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ON THE n-PARTITE TOURNAMENTS WITH EXACTLY $n-m+1$ CYCLES OF LENGTH m

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Abstract

Gutin and Rafiey [*Multipartite tournaments with small number of cycles*, Australas J. Combin. 34 (2006) 17–21] raised the following two problems: (1) Let $m \in \{3, 4, \ldots, n\}$. Find a characterization of strong n-partite tournaments having exactly $n-m+1$ cycles of length m; (2) Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length m for two values of m? In this paper, we discuss the strong n-partite tournaments D containing exactly $n-m+1$ cycles of length m for $4 \leq m \leq n-1$. We describe the substructure of such D satisfying a given condition and we also show that, under this condition, the second problem has a negative answer.

Keywords: multipartite tournaments, tournaments, cycles.

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1. Introduction

An *n-partite* or *multipartite tournament* is an orientation of a complete n-partite graph. A *tournament* is an n-partite tournament with exactly n vertices. A digraph D is *transitive* if, for every pair of arcs xy and yz in D such that $x \neq z$, the arc xz is also in D. It is easy to show that a tournament is transitive if and only if it is acyclic.

A digraph D is said to be *strong*, if for every pair of vertices x and y, D contains a path from x to y and a path from y to x. A directed path from x to y in D is denoted by an (x, y)-path. An l-cycle is a cycle of length l. A cycle or path in a digraph D is Hamiltonian if it includes all the vertices of D.

In 1966, Moon discussed the number of m-cycle in a strong tournament.

Theorem 1 (Moon [9]). *Let T be a strong tournament of order n. Then T contains at least $n-m+1$ cycles of length m for $3 \leq m \leq n$.*

The tournaments which gain the lower bound in Theorem 1 were characterized by Burzio and Demaria [2] for $m = 3$, Douglas [3] for $m = n$ and Las Vergnas [8] for $4 \leq m \leq n-1$. We list the result of Las Vergnas especially because we will use it to prove our main results.

Theorem 2 (Las Vergnas [8]). *Every strong tournament of order n having exactly $n-m+1$ cycles of given length m with $4 \leq m \leq n-1$ is isomorphic to Q_n, where Q_n is a tournament of order $n \geq 3$ obtained by reversing the arcs in the unique Hamiltonian path of a transitive tournament.*

In 2002, Volkmann extended Theorem 1 from tournaments to multipartite tournaments.

Theorem 3 (Volkmann [10]). *Let D be a strong n-partite tournament. Then D contains at least $n-m+1$ cycles of length m for $3 \leq m \leq n$.*

It is notable that the bound in Theorem 3 is sharp which can be seen in [4] for $n = 3$ and in [6] for $4 \leq m \leq n$. In addition, Gutin and Rafiey [6] raised the following two interesting and natural problems.

Problem 4 (Gutin and Rafiey [6]). *Given $m \in \{3, 4, \ldots, n\}$, find a characterization of strong n-partite tournaments having exactly $n-m+1$ cycles of length m.*

Problem 5 (Gutin and Rafiey [6]). *Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong n-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length m for two values of m?*

Problem 4 seems to be especially interesting for the case $m = n$ which was already solved by Gutin *et al.* in [7]. In this paper, we investigate strong n-partite tournaments D, which are not themselves tournaments and contain exactly $n-m+1$ cycles of length m for any given $4 \leq m \leq n-1$. We prove that if D has an $(n-1)$-cycle with no pair of vertices from the same partite set, then D must contain some given multipartite tournament as its subdigraph.

As far as Problem 5, Gutin and Rafiey [6] gave a negative answer for two values $n-1$ and n of m. This also implies that Problem 5 has a negative answer for $n = 4$. In this paper, we give a necessary condition to Problem 5 and show that if a strong n-partite tournament D, which is not itself a tournament, contains exactly $n-m+1$ cycles of length m for two values of $m \in \{4, 5, \ldots, n-1\}$, then there is no an $(n-1)$-cycle with no pair of vertices from the same partite set in D.

2. Terminology and Preliminaries

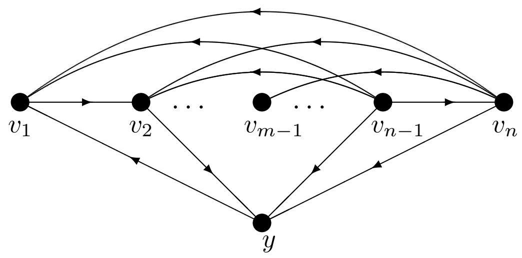
We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

Let D be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. We call the number of vertices of D the *order* of D. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D\langle A\rangle$. We use $V(D)\backslash V(A)$ to stand for the set of vertices which are in $V(D)$ but not in $V(A)$.

If xy is an arc in D, then we say that x *dominates* y and write $x \to y$. For two disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y, we say X *dominates* Y and write $X \to Y$. Furthermore, $X \Rightarrow Y$ denotes the property that there is no arc from Y to X.

The *out-neighborhood* $N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N^-(x)$ of a vertex x is the set of vertices dominating x. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x, respectively. The *global irregularity* of D is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. We denote by D^{-1} the inverse digraph of D.

In order to present our main results, we define a class of n-partite tournaments D_n of order $n+1$ as described in the following figure, where $3 \le m \le n$, $\{v_2, \ldots, v_{m-2}, v_m, \ldots, v_n\} \to y \to v_1$, y and v_{m-1} belong to the same partite set and $v_i \to v_j$ for all $1 < j+1 < i \le n$.



D_n

The following two theorems on cycles in strong n-partite tournaments are very useful to prove our main results.

Theorem 6 (Guo and Volkmann [5]). *Every partite set of a strong n-partite tournament, $n \ge 3$, contains a vertex which lies on an m-cycle for each $m \in \{3, 4, \ldots, n\}$.*

Theorem 7 (Gutin and Rafiey [6]). *Let D be a strong n-partite tournament containing exactly $n-m+1$ cycles of length m for some $m \in \{3, 4, \ldots, n\}$. Then every m-cycle of D has no pair of vertices from the same partite set.*

3. Main Results

Before presenting the main results, we first prove the following lemma.

Lemma 8. *Let D be a strong n-partite tournament, $n \geq 5$, containing exactly $n-m+1$ cycles of length m for some $3 \leq m \leq n-1$. If D has an $(n-1)$-cycle C with no pair of vertices from the same partite set, then the following statements hold.*

(a) *There are no two vertices u, w in $V(D)\backslash V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$.*

(b) *There exists a vertex $v \notin V(C)$ such that $D\langle V(C) \cup \{v\}\rangle$ is strong.*

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of D. Suppose, without loss of generality, that $C = v_1v_2 \cdots v_{n-1}v_1$ with $v_i \in V_i$, $i = 1, 2, \ldots, n-1$. By Theorem 1, $D\langle V(C)\rangle$ contains at least $(n-1)-m+1 = n-m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length m.

(a) Suppose to the contrary that there exist two vertices $u, w \in V(D)\backslash V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$. Obviously, $u \notin V_n$ or $w \notin V_n$. Assume, without loss of generality, that $w \in V_j$ for some $1 \leq j \leq n-1$. Since $n \geq 5$, there exist at least two vertices v_k and v_t, such that u, w, v_k and v_t are in different partite sets. If $m = 3$, then uwv_ku and uwv_tu are two m-cycles different from $C_1, C_2, \ldots, C_{n-m}$. This contradicts the fact that D contains exactly $n-m+1$ cycles of length m. If $m \geq 4$, then $uwv_{j-1}v_j \cdots v_{j+m-4}u$ (if $u \notin V_{j+m-4}$) or $uwv_{j+1}v_{j+2} \cdots v_{j+m-2}u$ (if $u \in V_{j+m-4}$) is an m-cycle with $w, v_j \in V_j$ or $u, v_{j+m-4} \in V_{j+m-4}$ (where all indices are modulo $n-1$). This is impossible by Theorem 7.

(b) Assume that there is no vertex $v \in V(D)\backslash V(C)$ such that $D\langle V(C) \cup \{v\}\rangle$ is strong. Let $S = \{x \in V(D)\backslash V(C) : C \Rightarrow x\}$ and $T = \{z \in V(D)\backslash V(C) : z \Rightarrow C\}$. Since D is strong, we have that S and T are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Thus, we have $C \Rightarrow u \rightarrow w \Rightarrow C$, which contradicts (a). ■

Theorem 9. *Let D be a strong n-partite tournament which is not itself a tournament and contains exactly $n-m+1$ cycles of length m for some $4 \leq m \leq n-1$. If D has an $(n-1)$-cycle C with no pair of vertices from the same partite set, then D contains some D_i or D_i^{-1} as its subdigraph for $i \in \{n-1, n\}$, where D_i is defined in Section 2.*

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of D and let $C = v_1v_2 \cdots v_{n-1}v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n-1$. By Theorem 1, $D\langle V(C)\rangle$ contains at least $n-m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length m. By Theorem 6, there exists a vertex in V_n, say x, which lies on an m-cycle C_{n-m+1} different from $C_1, C_2, \ldots, C_{n-m}$. We consider the following two cases.

Case 1. $D\langle V(C) \cup \{x\}\rangle$ is not strong. Since D contains exactly $n-m+1$ cycles of length m, we have that $D\langle V(C)\rangle$ contains exactly $n-m$ cycles of length

m. By Theorem 2, $D\langle V(C)\rangle$ is isomorphic to Q_{n-1}. So we may assume that $v_i \to v_j$ for all $1 < j+1 < i \le n-1$. Since $D\langle V(C) \cup \{x\}\rangle$ is not strong, we have that $C \to x$ or $x \to C$.

First we consider the case $C \to x$. Let $S = \{u \in V(D)\backslash(V(C) \cup \{x\}) : D\langle V(C) \cup \{u\}\rangle$ is strong$\}$. By Lemma 8(b), S is not empty. Since D is strong, there is a path from x to S. Let $P = x_1x_2\cdots x_t$ ($x_1 = x$) be such a path and assume that the P is of minimum length. That is, $x_t \in S$ and $D\langle V(C) \cup \{x_i\}\rangle$ is not strong for each $i \in \{1, 2, \ldots, t-1\}$. Since $C \to x_1$ and $x_1 \to x_2$, we have $x_2 \notin V(C)$. If $t > 2$, then by Lemma 8(a), we have $C \Rightarrow x_2$. Successively, we can get that $x_i \notin V(C)$ and $C \Rightarrow x_i$ for all $i \in \{2, 3, \ldots, t-1\}$ when $t > 2$.

If there exist two vertices v_i, v_j on C such that $x_t \to \{v_i, v_j\}$, then, when $t > m-1$, we have that $x_tv_ix_{t-(m-2)}x_{t-(m-3)}\cdots x_t$ (if $x_{t-(m-2)} \notin V_i$) or $x_tv_jx_{t-(m-2)}x_{t-(m-3)}\cdots x_t$ (if $x_{t-(m-2)} \in V_i$) is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction; when $t = m-1$, it is clear that $x_tv_ix_1\cdots x_t$ and $x_tv_jx_1\cdots x_t$ are two m-cycles different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction; when $t \le m-2$, it is easy to see that $x_tv_iv_{i+1}\cdots v_{i+(m-1-t)}x_1\cdots x_t$ and $x_tv_jv_{j+1}\cdots v_{j+(m-1-t)}x_1\cdots x_t$ are two m-cycles different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction.

Therefore, x_t has only one out-neighbor on C. We will show that $x_t \to v_1$. In fact, if $x_t \to v_i$ and $i \ge 2$, then we have that $x_tv_i\cdots v_{i+m-2}x_t$ (when $i+m-2 \le n-1$ and $x_t \notin V_{i+m-2}$) or $x_tv_i\cdots v_{i+m-3}v_{i-1}x_t$ (when $i+m-2 \le n-1$ and $x_t \in V_{i+m-2}$) or $x_tv_i\cdots v_{n-1}v_1\cdots v_{m-n+i-1}x_t$ (when $i+m-2 \ge n$ and $x_t \notin V_{m-n+i-1}$) or $x_tv_i\cdots v_{n-1}v_2\cdots v_{m-n+i}x_t$ (when $i+m-2 \ge n$ and $x_t \in V_{m-n+i-1}$) is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So we have $\{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow x_t$. Furthermore, if $v_{m-1} \to x_t$, then $x_tv_1v_2\cdots v_{m-1}x_t$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So $x_t \in V_{m-1}$. Let $x_t = y$. Then D contains D_{n-1} as its subdigraph.

For the case $x \to C$, by considering the inverse of D, it is easy to see that D contains D_{n-1}^{-1} as its subdigraph.

Case 2. $D\langle V(C) \cup \{x\}\rangle$ is strong. In this case, $D\langle V(C) \cup \{x\}\rangle$ is a strong tournament of order n. By Theorem 1, $D\langle V(C) \cup \{x\}\rangle$ contains at least $n-m+1$ cycles of length m. Note that D contains exactly $n-m+1$ cycles of length m. We have that $D\langle V(C) \cup \{x\}\rangle$ contains exactly $n-m+1$ cycles of length m. By Theorem 2, $D\langle V(C) \cup \{x\}\rangle$ is isomorphic to Q_n. So we may assume that $C' = v_1v_2\cdots v_nv_1$ is an n-cycle of $D\langle V(C) \cup \{x\}\rangle$ satisfying $v_i \in V_i$ and $v_i \to v_j$ for all $1 < j+1 < i \le n$. Obviously, $C_1 = v_1v_2\cdots v_mv_1, C_2 = v_2v_3\cdots v_{m+1}v_2, \ldots, C_{n-m+1} = v_{n-m+1}v_{n-m+2}\cdots v_nv_{n-m+1}$ are $n-m+1$ cycles of length m of D.

Claim 10. *There exists a vertex $y \in V(D)\backslash V(C')$ such that $D\langle V(C') \cup \{y\}\rangle$ is strong.*

Proof. Assume that there is no vertex $y \in V(D)\backslash V(C')$ such that $D\langle V(C')\cup\{y\}\rangle$ is strong. Let $S = \{x \in V(D)\backslash V(C') : C' \Rightarrow x\}$ and $T = \{z \in V(D)\backslash V(C') : z \Rightarrow V(C')\}$. Since D is strong, we have that S and T are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \to w$. Suppose that $u \in V_i$ and $w \in V_j$ for $1 \le i \ne j \le n$. Then $uwv_{j+1}v_{j+2}\cdots v_{j+m-2}u$ (if $i \ne j+m-2$) or $uwv_{j+2}v_{j+3}\cdots v_{j+m-1}u$ (if $i = j+m-2$) is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$. Note that D contains exactly $n-m+1$ cycles of length m. This is a contradiction. $\square$

By Claim 10, there are two vertices v_a, v_b $(1 \le a, b \le n)$, such that $v_a \to y \to v_b$. Assume that v_k is the first vertex from v_1 to v_n dominating y.

Claim 11. *$v_i \Rightarrow y$ for all $k \le i \le n$.*

Proof. Otherwise, there exists some index t such that either $v_t \to y \to v_{t+1}$ $(k \le t \le n-1)$ or $y, v_{t+1} \in V_{t+1}$ but $v_t \to y \to v_{t+2}$ $(k \le t \le n-2)$. We still assume that t is such a minimum index.

If $t \le n-m+1$, then either $v_tyv_{t+1}\cdots v_{t+m-2}v_t$ or $v_tyv_{t+2}\cdots v_{t+m-1}v_t$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $n-m+2 \le t \le n-2$, then either $v_tyv_{t+1}\cdots v_nv_{n-m+2}\cdots v_t$ or $v_tyv_{t+2}\cdots v_nv_{n-m+1}\cdots v_t$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $t = n-1$, then $y \to v_n$ and $v_{n-1}yv_nv_{n-m+2}\cdots v_{n-1}$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. $\square$

Claim 12. *$y \to v_1$.*

Proof. If $y \in V_1$, then $y \to v_2$ (otherwise, $k = 2$ and $\{v_2, v_3, \ldots, v_n\} \to y$ by Claim 11, which contradicts the assumption that $D\langle V(C') \cup \{y\}\rangle$ is strong). By Claim 11, we have $v_n \to y$. Therefore, $D\langle v_2, \ldots, v_n, y\rangle$ is a strong tournament. Then y is in an m-cycle of $D\langle v_2, \ldots, v_n, y\rangle$, which is different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

Therefore, $y \notin V_1$. If $v_1 \to y$, then $\{v_2, v_3, \ldots, v_n\} \Rightarrow y$ by Claim 11, which contradicts the assumption that $D\langle V(C')\cup\{y\}\rangle$ is strong. So we have $y \to v_1$. $\square$

By Claim 12, we have that $2 \le k \le n$ and $y \Rightarrow v_{m-1}$. Otherwise, $yv_1v_2\cdots v_{m-1}y$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $k = 2$, then by $m \ge 4$ and Claim 11, we have $y \in V_{m-1}$, and hence, $\{v_2, v_3, \ldots, v_n\} \Rightarrow y \to v_1$. Now, D contains D_n as its subdigraph.

If $2 < k < m-1$, then $y \in V_{m-1}$, $y \to v_2$ and $v_m \to y$. Thus, $yv_2\cdots v_my$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $m-1 \le k \le n-1$, then $1 \le k-m+2 \le k-2$ and $y \Rightarrow v_{k-m+2}$. Now $v_kyv_{k-m+2}\cdots v_k$ (if $y \to v_{k-m+2}$) or $v_{k+1}yv_{k-m+3}\ldots v_{k+1}$ (if $y \in V_{k-m+2}$) is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $k = n$, then $v_n \to y \Rightarrow \{v_1, v_2 \cdots v_{n-1}\}$ by the choice of k. It is easy to see that $y \in V_{n-m+2}$, as otherwise $yv_{n-m+2} \cdots v_n y$ is an m-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. Now D contains D_n^{-1} as its subdigraph. ■

Theorem 13. *Let D be a strong n-partite tournament, $n \geq 5$, which is not itself a tournament. If D contains an $(n-1)$-cycle with no pair of vertices from the same partite set, then D does not contain exactly $n - m + 1$ cycles of length m for two values of $m \in \{4, 5, \ldots, n-1\}$.*

Proof. Let m and m_1 be two distinct values from the set $\{4, 5, \ldots, n-1\}$ and assume that D has exactly $n-m+1$ cycles of length m. Let $V_1, V_2, \ldots, V_n$ be the partite sets of D. By Theorem 9, D contains some D_i or D_i^{-1} as its subdigraph for $i \in \{n-1, n\}$.

If D contains D_{n-1} (D_{n-1}^{-1}) as its subdigraph, then let $C = v_1 v_2 \cdots v_{n-1} v_1$ be an $(n-1)$-cycle of D_{n-1} $(D_{n-1}^{-1}))$ with $v_i \in V_i$ $(i = 1, 2, \ldots, n-1)$, $v_i \to v_j$ for all $1 < j+1 < i \leq n-1$, $y \in V_{m-1}$, $\{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow y \to v_1$ ($y \in V_{n-m+1}$ and $v_{n-1} \to y \Rightarrow \{v_1, v_2, \ldots, v_{n-2}\}$). By Theorem 1, $D\langle V(C)\rangle$ contains at least $(n-1) - m_1 + 1 = n - m_1$ cycles of length m_1. Note that $yv_1v_2 \cdots v_{m_1-1}y$ $(v_{n-1}yv_{n-(m_1-1)}v_{n-(m_1-2)} \cdots v_{n-1})$ is another m_1-cycle of $D_{n-1}(D_{n-1}^{-1}))$. In addition, there exists a vertex in V_n, say x, which is in an m_1-cycle of D different from the above m_1-cycles. Thus, D contains at least $n - m_1 + 2$ cycles of length m_1.

If D contains D_n (D_n^{-1}) as its subdigraph, then let $C = v_1 v_2 \cdots v_n v_1$ be an n-cycle of D_n (D_n^{-1}) with $v_i \in V_i$ $(i = 1, 2, \ldots, n)$, $v_i \to v_j$ for all $1 < j+1 < i \leq n$, $y \in V_{m-1}$, $\{v_2, v_3, \ldots, v_n\} \Rightarrow y \to v_1$ ($y \in V_{n-m+2}$, $v_n \to y \Rightarrow \{v_1, v_2, \ldots, v_{n-1}\}$). By Theorem 1, $D\langle V(C)\rangle$ contains at least $n - m_1 + 1$ cycles of length m_1. It is easy to see that $yv_1v_2 \cdots v_{m_1-1}y$ $(v_n y v_{n-(m_1-2)} v_{n-(m_1-3)} \cdots v_n)$ is another m_1-cycle of D_n (D_n^{-1}). Then D contains at least $n - m_1 + 2$ cycles of length m_1. The theorem is complete. ■

In 2004, Winzen [11] showed that an n-partite tournament D with $n \geq 4$ and $i_g(D) \leq 2$ contains a strong subtournament of order p for every $p \in \{3, 4, \ldots, n-1\}$. So D contains an $(n-1)$-cycle with no pair of vertices from the same partite set, which yields the following result.

Corollary 14. *If D is a strong n-partite tournament with $n \geq 5$ and $i_g(D) \leq 2$, which is not itself a tournament, then D does not contain exactly $n-m+1$ cycles of length m for two values of $m \in \{4, 5, \ldots, n-1\}$.*

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