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# ON THE $n$-PARTITE TOURNAMENTS WITH EXACTLY $n - m + 1$ CYCLES OF LENGTH $m$

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**Abstract**

Gutin and Rafiey [*Multipartite tournaments with small number of cycles*, Australas J. Combin. 34 (2006) 17–21] raised the following two problems: (1) Let $m \in \{3, 4, \ldots, n\}$. Find a characterization of strong $n$-partite tournaments having exactly $n-m+1$ cycles of length $m$; (2) Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong $n$-partite tournaments, which are not themselves tournaments, with exactly $n - m + 1$ cycles of length $m$ for two values of $m$? In this paper, we discuss the strong $n$-partite tournaments $D$ containing exactly $n - m + 1$ cycles of length $m$ for $4 \leq m \leq n - 1$. We describe the substructure of such $D$ satisfying a given condition and we also show that, under this condition, the second problem has a negative answer.

**Keywords:** multipartite tournaments, tournaments, cycles.

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## 1. Introduction

An *$n$-partite* or *multipartite tournament* is an orientation of a complete $n$-partite graph. A *tournament* is an $n$-partite tournament with exactly $n$ vertices. A digraph $D$ is *transitive* if, for every pair of arcs $xy$ and $yz$ in $D$ such that $x \neq z$, the arc $xz$ is also in $D$. It is easy to show that a tournament is transitive if and only if it is acyclic.

A digraph $D$ is said to be *strong*, if for every pair of vertices $x$ and $y$, $D$ contains a path from $x$ to $y$ and a path from $y$ to $x$. A directed path from $x$ to $y$ in $D$ is denoted by an $(x, y)$-path. An $l$-cycle is a cycle of length $l$. A cycle or path in a digraph $D$ is Hamiltonian if it includes all the vertices of $D$.

In 1966, Moon discussed the number of $m$-cycle in a strong tournament.

**Theorem 1** (Moon [9]). *Let $T$ be a strong tournament of order $n$. Then $T$ contains at least $n-m+1$ cycles of length $m$ for $3 \leq m \leq n$.*

The tournaments which gain the lower bound in Theorem 1 were characterized by Burzio and Demaria [2] for $m=3$, Douglas [3] for $m=n$ and Las Vergnas [8] for $4 \leq m \leq n-1$. We list the result of Las Vergnas especially because we will use it to prove our main results.

**Theorem 2** (Las Vergnas [8]). *Every strong tournament of order $n$ having exactly $n-m+1$ cycles of given length $m$ with $4 \leq m \leq n-1$ is isomorphic to $Q_n$, where $Q_n$ is a tournament of order $n \geq 3$ obtained by reversing the arcs in the unique Hamiltonian path of a transitive tournament.*

In 2002, Volkmann extended Theorem 1 from tournaments to multipartite tournaments.

**Theorem 3** (Volkmann [10]). *Let $D$ be a strong $n$-partite tournament. Then $D$ contains at least $n-m+1$ cycles of length $m$ for $3 \leq m \leq n$.*

It is notable that the bound in Theorem 3 is sharp which can be seen in [4] for $n=3$ and in [6] for $4 \leq m \leq n$. In addition, Gutin and Rafiey [6] raised the following two interesting and natural problems.

**Problem 4** (Gutin and Rafiey [6]). *Given $m \in \{3,4,\ldots,n\}$, find a characterization of strong $n$-partite tournaments having exactly $n-m+1$ cycles of length $m$.*

**Problem 5** (Gutin and Rafiey [6]). *Let $3 \leq m \leq n$ and $n \geq 4$. Are there strong $n$-partite tournaments, which are not themselves tournaments, with exactly $n-m+1$ cycles of length $m$ for two values of $m$?*

Problem 4 seems to be especially interesting for the case $m=n$ which was already solved by Gutin *et al.* in [7]. In this paper, we investigate strong $n$-partite tournaments $D$, which are not themselves tournaments and contain exactly $n-m+1$ cycles of length $m$ for any given $4 \leq m \leq n-1$. We prove that if $D$ has an $(n-1)$-cycle with no pair of vertices from the same partite set, then $D$ must contain some given multipartite tournament as its subdigraph.

As far as Problem 5, Gutin and Rafiey [6] gave a negative answer for two values $n-1$ and $n$ of $m$. This also implies that Problem 5 has a negative answer for $n=4$. In this paper, we give a necessary condition to Problem 5 and show that if a strong $n$-partite tournament $D$, which is not itself a tournament, contains exactly $n-m+1$ cycles of length $m$ for two values of $m \in \{4,5,\ldots,n-1\}$, then there is no an $(n-1)$-cycle with no pair of vertices from the same partite set in $D$.

## 2. Terminology and Preliminaries

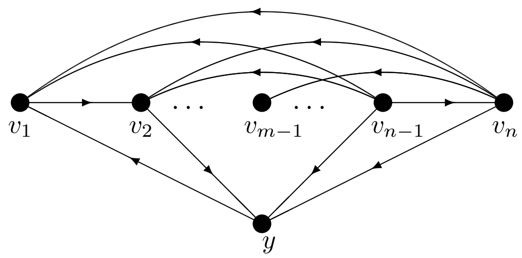
We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1].

Let $D$ be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. We call the number of vertices of $D$ the *order* of $D$. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D\langle A\rangle$. We use $V(D)\backslash V(A)$ to stand for the set of vertices which are in $V(D)$ but not in $V(A)$.

If $xy$ is an arc in $D$, then we say that $x$ *dominates* $y$ and write $x \to y$. For two disjoint subsets $X$ and $Y$ of $V(D)$, if every vertex of $X$ dominates every vertex of $Y$, we say $X$ *dominates* $Y$ and write $X \to Y$. Furthermore, $X \Rightarrow Y$ denotes the property that there is no arc from $Y$ to $X$.

The *out-neighborhood* $N^+(x)$ of a vertex $x$ is the set of vertices dominated by $x$ and the *in-neighborhood* $N^-(x)$ of a vertex $x$ is the set of vertices dominating $x$. The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of $x$, respectively. The *global irregularity* of $D$ is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. We denote by $D^{-1}$ the inverse digraph of $D$.

In order to present our main results, we define a class of $n$-partite tournaments $D_n$ of order $n + 1$ as described in the following figure, where $3 \le m \le n$, $\{v_2, \ldots, v_{m-2}, v_m, \ldots, v_n\} \to y \to v_1$, $y$ and $v_{m-1}$ belong to the same partite set and $v_i \to v_j$ for all $1 < j + 1 < i \le n$.




$D_n$

The following two theorems on cycles in strong $n$-partite tournaments are very useful to prove our main results.

**Theorem 6** (Guo and Volkmann [5]). *Every partite set of a strong $n$-partite tournament, $n \ge 3$, contains a vertex which lies on an $m$-cycle for each $m \in \{3, 4, \ldots, n\}$.*

**Theorem 7** (Gutin and Rafiey [6]). *Let $D$ be a strong $n$-partite tournament containing exactly $n - m + 1$ cycles of length $m$ for some $m \in \{3, 4, \ldots, n\}$. Then every $m$-cycle of $D$ has no pair of vertices from the same partite set.*

## 3. Main Results

Before presenting the main results, we first prove the following lemma.

**Lemma 8.** *Let $D$ be a strong $n$-partite tournament, $n \geq 5$, containing exactly $n-m+1$ cycles of length $m$ for some $3 \leq m \leq n-1$. If $D$ has an $(n-1)$-cycle $C$ with no pair of vertices from the same partite set, then the following statements hold.*

(a) *There are no two vertices $u, w$ in $V(D)\backslash V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$.*

(b) *There exists a vertex $v \notin V(C)$ such that $D\langle V(C) \cup \{v\}\rangle$ is strong.*

***Proof.*** Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$. Suppose, without loss of generality, that $C = v_1 v_2 \cdots v_{n-1} v_1$ with $v_i \in V_i$, $i = 1, 2, \ldots, n-1$. By Theorem 1, $D\langle V(C)\rangle$ contains at least $(n-1)-m+1 = n-m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$.

(a) Suppose to the contrary that there exist two vertices $u, w \in V(D)\backslash V(C)$ such that $C \Rightarrow u \rightarrow w \Rightarrow C$. Obviously, $u \notin V_n$ or $w \notin V_n$. Assume, without loss of generality, that $w \in V_j$ for some $1 \leq j \leq n-1$. Since $n \geq 5$, there exist at least two vertices $v_k$ and $v_t$, such that $u, w, v_k$ and $v_t$ are in different partite sets. If $m = 3$, then $uwv_ku$ and $uwv_tu$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$. This contradicts the fact that $D$ contains exactly $n-m+1$ cycles of length $m$. If $m \geq 4$, then $uwv_{j-1}v_j \cdots v_{j+m-4}u$ (if $u \notin V_{j+m-4}$) or $uwv_{j+1}v_{j+2} \cdots v_{j+m-2}u$ (if $u \in V_{j+m-4}$) is an $m$-cycle with $w, v_j \in V_j$ or $u, v_{j+m-4} \in V_{j+m-4}$ (where all indices are modulo $n-1$). This is impossible by Theorem 7.

(b) Assume that there is no vertex $v \in V(D)\backslash V(C)$ such that $D\langle V(C) \cup \{v\}\rangle$ is strong. Let $S = \{x \in V(D)\backslash V(C) : C \Rightarrow x\}$ and $T = \{z \in V(D)\backslash V(C) : z \Rightarrow C\}$. Since $D$ is strong, we have that $S$ and $T$ are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \rightarrow w$. Thus, we have $C \Rightarrow u \rightarrow w \Rightarrow C$, which contradicts (a). ■

**Theorem 9.** *Let $D$ be a strong $n$-partite tournament which is not itself a tournament and contains exactly $n-m+1$ cycles of length $m$ for some $4 \leq m \leq n-1$. If $D$ has an $(n-1)$-cycle $C$ with no pair of vertices from the same partite set, then $D$ contains some $D_i$ or $D_i^{-1}$ as its subdigraph for $i \in \{n-1, n\}$, where $D_i$ is defined in Section 2.*

***Proof.*** Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$ and let $C = v_1 v_2 \cdots v_{n-1} v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n-1$. By Theorem 1, $D\langle V(C)\rangle$ contains at least $n-m$ cycles $C_1, C_2, \ldots, C_{n-m}$ of length $m$. By Theorem 6, there exists a vertex in $V_n$, say $x$, which lies on an $m$-cycle $C_{n-m+1}$ different from $C_1, C_2, \ldots, C_{n-m}$. We consider the following two cases.

*Case* 1. $D\langle V(C) \cup \{x\}\rangle$ is not strong. Since $D$ contains exactly $n-m+1$ cycles of length $m$, we have that $D\langle V(C)\rangle$ contains exactly $n-m$ cycles of length

$m$. By Theorem 2, $D\langle V(C)\rangle$ is isomorphic to $Q_{n-1}$. So we may assume that $v_i \to v_j$ for all $1 < j+1 < i \leq n-1$. Since $D\langle V(C) \cup \{x\}\rangle$ is not strong, we have that $C \to x$ or $x \to C$.

First we consider the case $C \to x$. Let $S = \{u \in V(D)\backslash(V(C) \cup \{x\}) : D\langle V(C) \cup \{u\}\rangle$ is strong$\}$. By Lemma 8(b), $S$ is not empty. Since $D$ is strong, there is a path from $x$ to $S$. Let $P = x_1x_2\cdots x_t$ $(x_1 = x)$ be such a path and assume that the $P$ is of minimum length. That is, $x_t \in S$ and $D\langle V(C) \cup \{x_i\}\rangle$ is not strong for each $i \in \{1, 2, \ldots, t-1\}$. Since $C \to x_1$ and $x_1 \to x_2$, we have $x_2 \notin V(C)$. If $t > 2$, then by Lemma 8(a), we have $C \Rightarrow x_2$. Successively, we can get that $x_i \notin V(C)$ and $C \Rightarrow x_i$ for all $i \in \{2, 3, \ldots, t-1\}$ when $t > 2$.

If there exist two vertices $v_i, v_j$ on $C$ such that $x_t \to \{v_i, v_j\}$, then, when $t > m-1$, we have that $x_tv_ix_{t-(m-2)}x_{t-(m-3)}\cdots x_t$ (if $x_{t-(m-2)} \notin V_i$) or $x_tv_jx_{t-(m-2)}x_{t-(m-3)}\cdots x_t$ (if $x_{t-(m-2)} \in V_i$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction; when $t = m-1$, it is clear that $x_tv_ix_1\cdots x_t$ and $x_tv_jx_1\cdots x_t$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction; when $t \leq m-2$, it is easy to see that $x_tv_iv_{i+1}\cdots v_{i+(m-1-t)}x_1\cdots x_t$ and $x_tv_jv_{j+1}\cdots v_{j+(m-1-t)}x_1\cdots x_t$ are two $m$-cycles different from $C_1, C_2, \ldots, C_{n-m}$, a contradiction.

Therefore, $x_t$ has only one out-neighbor on $C$. We will show that $x_t \to v_1$. In fact, if $x_t \to v_i$ and $i \geq 2$, then we have that $x_tv_i\cdots v_{i+m-2}x_t$ (when $i+m-2 \leq n-1$ and $x_t \notin V_{i+m-2}$) or $x_tv_i\cdots v_{i+m-3}v_{i-1}x_t$ (when $i+m-2 \leq n-1$ and $x_t \in V_{i+m-2}$) or $x_tv_i\cdots v_{n-1}v_1\cdots v_{m-n+i-1}x_t$ (when $i+m-2 \geq n$ and $x_t \notin V_{m-n+i-1}$) or $x_tv_i\cdots v_{n-1}v_2\cdots v_{m-n+i}x_t$ (when $i+m-2 \geq n$ and $x_t \in V_{m-n+i-1}$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So we have $\{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow x_t$. Furthermore, if $v_{m-1} \to x_t$, then $x_tv_1v_2\cdots v_{m-1}x_t$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. So $x_t \in V_{m-1}$. Let $x_t = y$. Then $D$ contains $D_{n-1}$ as its subdigraph.

For the case $x \to C$, by considering the inverse of $D$, it is easy to see that $D$ contains $D_{n-1}^{-1}$ as its subdigraph.

*Case* 2. $D\langle V(C) \cup \{x\}\rangle$ is strong. In this case, $D\langle V(C) \cup \{x\}\rangle$ is a strong tournament of order $n$. By Theorem 1, $D\langle V(C) \cup \{x\}\rangle$ contains at least $n-m+1$ cycles of length $m$. Note that $D$ contains exactly $n-m+1$ cycles of length $m$. We have that $D\langle V(C) \cup \{x\}\rangle$ contains exactly $n-m+1$ cycles of length $m$. By Theorem 2, $D\langle V(C) \cup \{x\}\rangle$ is isomorphic to $Q_n$. So we may assume that $C' = v_1v_2\cdots v_nv_1$ is an $n$-cycle of $D\langle V(C) \cup \{x\}\rangle$ satisfying $v_i \in V_i$ and $v_i \to v_j$ for all $1 < j+1 < i \leq n$. Obviously, $C_1 = v_1v_2\cdots v_mv_1, C_2 = v_2v_3\cdots v_{m+1}v_2, \ldots, C_{n-m+1} = v_{n-m+1}v_{n-m+2}\cdots v_nv_{n-m+1}$ are $n-m+1$ cycles of length $m$ of $D$.

**Claim 10.** *There exists a vertex $y \in V(D)\backslash V(C')$ such that $D\langle V(C') \cup \{y\}\rangle$ is strong.*

***Proof.*** Assume that there is no vertex $y \in V(D)\backslash V(C')$ such that $D\langle V(C')\cup\{y\}\rangle$ is strong. Let $S = \{x \in V(D)\backslash V(C') : C' \Rightarrow x\}$ and $T = \{z \in V(D)\backslash V(C') : z \Rightarrow V(C')\}$. Since $D$ is strong, we have that $S$ and $T$ are non-empty and there are vertices $u \in S$ and $w \in T$ such that $u \to w$. Suppose that $u \in V_i$ and $w \in V_j$ for $1 \leq i \neq j \leq n$. Then $uwv_{j+1}v_{j+2}\cdots v_{j+m-2}u$ (if $i \neq j+m-2$) or $uwv_{j+2}v_{j+3}\cdots v_{j+m-1}u$ (if $i = j+m-2$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$. Note that $D$ contains exactly $n-m+1$ cycles of length $m$. This is a contradiction. □

By Claim 10, there are two vertices $v_a, v_b$ $(1 \leq a, b \leq n)$, such that $v_a \to y \to v_b$. Assume that $v_k$ is the first vertex from $v_1$ to $v_n$ dominating $y$.

**Claim 11.** *$v_i \Rightarrow y$ for all $k \leq i \leq n$.*

***Proof.*** Otherwise, there exists some index $t$ such that either $v_t \to y \to v_{t+1}$ $(k \leq t \leq n-1)$ or $y, v_{t+1} \in V_{t+1}$ but $v_t \to y \to v_{t+2}$ $(k \leq t \leq n-2)$. We still assume that $t$ is such a minimum index.

If $t \leq n-m+1$, then either $v_t y v_{t+1}\cdots v_{t+m-2}v_t$ or $v_t y v_{t+2}\cdots v_{t+m-1}v_t$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $n-m+2 \leq t \leq n-2$, then either $v_t y v_{t+1}\cdots v_n v_{n-m+2}\cdots v_t$ or $v_t y v_{t+2}\cdots v_n v_{n-m+1}\cdots v_t$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $t = n-1$, then $y \to v_n$ and $v_{n-1} y v_n v_{n-m+2}\cdots v_{n-1}$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. □

**Claim 12.** *$y \to v_1$.*

***Proof.*** If $y \in V_1$, then $y \to v_2$ (otherwise, $k = 2$ and $\{v_2, v_3, \ldots, v_n\} \to y$ by Claim 11, which contradicts the assumption that $D\langle V(C')\cup\{y\}\rangle$ is strong). By Claim 11, we have $v_n \to y$. Therefore, $D\langle v_2, \ldots, v_n, y\rangle$ is a strong tournament. Then $y$ is in an $m$-cycle of $D\langle v_2, \ldots, v_n, y\rangle$, which is different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

Therefore, $y \notin V_1$. If $v_1 \to y$, then $\{v_2, v_3, \ldots, v_n\} \Rightarrow y$ by Claim 11, which contradicts the assumption that $D\langle V(C')\cup\{y\}\rangle$ is strong. So we have $y \to v_1$. □

By Claim 12, we have that $2 \leq k \leq n$ and $y \Rightarrow v_{m-1}$. Otherwise, $yv_1v_2\cdots v_{m-1}y$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $k = 2$, then by $m \geq 4$ and Claim 11, we have $y \in V_{m-1}$, and hence, $\{v_2, v_3, \ldots, v_n\} \Rightarrow y \to v_1$. Now, $D$ contains $D_n$ as its subdigraph.

If $2 < k < m-1$, then $y \in V_{m-1}$, $y \to v_2$ and $v_m \to y$. Thus, $yv_2\cdots v_m y$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $m-1 \leq k \leq n-1$, then $1 \leq k-m+2 \leq k-2$ and $y \Rightarrow v_{k-m+2}$. Now $v_k y v_{k-m+2}\cdots v_k$ (if $y \to v_{k-m+2}$) or $v_{k+1} y v_{k-m+3}\ldots v_{k+1}$ (if $y \in V_{k-m+2}$) is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction.

If $k = n$, then $v_n \to y \Rightarrow \{v_1, v_2 \cdots v_{n-1}\}$ by the choice of $k$. It is easy to see that $y \in V_{n-m+2}$, as otherwise $yv_{n-m+2} \cdots v_n y$ is an $m$-cycle different from $C_1, C_2, \ldots, C_{n-m+1}$, a contradiction. Now $D$ contains $D_n^{-1}$ as its subdigraph. ■

**Theorem 13.** *Let $D$ be a strong $n$-partite tournament, $n \geq 5$, which is not itself a tournament. If $D$ contains an $(n-1)$-cycle with no pair of vertices from the same partite set, then $D$ does not contain exactly $n-m+1$ cycles of length $m$ for two values of $m \in \{4, 5, \ldots, n-1\}$.*

***Proof.*** Let $m$ and $m_1$ be two distinct values from the set $\{4, 5, \ldots, n-1\}$ and assume that $D$ has exactly $n-m+1$ cycles of length $m$. Let $V_1, V_2, \ldots, V_n$ be the partite sets of $D$. By Theorem 9, $D$ contains some $D_i$ or $D_i^{-1}$ as its subdigraph for $i \in \{n-1, n\}$.

If $D$ contains $D_{n-1}$ ($D_{n-1}^{-1}$) as its subdigraph, then let $C = v_1 v_2 \cdots v_{n-1} v_1$ be an $(n-1)$-cycle of $D_{n-1}$ ($D_{n-1}^{-1}$)) with $v_i \in V_i$ $(i = 1, 2, \ldots, n-1)$, $v_i \to v_j$ for all $1 < j+1 < i \leq n-1$, $y \in V_{m-1}$, $\{v_2, v_3, \ldots, v_{n-1}\} \Rightarrow y \to v_1$ ($y \in V_{n-m+1}$ and $v_{n-1} \to y \Rightarrow \{v_1, v_2, \ldots, v_{n-2}\}$). By Theorem 1, $D\langle V(C)\rangle$ contains at least $(n-1) - m_1 + 1 = n - m_1$ cycles of length $m_1$. Note that $yv_1 v_2 \cdots v_{m_1-1} y$ ($v_{n-1} y v_{n-(m_1-1)} v_{n-(m_1-2)} \cdots v_{n-1}$) is another $m_1$-cycle of $D_{n-1}(D_{n-1}^{-1})$). In addition, there exists a vertex in $V_n$, say $x$, which is in an $m_1$-cycle of $D$ different from the above $m_1$-cycles. Thus, $D$ contains at least $n - m_1 + 2$ cycles of length $m_1$.

If $D$ contains $D_n$ ($D_n^{-1}$) as its subdigraph, then let $C = v_1 v_2 \cdots v_n v_1$ be an $n$-cycle of $D_n$ ($D_n^{-1}$) with $v_i \in V_i$ $(i = 1, 2, \ldots, n)$, $v_i \to v_j$ for all $1 < j+1 < i \leq n$, $y \in V_{m-1}$, $\{v_2, v_3, \ldots, v_n\} \Rightarrow y \to v_1$ ($y \in V_{n-m+2}$, $v_n \to y \Rightarrow \{v_1, v_2, \ldots, v_{n-1}\}$). By Theorem 1, $D\langle V(C)\rangle$ contains at least $n - m_1 + 1$ cycles of length $m_1$. It is easy to see that $yv_1 v_2 \cdots v_{m_1-1} y$ ($v_n y v_{n-(m_1-2)} v_{n-(m_1-3)} \cdots v_n$) is another $m_1$-cycle of $D_n$ ($D_n^{-1}$). Then $D$ contains at least $n - m_1 + 2$ cycles of length $m_1$. The theorem is complete. ■

In 2004, Winzen [11] showed that an $n$-partite tournament $D$ with $n \geq 4$ and $i_g(D) \leq 2$ contains a strong subtournament of order $p$ for every $p \in \{3, 4, \ldots, n-1\}$. So $D$ contains an $(n-1)$-cycle with no pair of vertices from the same partite set, which yields the following result.

**Corollary 14.** *If $D$ is a strong $n$-partite tournament with $n \geq 5$ and $i_g(D) \leq 2$, which is not itself a tournament, then $D$ does not contain exactly $n-m+1$ cycles of length $m$ for two values of $m \in \{4, 5, \ldots, n-1\}$.*

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