_2)$  and  $w_1x_2 \in E^*(G)$ , contradicting Lemma 26. Thus, the part of Theorem 11

regarding implicit  $K_1 \cup P_3$ -c-heavy graphs is finished by these contradictions. The validity of the remaining part of Theorem 11 will be completed in the following.

*Case 4.*  $G$  satisfies the assumptions of Theorem 17.

**Claim 36.**  $x_1x_2 \in E(G)$ .

**Proof.** Suppose that  $x_1x_2 \notin E(G)$ . By the choice of  $P$ , Claim 27 and Claim 32 we have that  $P' = y_1y_1^-x_1u_1u_2 \dots u_r x_2 y_2^- y_2$  is an induced  $P_{r+6}$ , where  $r \geq 1$ . Let  $y_1y_1^-x_1u_1v_5v_6v_7$  be the path induced by the first seven vertices of  $P'$ . Since  $u_1$  is implicit light, it follows from the assumptions of Theorem 17 that for some  $a \in \{y_1, y_1^-\}$  and  $b \in \{v_6, v_7\}$  the inequality  $id(a) + id(b) \geq n$  holds. Since  $b \in V(P) \cup \{x_2, y_2^-, y_2\}$ , this contradicts Claim 32. ■

We complete the proof by considering two cases, depending on the value of  $r$ . When  $r \geq 2$ , we can use the method of the proof of Case 2 in [9] completely, because the proof does not involve any heavy subgraphs other than the claw. Here we omit the proof and consider the case when  $r = 1$ .

Suppose that  $r = 1$ . Then the set  $\{y_1, y_1^-, x_1, u_1, x_2, y_2^-, y_2\}$  induces a  $D$ . Since the vertex  $u_1$  is implicit light, Claim 32 implies that  $G$  does not satisfy the conditions imposed on induced deers in Theorem 17. Hence, it satisfies the conditions imposed on  $H$ .

Observe that  $\{x_1^-, x_1^+, x_1, u_1, x_2\}$  induces an  $H$ . Now it follows from Claim 31 and Claim 32 that both vertices  $x_1^-$  and  $x_1^+$  are implicit heavy. Similarly as in Case 2, this implies that both  $x_2^-$  and  $x_2^+$  are implicit light and  $x_2^-x_2^+ \in E(G)$ . But now the set  $\{u_1, x_1, x_2, x_2^-, x_2^+\}$  induces an  $H$ . By Claim 31 and Claim 32, this contradicts the assumptions of Theorem 17. This final contradiction completes the proof of Theorem 17.

Observe that every 2-connected implicit-claw-heavy graph that is implicit  $S$ -c-heavy for  $S$  being one of  $K_1 \cup K_2$ ,  $2K_1 \cup K_2$ ,  $K_1 \cup 2K_2$ ,  $K_2 \cup K_2$ ,  $K_2 \cup P_3$ ,  $K_1 \cup P_4$ ,  $K_2 \cup P_4$ ,  $P_4$ ,  $P_5$  and  $P_6$  satisfies the assumptions of Theorem 17. Hence, together with Case 3, Case 4 completes also the proof of Theorem 11.

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### REFERENCES

- [1] P. Bedrossian, Forbidden Subgraph and Minimum Degree Conditions for Hamiltonicity, Ph.D. Thesis (Memphis State University, USA, 1991).

- [2] P. Bedrossian, G. Chen and R.H. Schelp, *A generalization of Fan's condition for Hamiltonicity, pancyclicity and Hamiltonian connectedness*, Discrete Math. **115** (1993) 39–50.  
doi:10.1016/0012-365X(93)90476-A
- [3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, GTM **244** (Springer, Berlin, 2008).
- [4] H.J. Broersma, Z. Ryjáček and I. Schiermeyer, *Dirac's minimum condition restricted to claws*, Discrete Math. **167/168** (1997) 155–166.  
doi:10.1016/S0012-365X(96)00224-5
- [5] H.J. Broersma and H.J. Veldman, *Restrictions on induced subgraphs ensuring Hamiltonicity or pancyclicity of  $K_{1,3}$ -free graphs*, in: Contemporary Methods in Graph Theory, (BI Wissenschaftsverlag, Mannheim, 1990) 181–194.
- [6] J. Brousek, *Forbidden triples for Hamiltonicity*, Discrete Math. **251** (2002) 71–76.  
doi:10.1016/S0012-365X(01)00326-0
- [7] R. Čada, *Degree conditions on induced claws*, Discrete Math. **308** (2008) 5622–5631.  
doi:10.1016/j.disc.2007.10.026
- [8] J. Cai and H. Li, *Hamilton cycles in implicit 2-heavy graphs*, Graphs Combin. **32** (2016) 1329–1337.  
doi:10.1007/s00373-015-1669-4
- [9] J. Cai and Y. Zhang, *Fan-type implicit-heavy subgraphs for Hamiltonicity of implicit claw-heavy graphs*, Inform. Process. Lett. **116** (2016) 668–673.  
doi:10.1016/j.ipl.2016.06.012
- [10] G. Chen, B. Wei and X. Zhang, *Forbidden graphs and Hamiltonian cycles*, preprint, 1995.
- [11] G. Chen, B. Wei and X. Zhang, *Degree-light-free graphs and Hamiltonian cycles*, Graphs Combin. **17** (2001) 409–434.  
doi:10.1007/s003730170018
- [12] B. Chen and S. Zhang, *An implicit degree condition for long cycles in 2-connected graphs*, Appl. Math. Lett. **19** (2006) 1148–1151.  
doi:10.1016/j.aml.2005.12.006
- [13] V. Chvátal and P. Erdős, *A note on Hamiltonian circuits*, Discrete Math. **2** (1972) 111–113.  
doi:10.1016/0012-365X(72)90079-9
- [14] D. Duffus, R.J. Gould and M.S. Jacobson, *Forbidden subgraphs and the Hamiltonian theme*, Proceedings of the 4th International Conference on the Theory and Applications of Graphs, Kalamazoo, 1980 (Wiley, New York, 1981) 297–316.
- [15] G. Fan, *New sufficient conditions for cycles in graphs*, J. Combin. Theory Ser. B **37** (1984) 221–227.  
doi:10.1016/0095-8956(84)90054-6

- [16] R. Faudree, Z. Ryjáček and I. Schiermeyer, *Forbidden subgraphs and cycle extendability*, J. Combin. Math. Combin. Comput. **19** (1995) 109–128.
- [17] R.J. Faudree and R.J. Gould, *Characterizing forbidden pairs for Hamiltonian properties*, Discrete Math. **173** (1997) 45–60.  
doi:10.1016/S0012-365X(96)00147-1
- [18] S. Goodman and S.T. Hedetniemi, *Sufficient conditions for a graph to be Hamiltonian*, J. Combin. Theory Ser. B **16** (1974) 175–180.  
doi:10.1016/0095-8956(74)90061-6
- [19] R.J. Gould and M.S. Jacobson, *Forbidden subgraphs and Hamiltonian properties of graphs*, Discrete Math. **42** (1982) 189–196.  
doi:10.1016/0012-365X(82)90216-3
- [20] Z. Hu, *A generalization of fan's condition and forbidden subgraph conditions for Hamiltonicity*, Discrete Math. **196** (1999) 167–175.  
doi:10.1016/S0012-365X(98)00200-3
- [21] X. Huang, *Hamilton cycles in implicit claw-heavy graphs*, Inform. Process. Lett. **114** (2014) 676–679.  
doi:10.1016/j.ipl.2014.06.007
- [22] X. Huang, *Implicit degree condition for Hamiltonicity of 2-heavy graphs*, Discrete Appl. Math. **219** (2017) 126–131.  
doi:10.1016/j.dam.2016.10.031
- [23] B. Li and B. Ning, *Heavy subgraphs, stability and Hamiltonicity*, Discuss. Math. Graph Theory **37** (2017) 691–710.  
doi:10.7151/dmgt.1967
- [24] B. Li, B. Ning and S. Zhang, *Degree conditions restricted to induced paths for Hamiltonicity of claw-heavy graphs*, Acta Math. Sin. (Engl. Ser.) **33** (2017) 301–310.  
doi:10.1007/s10114-016-5735-5
- [25] B. Li, Z. Ryjáček, Y. Wang and S. Zhang, *Pairs of heavy subgraphs for Hamiltonicity of 2-connected graphs*, SIAM J. Discrete Math. **26** (2012) 1088–1103.  
doi:10.1137/11084786X
- [26] B. Li and P. Vrána, *Forbidden pairs of disconnected graphs implying Hamiltonicity*, J. Graph Theory **84** (2017) 249–261.  
doi:10.1002/jgt.22024
- [27] G. Li, B. Wei and T. Gao, *A structural method for Hamiltonian graphs*, Australas. J. Combin. **11** (1995) 257–262.
- [28] H. Li, W. Ning and J. Cai, *An implicit degree condition for cyclability in graphs*, Lecture Notes in Comput. Sci. **6681** (2011) 82–89.  
doi:10.1007/978-3-642-21204-8\_12
- [29] B. Ning, *Fan-type degree condition restricted to triples of induced subgraphs ensuring Hamiltonicity*, Inform. Process. Lett. **113** (2013) 823–826.  
doi:10.1016/j.ipl.2013.07.014

- [30] B. Ning and S. Zhang, *Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs*, Discrete Math. **313** (2013) 1715–1725.  
doi:10.1016/j.disc.2013.04.023
- [31] O. Ore, *Note on Hamilton circuits*, Amer. Math. Monthly **67** (1960) 55.  
doi:10.2307/2308928
- [32] Y. Zhu, H. Li and X. Deng, *Implicit-degrees and circumferences*, Graphs Combin. **5** (1989) 283–290.  
doi:10.1007/BF01788680

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