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LINEAR LIST COLORING OF SOME SPARSE GRAPHS

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Abstract

A linear k-coloring of a graph is a proper k-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. A graph G is linearly L-colorable if there is a linear coloring c of G for a given list assignment $L = \{L(v) : v \in V(G)\}$ such that $c(v) \in L(v)$ for all $v \in V(G)$, and G is linearly k-choosable if G is linearly L-colorable for any list assignment with $|L(v)| \geq k$. The smallest integer k such that G is linearly k-choosable is called the linear list chromatic number, denoted by $lc_l(G)$. It is clear that $lc_l(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ for any graph G with maximum degree $\Delta(G)$. The maximum average degree of a graph G, denoted by $mad(G)$, is the maximum of the average degrees of all subgraphs of G. In this note, we shall prove the following. Let G be a graph, (1) if $mad(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$, then $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$; (2) if $mad(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$, then $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$; (3) if $mad(G) < \frac{20}{7}$ and $\Delta(G) \geq 5$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

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1. Introduction

All graphs considered here are finite, simple and undirected. For a graph G, denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the vertex set, edge set, the minimum degree and the maximum degree, respectively. For a vertex $v \in V(G)$, let $N(v)$ and $d(v)$ be the neighborhood and the degree of v in G, respectively. The closed neighborhood of a vertex $v \in V(G)$, denoted by $N[v]$, is defined to be $N(v) \cup v$. A k-vertex (k^--vertex and k^+-vertex, respectively) is a vertex with degree k (at most k and at least k, respectively). A 2-vertex $v \in V(G)$ is called an (a, b)-vertex if it is adjacent to an a-vertex and a b-vertex, and an (a, b^+)-vertex is defined similarly. The maximum average degree $mad(G)$ of a graph G is defined as $mad(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}$, where $H \subseteq G$ signified that H is a subgraph of G.

A *proper k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E(G)$. A *linear k-coloring* of a graph is a proper k-coloring of the graph such that any subgraph induced by the vertices of any pair of color classes is a union of vertex-disjoint paths. The *linear chromatic number* $lc(G)$ of a graph G is the smallest number k such that G has a linear k-coloring. A graph G is *linearly L-colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$, there exists a linear coloring c of G such that $c(v) \in L(v)$ for all $v \in V(G)$. If G is linearly L-colorable for any list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be *linearly k-choosable*. The smallest integer k such that the graph G is linearly k-choosable is called the *linear list chromatic number*, denoted by $lc_l(G)$. The concept of linear coloring was first introduced by Yuster [8], and linear list colorings were first investigated by Esperet, Montassier and Raspaud [4].

It is clear that the linear chromatic number $lc(G)$ of a graph G with maximum degree $\Delta(G)$ has a trivial lower bound $lc(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $lc_l(G) \geq lc(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. Esperet *et al.* [4] proved that trees with maximum degree $\Delta(G)$ satisfy $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. This equality suggests that the linear list chromatic numbers of sparse graphs (with $mad(G) < 3$) might be close to the trivial lower bound. Cranston and Yu [1] asked: Does there exist a constant C such that every sparse graph G satisfies $lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? Some authors have proved that for the class of some sparse graphs, such constant C exists and is close to or equal to 1. We list the currently known results about this subject as follows.

Theorem 1. *Let G be a graph.*

(i) (Esperet *et al.* [4]) *If $mad(G) < \frac{8}{3}$, then $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.*

(ii) (Wang and Wu [7]) *If $mad(G) < \frac{14}{5}$, then $lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.*

(iii) (Cranston and Yu [1]) *If* $mad(G) < 3$ *and* $\Delta(G) \geq 9$, *then* $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

(iv) (Cranston and Yu [1]) *If* $mad(G) < \frac{12}{5}$ *and* $\Delta(G) \geq 3$, *then* $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

A *planar graph* is a graph that can be drawn on the Euclidean plane such that its edges meet at their ends only. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle of G. For a planar graph G with girth g, we have $mad(G) < \frac{2g}{g-2}$ by Euler's formula. So we can get some results from above results. Li, Wang and Raspaud [5] also asked: Is there a constant C such that every planar graph G has $lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + C$? About this question, there are some other results as follows.

Theorem 2. *Let* G *be a planar graph.*

(i) (Cranston and Yu [1]) *If* $g(G) \geq 5$, *then* $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 4$.

(ii) (Dong *et al.* [2]) *If* $g(G) \geq 6$, *then* $lc(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 3$.

(iii) (Dong and Lin [3]) *If* $g(G) \geq 6$ *and* $\Delta(G) \geq 39$, *then* $lc(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

In this paper, we prove the following results.

Theorem 3. *Let* G *be a graph.*

(1) *If* $mad(G) < \frac{8}{3}$ *and* $\Delta(G) \geq 7$, *then* $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

(2) *If* $mad(G) < \frac{18}{7}$ *and* $\Delta(G) \geq 5$, *then* $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

(3) *If* $mad(G) < \frac{20}{7}$ *and* $\Delta(G) \geq 5$, *then* $lc_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

Then the following results about planar graphs are implied immediately from Theorem 3(1) and (2), respectively.

Theorem 4. *Let* G *be a planar graph.*

(1) *If* $g(G) \geq 8$ *and* $\Delta(G) \geq 7$, *then* $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

(2) *If* $g(G) \geq 9$ *and* $\Delta(G) \geq 5$, *then* $lc_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$.

We will prove the three results of Theorem 3 by contradiction in the following three sections, respectively. For convenience, we introduce some notations that will be used. Let c be a coloring of G; we use $c(v)$ to denote the color of v in c, and $c(S) = \{c(v) : v \in S\}$ for $S \subset V(G)$. Let $c_i(v)$ be the set of colors appeared i times in $N(v)$. For a vertex $v \in V(G)$, let $n_2(v)$ for clarity be the number of 2-vertices in $N(v)$.

2. Graphs with $mad(G) < \frac{8}{3}$ and $\Delta(G) \geq 7$

In order to prove Theorem 3(1), we prove the following result instead, which implies Theorem 3(1) immediately.

Theorem 5. *Let $M \geq 7$ be an integer. If G is a graph with $mad(G) < \frac{8}{3}$ and $\Delta(G) \leq M$, then $lc_l(G) = \lceil \frac{M}{2} \rceil + 1$.*

Proof. By contradiction, we suppose that Theorem 5 is false. Let G be a counterexample with the fewest vertices, and L the list assignment of size $\lceil \frac{M}{2} \rceil + 1$ such that G has no linear L-coloring. Let H be a proper subgraph of G. Clearly, $mad(H) < \frac{8}{3}$ and $\Delta(H) \leq M$. By the choice of G, we have $lc_l(H) = \lceil \frac{M}{2} \rceil + 1$, while $lc_l(G) > \lceil \frac{M}{2} \rceil + 1$. In the proof we need some structural lemmas, Lemma 6 is well-known. ■

Lemma 6. *The graph G is connected, and $\delta(G) \geq 2$.*

Lemma 7 ([3] Lemma 2.2)**.** *Let v be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 1$.*

Lemma 8. *Let v be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then v_1, v_2, v_3 must be $(3, 6^+)$-vertices.*

Proof. Assume that v_1 is a $(3, 5^-)$-vertex, and u_i is the neighbor of v_i other than v, where $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then G' has a linear L-coloring c by the minimality of G. If $c(v_2) \neq c(v_3)$, we can extend the linear L-coloring c of G' to v_1 since $|L(v_1)\backslash\{c(u_1), c_2(u_1)\}| \geq 2$. Then we can color v with a color in $L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$ when $c(v_1) \notin \{c(v_2), c(v_3)\}$, or $L(v)\backslash\{c(u_1), c(v_2), c(v_3)\}$ when $c(v_1) \in \{c(v_2), c(v_3)\}$. Clearly, there will be no bi-colored cycles created, and we get a linear list coloring of G. If $c(v_2) = c(v_3)$, we can extend the linear L-coloring c of G' to v_1 since $|L(v_1)\backslash\{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. Finally, we can color v with a color in $L(v)\backslash\{c(v_1), c(v_2), c(u_2), c(u_3)\}$, which ensure that no bi-colored cycle passes vv_2u_2 or vv_3u_3. Thus, we also get a linear list coloring of G extended from the linear L-coloring of G'. A contradiction. ■

Lemma 9. *Let v be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If v_1, v_2, v_3, v_4 are $(5, 3)$-vertices, then v_5 must be a $(5, 4^+)$-vertex.*

Proof. Suppose to the contrary, let v_5 be a $(5, 3)$-vertex, and u_i be the neighbor of v_i other than v for $i \in \{1, 2, \ldots, 5\}$.

Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. There exist at least $|L(v_1)\backslash\{c(u_1), c(N(u_1))\}| \geq 2$ available colors for v_1. Since $|L(v_2)\backslash\{c(v_1), c(u_2), c(N(u_2))\}| \geq 1$ and $|L(v_3)\backslash\{c(v_1), c(v_2), c(u_3), c_2(u_3)\}| \geq 1$, we can extend the coloring c of G' to v_1, v_2, v_3 such that $|\{c(v_1), c(v_2), c(v_3)\}| = 3$.

Notice that there will be no bi-colored cycle passing vv_1u_1 or vv_2u_2. Then we color v_4 with a color in $L(v_4)\backslash\{c(u_4), c(N(u_4))\}$, and no bi-colored cycle will pass vv_4u_4. Finally, we extend the coloring c to v_5 and v in two different cases.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 4$, we can linearly color v with a color in $L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_4)\}$, and color v_5 such that no bi-colored cycle passes vv_5u_5 as $|L(v_5)\backslash\{c(u_5), c(v), c(N(u_5))\}| \geq 1$. So we get a linear L-coloring of G.

If $|\{c(v_1), c(v_2), c(v_3), c(v_4)\}| = 3$, we can color v_5 such that no bi-colored cycle passing vv_5u_5 since $|L(v_5)\backslash\{c(v_4), c(u_5), c(N(u_5))\}| \geq 1$, and color v with a color in $L(v) \setminus \{c(v_1), c(v_2), c(v_3), c(v_5)\}$. Thus, we get a linear L-coloring of G.

Therefore, we can extend the linear L-coloring c of G' to G, a contradiction. ■

Lemma 10. *Let v be a 7-vertex with $N(v) = \{v_1, v_2, \ldots, v_7\}$ and $n_2(v) = 7$. If $v_1, v_2, \ldots, v_5$ are $(7, 2)$-vertices, then at least one of v_6 and v_7 is a $(7, 4^+)$-vertex.*

Proof. Assume that v_6 and v_7 are $(7, 3^-)$-vertices, and u_i is the neighbor of v_i other than v for $i = 1, 2, \ldots, 7$. Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. First, we extend the linear L-coloring c of G' to v_7 and v_6 such that $c(v_6) \neq c(v_7)$ and no bi-colored cycle passes vv_7u_7 or vv_6u_6 since $|L(v_7)\backslash\{c(u_7), c(N(u_7))\}| \geq 2$ and $|L(v_6)\backslash\{c(u_6), c(N(u_6)), c(v_7)| \geq 1$. Next, we can color v_5 such that $c(v_5) \notin \{c(v_6), c(v_7)\}$ and no bi-colored cycle passes vv_5u_5 since $|L(v_5)\backslash\{c(u_5), c(N(u_5)), c(v_6), c(v_7)\}| \geq 1$. Then we can color v_4 with $c(v_4) \notin \{c(v_5), c(v_6), c(v_7)\}$ since $|L(v_4)\backslash\{c(u_4), c(v_5), c(v_6), c(v_7)\}| \geq 1$. Notice that $|\{c(v_4), c(v_5), c(v_6), c(v_7)\}| = 4$. Then we color v with a color in $L(v)\backslash\{c(v_7), c(v_6), c(v_5), c(v_4)\}$. Since $|L(v_3)\backslash\{c(u_3), c(v), c(N(u_3))\}| \geq 2$ and $|L(v_2)\backslash\{c(u_2), c(v), c_2(v), c(N(u_2))\}| \geq 1$ ($|c_2(v)| \leq 1$ now), we can color v_3, v_2 in order such that no bi-colored cycle passes vv_3u_3 or vv_2u_2. Finally, in order to avoid bi-colored cycles passing vv_1u_1, we can color v_1 with a color in $L(v_1)\backslash \{c(u_1), c(v), c_2(v)\}$ ($|c_2(v)| \leq 2$ now) when $c(u_1) \neq c(v)$, or color v_1 with a color in $L(v_1)\backslash\{c(u_1), c_2(v), c(N(u_1))\}$ when $c(u_1) = c(v)$. Thus, we get a linear list coloring of G extended from the linear list coloring c of G', a contradiction. ■

To complete our proof of Theorem 5, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function ω on $V(G)$ by $\omega(v) = d(v) - \frac{8}{3}$ for every $v \in V(v)$. Since $mad(G) < \frac{8}{3}$, the sum of the initial charge is negative. If we can make suitable discharging rules to redistribute charges among vertices so that the final charge $\omega'(v)$ of every vertex $v \in V(G)$ is nonnegative, then we get a contradiction. The discharging rules are as follows.

R1. Every 8^+-vertex sends $\frac{2}{3}$ to each adjacent 2-vertex.

R2. Every 7-vertex sends $\frac{2}{3}$ to each adjacent $(7, 2)$-vertex, $\frac{5}{9}$ to each adjacent $(7, 3)$-vertex, and $\frac{1}{3}$ to each adjacent $(7, 4^+)$-vertex.

R3. Every 6-vertex sends $\frac{5}{9}$ to each adjacent 2-vertex.

R4. Every 5-vertex sends $\frac{1}{2}$ to each adjacent $(5,3)$-vertex, $\frac{1}{3}$ to each adjacent $(5,4^+)$-vertex.

R5. Every 4-vertex sends $\frac{1}{3}$ to each adjacent 2-vertex.

R6. Every 3-vertex sends $\frac{1}{6}$ to each adjacent $(3,5)$-vertex, and $\frac{1}{9}$ to each adjacent $(3,6^+)$-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V(G)$.

Let v be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 7$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{2}{3} = 2 - \frac{8}{3} + \frac{2}{3} = 0$. If $d(x) = 3$, then $d(y) \geq 5$ by Lemma 7. Thus $\omega'(v) \geq \omega(v) + \frac{1}{6} + \frac{1}{2} = 2 - \frac{8}{3} + \frac{2}{3} = 0$ or $\omega'(v) \geq \omega(v) + \frac{1}{9} + \frac{5}{9} = 2 - \frac{8}{3} + \frac{2}{3} = 0$ by R6, R2, R3, and R4. Otherwise, $d(x) \geq 4$ and $d(y) \geq 5$, we have $\omega'(v) \geq \omega(v) + \frac{1}{3} + \frac{1}{3} = 2 - \frac{8}{3} + \frac{2}{3} = 0$ by R5, R2, R3, and R4.

Let v be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3,6^+)$-vertices by Lemma 8. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{9} = 3 - \frac{8}{3} - \frac{1}{3} = 0$ by R6. If $n_2(v) \leq 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{6} = 3 - \frac{8}{3} - \frac{1}{3} = 0$ by R6.

Let v be a 4-vertex. Then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{3} = 4 - \frac{8}{3} - \frac{4}{3} = 0$ by R5.

Let v be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 5 - \frac{8}{3} - 2 > 0$ by R4. If $n_2(v) = 5$, then there are at most four $(3,5)$-vertices in $N(v)$ by Lemma 9. Thus $\omega'(v) = \omega(v) - 4 \times \frac{1}{2} - \frac{1}{3} = 5 - \frac{8}{3} - \frac{7}{3} = 0$ by R4.

Let v be a 6-vertex. Then $\omega'(v) \geq \omega(v) - 6 \times \frac{5}{9} = 6 - \frac{8}{3} - \frac{10}{3} = 0$ by R3.

Let v be a 7-vertex. If $n_2(v) \leq 6$, then $\omega'(v) \geq \omega(v) - 6 \times \frac{2}{3} = 7 - \frac{8}{3} - 4 > 0$ by R2. When $n_2(v) = 7$, if there are no more than four $(7,2)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{2}{3} - 3 \times \frac{5}{9} = 7 - \frac{8}{3} - \frac{13}{3} = 0$ by R2; if there are five $(7,2)$-vertices in $N(v)$, then at least one of the other neighbors is a $(7,4^+)$-vertex from Lemma 10, and $\omega'(v) = \omega(v) - 6 \times \frac{2}{3} - \frac{1}{3} = 7 - \frac{8}{3} - \frac{13}{3} = 0$ by R2.

Finally, if $d(v) \geq 8$, then $\omega'(v) \geq \omega(v) - \frac{2}{3} \times d(v) = \frac{d(v)}{3} - \frac{8}{3} = \frac{d(v)-8}{3} \geq 0$ by R1.

Thus, we get the desired contradiction, and Theorem 5 is proved. ■

It is interesting that Cranston and Yu [1] cited an example ($mad(K_{2,3}) = \frac{12}{5}$ and $lc(K_{2,3}) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$) to illustrate that the bound in Theorem 1(iv) is sharp. Similarly, the graph $K_{2,4}$ satisfies $lc(K_{2,4}) \geq \lceil \frac{\Delta}{2} \rceil + 2$, $\Delta(K_{2,4}) = 4$ and $mad(K_{2,4}) = \frac{8}{3}$. So the hypothesis about $\Delta(G)$ in Theorem 3(1) is essential, and we suspect it can be replaced by $\Delta(G) \geq 5$.

3. Graphs with $mad(G) < \frac{18}{7}$ and $\Delta(G) \geq 5$

For Theorem 3(2), we prove the following result instead.

Theorem 11. *Let $M \geq 5$ be an integer. If G is a graph with $mad(G) < \frac{18}{7}$ and $\Delta(G) \leq M$, then $lc_l(G) = \lceil \frac{M}{2} \rceil + 1$.*

Proof. By contradiction, we suppose that Theorem 11 is false. Let G be a counterexample with the fewest vertices and L be a list assignment of size $\lceil \frac{M}{2} \rceil + 1 \geq 4$ such that G has no linear L-coloring. In the proof we need some structural lemmas, and it is clear that Lemma 6 and Lemma 7 are also true. ■

Lemma 12. *Let v be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$ and $n_2(v) = 3$. Then v_1, v_2, v_3 must be $(3, 5^+)$-vertices.*

Proof. Assume that v_1 is a $(3, 4^-)$-vertex, and u_i is the neighbor of v_i other than v for $i = 1, 2, 3$. Let $G' = G - \{v, v_1\}$. Then G' has a linear L-coloring c by the minimality of G. If $c(v_2) \neq c(v_3)$, there exist at least $|L(v_1)\backslash\{c(u_1), c_2(u_1)\}| \geq 2$ colors available for v_1. If there is an available color $\alpha \notin \{c(v_2), c(v_3)\}$ for v_1, then let $c(v_1) = \alpha$ and $c(v) \in L(v)\backslash\{c(v_1), c(v_2), c(v_3)\}$. If the available colors for v_1 are exactly $c(v_2)$ and $c(v_3)$, then let $c(v_1) = c(v_2)$ and $c(v) \in L(v)\backslash\{c(v_1), c(u_2), c(v_3)\}$. It is similar for $c(v_1) = c(v_3)$. Thus we get a linear list coloring of G extended from the linear L-coloring c of G'. If $c(v_2) = c(v_3)$, we can extend the linear list coloring c of G' to v_1 since $|L(v_1)\backslash\{c(u_1), c_2(u_1), c(v_2)\}| \geq 1$. There is at least $|L(v)\backslash\{c(v_1), c(v_2), c(u_2)| \geq 1$ color available for v. Thus, we also get a linear list coloring of G. A contradiction. ■

Lemma 13. *Let v be a 3-vertex with $N(v) = \{v_1, v_2, v_3\}$. If v_1 and v_2 are $(3, 3)$-vertices, then v_3 must be a 4^+-vertex.*

Proof. Assume that v_3 is a 3^--vertex, and u_i is the neighbor of v_i other than v for $i = 1, 2$. Let $G' = G - \{v, v_1, v_2\}$. Then G' has a linear L-coloring c by the minimality of G. We can extend the linear L-coloring c to v_1 such that no bi-colored cycle passes vv_1u_1 since $|L(v_1)\backslash\{c(u_1), c(N(u_1))\}| \geq 1$.

If $c(v_1) = c(v_3)$. Since $|L(v_2)\backslash\{c(v_1), c(u_2), c_2(u_2)\}| \geq 1$, we can extend the coloring c to v_2. Finally, we can color v with a color in $L(v)\backslash\{c(v_1), c(v_2), c_2(v_3)\}$ when $|c_2(v_3)| = 1$, or in $L(v)\backslash\{c(v_1), c(v_2), c(u_2)\}$ when $|c_2(v_3)| = 0$. It is clear that no bi-colored cycle passes v_2vv_3. Then we get a linear list coloring of G.

If $c(v_1) \neq c(v_3)$. There is at least $|L(v)\backslash\{c(v_1), c(v_3), c_2(v_3)\}| \geq 1$ color available for v. Finally, we can color v_2 with a color in $L(v_2)\backslash\{c(v), c(u_2), c_2(u_2)\}$ when $c(v) \neq c(u_2)$, or in $L(v_2)\backslash\{c(u_2), c(N(u_2))\}$ when $c(v) = c(u_2)$. In this process, there will be no bi-colored cycle passing vv_2. Thus, we also get a linear list coloring of G extended from the linear L-coloring c of G'. A contradiction. ■

Lemma 14. *Let v be a 4-vertex with $n_2(v) = 4$ in G. Then there are at most two $(4, 3)$-vertices in $N(v)$.*

Proof. Let $N(v) = \{v_1, \ldots, v_4\}$, and u_i be the other neighbor of v_i for $i = 1, \ldots, 4$. Assume that v_1, v_2 and v_3 are $(4, 3)$-vertices. Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. We can extend the linear

L-coloring c of G' to v_4 since $|L(v_4)\backslash\{c(u_4), c_2(u_4)\}| \geq 1$. We can continue to extend to v_3 with $c(v_3) \neq c(v_4)$ since $|L(v_3)\backslash\{c(u_3), c_2(u_3), c(v_4)\}| \geq 1$. Then we color v_2 with a color in $L(v_2)\backslash\{c(u_2), c(N(u_2))\}$. Notice that no bi-colored cycle passes vv_2u_2 or v_3vv_4. This signifies that any bi-colored cycle in G if there will be must passes v_1. Finally, we will extend the coloring c to v_1 and v in two different cases.

If $c(v_2) \notin \{c(v_3), c(v_4)\}$, we can choose a color from $\{L(v)\backslash\{c(v_2), c(v_3), c(v_4)\}$ for v. Then there is at least $|L(v_1)\backslash\{c(v), c(u_1), c_2(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\backslash\{c(u_1), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$ color available for v_1, which ensure no bi-colored cycle passes vv_1u_1. So we get a linear list coloring of G.

If $c(v_2) \in \{c(v_3), c(v_4)\}$, suppose $c(v_2) = c(v_3)$ (similarly for $c(v_2) = c(v_4)$). If $|c_2(u_1)| = 1$, we color v_1 with a color in $L(v_1)\backslash\{c(v_2), c(u_1), c_2(u_1)\}$, and no bi-colored cycle passes vv_1. If $|c_2(u_1)| = 0$, we color v_1 with a color in $L(v_1)\backslash\{c(v_2), c(u_1), c(v_4)\}$, which ensure that no bi-colored cycle passes v_1vv_4. Then we color v with a color in $L(v)\backslash\{c(v_1), c(v_2), c(v_4)\}$. Notice that no bi-colored cycle passes v_1vv_3 since $c(v_1) \neq c(v_3)$. Thus, we also get a linear list coloring c of G. A contradiction. ∎

Lemma 15. *Let v be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$. If v_1, v_2, v_3, v_4 are four $(5, 2)$-vertices, then v_5 must be a 3^+-vertex.*

Proof. Assume that v_5 is a 2-vertex, and u_i is the neighbor of v_i other than v for $i = 1, 2, \ldots, 5$. Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. We can extend the L-coloring c of G' to v_5 since $|L(v_5)\backslash\{c(u_5), c_2(u_5)\}| \geq 1$, and continue to v_4 such that $c(v_4) \neq c(v_5)$ and no bi-colored cycle passes v_4u_4 since $|L(v_4)\backslash\{c(u_4), c(N(u_4)), c(v_5)\}| \geq 1$, then to v_3 with $c(v_3) \notin \{c(v_4), c(v_5)\}$ since $|L(v_3)\backslash\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. We can color v with a color in $L(v)\backslash\{c(v_5), c(v_4), c(v_3)\}$, and color v_2 such that no bi-colored cycle passes vv_2u_2 since $|\{L(v_2)\backslash\{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color v_1 linearly since $|L(v_1)\backslash\{c(v), c_2(v), c(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\backslash\{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_5)$, there will be no bi-colored cycle created. Thus we can extend the linear L-coloring c of G' to G. A contradiction. ∎

Lemma 16. *Let v be a 5-vertex with $N(v) = \{v_1, \ldots, v_5\}$ and $n_2(v) = 5$. If v_1, v_2, v_3 are $(5, 2)$-vertices, then at least one of v_4 and v_5 is a $(5, 4^+)$-vertex.*

Proof. Assume that v_4 and v_5 are $(5, 3^-)$-vertices, and u_i is the neighbor of v_i other than v for $i = 1, \ldots, 5$. Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. We can extend the coloring c of G' to v_5 such that no bi-colored cycle passes vv_5u_5 since $|L(v_5)\backslash\{c(u_5), c(N(u_5))\}| \geq 1$, and continue to v_4 such that $c(v_4) \neq c(v_5)$ as $|L(v_4)\backslash\{c(u_4), c_2(u_4), c(v_5)| \geq 1$, then to v_3 with $c(v_3) \notin \{c(v_5), c(v_4)\}$ since $|L(v_3)\backslash\{c(u_3), c(v_4), c(v_5)\}| \geq 1$. Now we can color v with a color in $L(v)\backslash\{c(v_5), c(v_4), c(v_3)\}$, and color v_2 such that no bi-colored cycle passes vv_2u_2 since $|L(v_2)\backslash\{c(v), c(u_2), c(N(u_2))\}| \geq 1$. Finally, we can color v_1 linearly since $|L(v_1)\backslash\{c(v), c_2(v), c(u_1)\}| \geq 1$ when $c(v) \neq c(u_1)$, or $|L(v_1)\backslash\{c(v), c_2(v), c(N(u_1))\}| \geq 1$ when $c(v) = c(u_1)$. Note that $c(v_3) \neq c(v_4)$, there will be no bi-colored cycle created. Thus, we can extend the linear L-coloring c of G' to G. A contradiction. ■

We will derive a contradiction by a discharging procedure proceeded in G to complete the proof of Theorem 11. In the discharging procedure, the initial charge function ω is defined as $\omega(v) = d(v) - \frac{18}{7}$ for every vertex $v \in V(G)$, and the discharging rules are as follows.

R1. Every 6^+-vertex sends $\frac{4}{7}$ to each adjacent 2-vertex or 3-vertex.

R2. Every 5-vertex sends $\frac{4}{7}$ to each adjacent $(5,2)$-vertex, $\frac{3}{7}$ to each adjacent $(5,3)$-vertex, $\frac{2}{7}$ to each adjacent $(5,4^+)$-vertex, $\frac{1}{7}$ to each adjacent 3-vertex.

R3. Every 4-vertex sends $\frac{3}{7}$ to each adjacent $(4,3)$-vertex, $\frac{2}{7}$ to each adjacent $(4,4^+)$-vertex, $\frac{1}{7}$ to each adjacent 3-vertex.

R4. Every 3-vertex sends $\frac{2}{7}$ to each adjacent $(3,3)$-vertex, $\frac{1}{7}$ to each adjacent $(3,4^+)$-vertex.

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V$.

If $d(v) \geq 6$, then $\omega'(v) \geq \omega(v) - \frac{4}{7} \times d(v) = \frac{3d(v)}{7} - \frac{18}{7} = \frac{3d(v)-18}{7} \geq 0$ by R1.

Let v be a 5-vertex. If $n_2(v) \leq 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{4}{7} - \frac{1}{7} = 5 - \frac{18}{7} - \frac{16}{7} - \frac{1}{7} = 0$ by R2. When $n_2(v) = 5$, there are at most three $(2,5)$-vertices in $N(v)$ by Lemma 15. If there are two or less $(2,5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{4}{7} - 3 \times \frac{3}{7} = 5 - \frac{18}{7} - \frac{8}{7} - \frac{9}{7} = 0$ by R2. If there are three $(2,5)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{4}{7} - \frac{3}{7} - \frac{2}{7} = 5 - \frac{18}{7} - \frac{12}{7} - \frac{5}{7} = 0$ by Lemma 16 and R2.

Let v be a 4-vertex. If $n_2(v) \leq 3$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{3}{7} - \frac{1}{7} = 4 - \frac{18}{7} - \frac{9}{7} - \frac{1}{7} = 0$ by R3. If $n_2(v) = 4$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{2}{7} = 4 - \frac{18}{7} - \frac{6}{7} - \frac{4}{7} = 0$ by Lemma 14 and R3.

Let v be a 3-vertex. If $n_2(v) = 3$, then the vertices in $N(v)$ must be $(3,5^+)$-vertices by Lemma 12. Thus $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0$ by R4. If $n_2(v) = 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{2}{7} + \frac{1}{7} = 3 - \frac{18}{7} - \frac{4}{7} + \frac{1}{7} = 0$ by Lemma 13 and all discharging rules, or $\omega'(v) \geq \omega(v) - \frac{2}{7} - \frac{1}{7} = 3 - \frac{18}{7} - \frac{3}{7} = 0$. If $n_2(v) \leq 1$, then $\omega'(v) \geq \omega(v) - \frac{2}{7} = 3 - \frac{18}{7} - \frac{2}{7} > 0$ by R4.

Finally, let v be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. If $d(x) = 2$, then $d(y) \geq 5$ by Lemma 7. By R1 and R2, $\omega'(v) = \omega(v) + \frac{4}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$. When $d(x) = 3$, we have $\omega'(v) = \omega(v) + 2 \times \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ if $d(y) = 3$, and $\omega'(v) = \omega(v) + \frac{1}{7} + \frac{3}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ if $d(y) \geq 4$. Otherwise, $d(x) \geq 4$ and $d(y) \geq 4$, we have $\omega'(v) \geq \omega(v) + \frac{2}{7} + \frac{2}{7} = 2 - \frac{18}{7} + \frac{4}{7} = 0$ by R1, R2, and R3. We get the desired contradiction, and Theorem 11 is proved. ■

Similarly, the condition $\Delta(G)$ in Theorem 3(2) must be $\Delta(G) \geq 4$.

4. GRAPHS WITH $mad(G) < \frac{20}{7}$ AND $\Delta(G) \geq 5$

Cranston and Yu [1] conjectured that the hypothesis $\Delta(G) \geq 9$ of Theorem 1(iii) can be replaced by $\Delta(G) \geq 7$, even $\Delta(G) \geq 5$. Now, we prove Theorem 3(3) to support their conjecture. In order to prove Theorem 3(3), we prove the following theorem which implies Theorem 3(3) immediately.

Theorem 17. *Let $M \geq 5$ be an integer. If G is a graph with $mad(G) < \frac{20}{7}$ and $\Delta(G) \leq M$, then $lc_l(G) \leq \lceil \frac{M}{2} \rceil + 2$.*

Proof. Let G be a counterexample of the fewest vertices with $mad(G) < \frac{20}{7}$ and $5 \leq \Delta(G) \leq 8$ (Theorem 17 is true for graphs G with $\Delta(G) \geq 9$ by Theorem 1(iii)). There exists an assignment L with $|L| \geq \lceil \frac{M}{2} \rceil + 2 \geq 5$ such that G is not linearly L-choosable, but H has a linear L-coloring, where H is any proper subgraph of G. Clearly, G is connected and $\delta(G) \geq 2$. In the proof we need some structural lemmas. ∎

Lemma 18. *Let v be a 2-vertex with $N(v) = \{v_1, v_2\}$. Then $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \geq \lceil \frac{M}{2} \rceil + 2$.*

Proof. Assume $\left\lceil \frac{d(v_1)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil \leq \lceil \frac{M}{2} \rceil + 1$. Let $G' = G - v$. Then G' has a linear L-coloring c by the minimality of G. If $c(v_1) \neq c(v_2)$, we can color v with any color in $L(v) \setminus \{c(v_1), c(v_2), c_2(v_1), c_2(v_2)\}$. Then the number of available colors for v is at least $\lceil \frac{M}{2} \rceil + 2 - \left(2 + \left\lfloor \frac{d(v_1)-1}{2} \right\rfloor + \left\lfloor \frac{d(v_2)-1}{2} \right\rfloor\right) = \lceil \frac{M}{2} \rceil + 2 - \left(\left\lceil \frac{d(v_2)}{2} \right\rceil + \left\lceil \frac{d(v_2)}{2} \right\rceil\right) \geq 1$. Clearly, there will be no bi-colored cycle created. So we extend the linear L-coloring c of G' to G. Now we suppose $c(v_1) = c(v_2)$. In order to color v linearly and avoid bi-colored cycles created, the forbidden color set for v contains the color $c(v_1)$, the colors appearing twice in $N(v_1)$ or $N(v_2)$, and the colors appearing in both $N(v_1)$ and $N(v_2)$. So at most $1 + |c_2(v_1) \cup c_2(v_2)| + |c_1(v_1) \cap c_1(v_2)| \leq \left\lceil \frac{d(v_1)+d(v_2)}{2} \right\rceil \leq \left\lceil \frac{d(x)}{2} \right\rceil + \left\lceil \frac{d(y)}{2} \right\rceil \leq \lceil \frac{M}{2} \rceil + 1$ colors are forbidden for v. Thus, we also can get a linear L-coloring of G. A contradiction. ∎

Lemma 19. *Let v be a 3-vertex of G with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 2$, then $d(v_2) \geq 3$ and $\left\lfloor \frac{d(v_2)+d(v_3)}{2} \right\rfloor \geq \lceil \frac{M}{2} \rceil + 1$.*

Proof. We prove $d(v_2) \geq 3$ first. To the contrary, we assume $d(v_2) = 2$. Let $G' = G - \{v, v_1, v_2\}$. Then G' has a linear L-coloring c by the minimality of G. The neighbors of v_1 and v_2 other than v are denoted by u_1 and u_2, respectively. We can extend the coloring c of G' to v_1 such that $c(v_1) \neq c(v_3)$ since

$|L(v_1) \setminus \{c(u_1), c_2(u_1), c(v_3)\}| \geq 1$, which ensures that no bi-colored cycle passes v_1vv_3. We can continue to extend to v since $|L(v) \setminus \{c(v_1), c(v_3), c_2(v_3)\}| \geq 1$. If $c(v) \neq c(u_2)$, which means that no bi-colored cycle passes vv_2u_2, we can color v_2 linearly since $|L(v_2)\setminus\{c(v), c(u_2), c_2(u_2)\}| \geq 1$. When $c(v) = c(u_2)$, the number of available colors for v_2 is at least $|L(v_2)\setminus\{c(u_2), c_2(u_2)\}| \geq 2$. If there is an available color $\alpha \notin \{c(v_1), c(v_3)\}$ for v_2, then we color v_2 with α. Now we assume that the available colors for v_2 are exactly $c(v_1)$ and $c(v_3)$. Notice that $|c_2(u_2)| = \lfloor \frac{M-1}{2} \rfloor$ and $|c_1(u_2)| \leq 1$ now. To avoid bi-colored cycle created, the number of forbidden colors for v_2 is at most $|\{c(u_2), c(N(u_2))\}| = 1 + |c_2(u_2)| + |c_1(u_2)| \leq 1 + \lfloor \frac{M-1}{2} \rfloor + 1 = \lceil \frac{M}{2} \rceil + 1$, so we can color v_2 linearly. Thus, we get a linear L-coloring of G extended from the linear L-coloring c of G'. A contradiction.

Now, we prove the inequality. Suppose to the contrary that, we have $\left\lfloor \frac{d(v_2)+d(v_3)}{2} \right\rfloor \leq \lceil \frac{M}{2} \rceil$, and u_1 is the neighbor of v_1 other than v. Let $G' = G - v_1$, then G' has a linear L-coloring c by the minimality of G.

Case 1. $c(v_2) \neq c(v_3)$. If $c(v) \neq c(u_1)$, then we can extend the coloring c to v_1 to get a linear L-coloring of G since $|L(v_1)\setminus\{c(v), c(u_1), c_2(u_1)\}| \geq 1$. If $c(v) = c(u_1)$, the number of available colors for v_1 is at least $|L(v_1)\setminus\{c(v), c_2(u_1)\}| \geq 2$. If there is a color $\alpha \notin \{c(v_2), c(v_3)\}$ available for v, then we can extend c from G' to G by coloring v_1 with α. Now we assume that $L(v_1)\setminus\{c(v), c_2(u_1)\} = \{c(v_2), c(v_3)\}$. Notice that $|c_2(u_1)| = \lfloor \frac{M-1}{2} \rfloor$ now. Then $c(v_2)$ and $c(v_3)$ appears at most once in $N(u_1)$, but both of them could not appear in $N(u_1)$ at the same time (otherwise $|L(v_1) \setminus \{c(v), c_2(u_1)\}| \geq \lceil \frac{M}{2} \rceil + 2 - (1 + \lfloor \frac{M-3}{2} \rfloor) \geq 3$). So we color v_1 with $c(v_3)$ if $c(v_2)$ appears in $N(u_1)$, otherwise color v_1 with $c(v_2)$. Then there will be no bi-colored cycle created. Thus, we get a linear L-coloring of G extended from the linear L-coloring c of G'.

Case 2. $c(v_2) = c(v_3)$. If $c(v) = c(u_1)$, then we can color v_1 linearly since $|L(v_1)\setminus\{c(v_2), c(v), c_2(u_1)\}| \geq 1$, and no bi-colored cycle created. Now, suppose $c(v) \neq c(u_1)$. We erase the color of v first, then we can extend the list coloring c to v_1 since $|L(v_1)\setminus\{c(v_2), c(u_1), c_2(u_1)\}| \geq 1$. To avoid bi-colored cycle created, the number of forbidden colors for v is at most $2+|c_2(v_2)\cup c_2(v_3)|+|c_1(v_2)\cap c_1(v_3)| \leq 2+\left\lfloor \frac{d(v_2)-1+d(v_3)-1}{2} \right\rfloor = 1+\left\lfloor \frac{d(v_2)+d(v_3)}{2} \right\rfloor \leq \lceil \frac{M}{2} \rceil+1$. We also can extend the linear L-coloring c of G' to G. A contradiction. ∎

Lemma 20. *Let v be a 4-vertex in G. Then $n_2(v) \leq 3$.*

Proof. Let $N(v) = \{v_1, \ldots, v_4\}$. Suppose to the contrary, let $n_2(v) = 4$, and u_i be the neighbor of v_i other than v for $i = 1, \ldots, 4$. Let $G' = G - N[v]$. Then G' has a linear L-coloring c by the minimality of G. Since $|L(v_i)\setminus\{c(u_i), c_2(u_i)\}| \geq 2$, we can color v_i linearly with at least two different colors for $i = 1, 2, 3, 4$. Finally, we can color v with a color in $L(v)\setminus c(N(v))$ if $|c(N(v)) = 4|$, or in

$L(v)\backslash\{c(N(v)), c(u_i)\}$ if $c(v_i) = c(v_j)$ for $1 \le i < j \le 4$. And no bi-colored cycle appears in this process. Thus, we get a linear list coloring of G extended from the linear L-coloring c of G', a contradiction. ∎

Lemma 21. *Let v be a 4-vertex with $N(v) = \{v_1, \ldots, v_4\}$. If $d(v_1) = d(v_2) = d(v_3) = 2$ and v_3 is a $(4,5)$-vertex, then v_4 must be a 4^+-vertex.*

Proof. Suppose to the contrary, let v_4 be a 3^--vertex, and u_i be the neighbor of v_i other than v for $i = 1, 2, 3$. Let $G' = G - \{v, v_1, v_2, v_3\}$, then G' has a linear L-coloring c by the minimality of G. We can extend the coloring c of G' to v_1 with $c(v_1) \ne c(v_4)$ since $|L(v_1)\backslash\{c(u_1), c_2(u_1), c(v_4)\}| \ge 1$. Then there are at least $|L(v_2)\backslash\{c(u_2), c_2(u_2)\}| \ge 2$ colors available for v_2.

If there is a color $\alpha \notin \{c(v_1), c(v_4)\}$ available for v_2, let $c(v_2) = \alpha$. If $|c_2(v_4)| = 1$, we can choose a color for v in $L(v)\backslash\{c(v_1), c(v_2), c(v_4), c_2(v_4)\}$, and there will be no bi-colored cycle created passing vv_4. Then, we color v_3 with a color in $L(v_3)\backslash\{c(v), c(u_3), c_2(u_3)\}$ if $c(v) \ne c(u_3)$. When $c(v) = c(u_3)$, in order to color v_3 linearly (no bi-colored cycle created), we must forbidden $c(v)$, $c_2(N(u_3))$ and $\{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))$. Notice that $d(u_3) = 5$, then $|c_2(N(u_3)) \cup (\{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3)))| \le 3$. So we can color v_3 linearly with a color in $L(v_3) \setminus \{c(v), c_2(N(u_3)), \{c(v_1), c(v_2), c(v_4)\} \cap c_1(N(u_3))\}$. Thus, we get a linear list coloring of G.

Suppose the available color set for v_2 is exactly $\{c(v_1), c(v_4)\}$. Notice that $|c_2(u_2)| = \lfloor \frac{M-1}{2} \rfloor$ now. We color v_2 with $c(v_4)$ first. If $|c_2(u_3)| = 2$, we color v_3 with a color in $L(v_3)\backslash\{c(v_4), c(u_3), c_2(u_3)\}$. If $|c_2(u_3)| \le 1$, we color v_3 with a color in $L(v_3)\backslash\{c(v_4), c(v_1), c(u_3), c_2(u_3)\}$. Notice $d(u_3) = 5$, then no bi-colored cycle passes v_3u_3. Finally, we can color v with a color in $L(v)\backslash\{c(v_1), c(v_3), c(v_4), c_2(v_4)\}$ if $|c_2(v_4)| = 1$, or in $L(v)\backslash\{c(v_1), c(v_3), c(v_4), c(u_2)\}$ if $|c_2(v_4)| = 0$. Clearly, there will be no bi-colored cycle passing vv_4. Then we extend the linear L-coloring c of G' to G, a contradiction. ∎

Lemma 22. *Let v be a 4-vertex with $N(v) = \{v_1, \ldots, v_4\}$. If $d(v_1) = d(v_2) = d(v_3) = 2$ and v_2, v_3 are $(4,5)$-vertices, then v_4 must be a 5^+-vertex.*

Proof. Suppose to the contrary, let v_4 be a 4^--vertex, and u_i be the neighbor of v_i other than v for $i = 1, 2, 3$. Let $G' = G - \{v, v_1, v_2, v_3\}$. Then G' has a linear L-coloring c by the minimality of G. We can extend the coloring c of G' to v_1 with $c(v_1) \ne c(v_4)$ since $|L(v_1)\backslash\{c(u_1), c_2(u_1), c(v_4)\}| \ge 1$. Then there are at least $|L(v_2)\backslash\{c(u_2), c_2(u_2)\}| \ge 2$ colors available for v_2.

If there is an available color $\alpha \notin \{c(v_1), c(v_4)\}$ for v_2, let $c(v_2) = \alpha$. If $|c_2(v_4)| = 1$, we color v with a color in $L(v)\backslash\{c(v_1), c(v_2), c(v_4), c_2(v_4)\}$. Then, we color v_3 with a color in $L(v_3)\backslash\{c(v), c(u_3), c_2(u_3)\}$ if $c(v) \ne c(u_3)$. If $c(v) = c(u_3)$, we color v_3 with a color in $L(v_3)\backslash\{c(u_3), c_2(u_3)\}$, $L(v_3)\backslash\{c(u_3), c_1(u_3), c_2(u_3)\}$ or $L(v_3)\backslash\{c(u_3), c(v_1), c(v_2), c(v_4)\}$ when $|c_2(u_3)| = 2$, $|c_2(u_3)| = 1$ or $|c_2(u_3)| = 0$, respectively. Notice that v_3 is a $(4,5)$-vertex, it means $d(u_3) = 5$, then there will be no bi-colored cycle passing v_3u_3. Then we get a linear list coloring of G. If $|c_2(v_4)| = 0$, we choose a color for v in $L(v)\backslash\{c(v_1), c(v_2), c(v_4), c(u_3)\}$, then we can color v_3 linearly since $|L(v_3)\backslash\{c(v), c(u_3), c_2(u_3)\}| \geq 1$. We also get a linear list coloring of G.

When the available color set for v_2 is exactly $\{c(v_1), c(v_4)\}$ (notice that $|c_2(u_2)| = 2$, and there will be no bi-colored cycle passing v_2u_2), we can color v_2 with $c(v_4)$. If $|c_2(u_3)| = 2$, we color v_3 with a color in $L(v_3)\backslash\{c(v_4), c(u_3), c_2(u_3)\}$; if $|c_2(u_3)| \leq 1$, we color v_3 with a color in $L(v_3)\backslash\{c(v_4), c(v_1), c(u_3), c_2(u_3)\}$. Notice $d(u_3) = 5$, there will be no bi-colored cycle passing v_3u_3. Finally, we can color v with a color in $L(v)\backslash\{c(v_1), c(v_3), c(v_4), c_2(v_4)\}$ if $|c_2(v_4)| = 1$, or in $L(v)\backslash\{c(v_1), c(v_3), c(v_4)\}$ if $|c_2(v_4)| = 0$. Then we extend the linear L-coloring c of G' to G, a contradiction. ■

To complete our proof of Theorem 17, it suffices to derive a contradiction by a discharging procedure. We define the initial charge function ω on $V(G)$ by $\omega(v) = d(v) - \frac{20}{7}$ for every $v \in V(G)$. The discharging rules are as follows.

R1. Every 5^+-vertex sends $\frac{d(v)-\frac{20}{7}}{d(v)}$ to each adjacent vertex.

R2. Every 4-vertex sends $\frac{3}{7}$ to each adjacent $(4,5)$-vertex, $\frac{1}{3}$ to each adjacent $(4,6)$-vertex, $\frac{1}{7}$ to each adjacent 3-vertex;

R3. Every 3-vertex sends $\frac{3}{7}$ to each adjacent 2-vertex (if it has one).

Now we are going to show that $\omega'(v) \geq 0$ for all $v \in V(G)$. We only need to check the final charges of 4^--vertices from the discharging rules.

Let v be a 4-vertex in G. Then $n_2(v) \leq 3$ by Lemma 20. If $n_2(v) \leq 2$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{3}{7} - 2 \times \frac{1}{7} = 0$ by R2. When $n_2(v) = 3$, if three are three $(4,6)$-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - 3 \times \frac{1}{3} - \frac{1}{7} = 0$; if there is only one $(4,5)$-vertex in $N(v)$, then $\omega'(v) \geq \omega(v) - 2 \times \frac{1}{3} - \frac{3}{7} > 0$ by Lemma 21 and R2; if there are two or more $(4,5)$-vertices in $N(v)$, we have $\omega'(v) \geq \omega(v) - 3 \times \frac{3}{7} + \frac{3}{7} > 0$ by Lemma 22 and R2.

Let v be a 3-vertex in G. Then $n_2(v) \leq 1$ by Lemma 19. If $n_2(v) = 0$, then $\omega'(v) = \omega(v) = 3 - \frac{20}{7} > 0$. When $n_2(v) = 1$, if there is a 3-vertex in $N(v)$, we have $\omega'(v) \geq \omega(v) - \frac{3}{7} + \frac{3}{7} > 0$ by Lemma 19 and R3; if there are two 4^+-vertices in $N(v)$, then $\omega'(v) \geq \omega(v) - \frac{3}{7} + 2 \times \frac{1}{7} = 0$.

Let v be a 2-vertex with $N(v) = \{x, y\}$ and $d(x) \leq d(y)$. Clearly, $d(x) \geq 3$ by Lemma 18. If $d(x) = 3$, then $d(y) \geq 5$ by Lemma 19, so $\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0$ by R3 and R1. If $d(x) = 4$, then $d(y) \geq 5$ by Lemma 18, so $\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0$, or $\omega'(v) \geq \omega(v) + \frac{1}{3} + \frac{11}{21} = 2 - \frac{20}{7} + \frac{6}{7} = 0$. Otherwise, $d(x) \geq 5$ and $d(y) \geq 5$, we have $\omega'(v) \geq \omega(v) + \frac{3}{7} + \frac{3}{7} = 2 - \frac{20}{7} + \frac{6}{7} = 0$.

In summary, the proof of Theorem 3 is completed. ■

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