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DOMINATION NUMBER, INDEPENDENT DOMINATION NUMBER AND 2-INDEPENDENCE NUMBER IN TREES

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Abstract

For a graph G, let $\gamma(G)$ be the domination number, $i(G)$ be the independent domination number and $\beta_2(G)$ be the 2-independence number. In this paper, we prove that for any tree T of order $n \geq 2$, $4\beta_2(T) - 3\gamma(T) \geq 3i(T)$, and we characterize all trees attaining equality. Also we prove that for every tree T of order $n \geq 2$, $i(T) \leq \frac{3\beta_2(T)}{4}$, and we characterize all extreme trees.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v) \setminus S$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. We denote the set of all leaves of a tree T by $L(T)$. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We denote the set of leaves adjacent to a vertex v by L_v.

A set S of vertices in a graph G is a *dominating set* if every vertex of $V \setminus S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set of minimum cardinality of G is called a $\gamma(G)$*-set*. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [10, 11].

A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$, where $G[S]$ is the subgraph induced by S. The *independent domination number* (respectively, the independence number) of G denoted by $i(G)$ (respectively, $\beta(G)$) is the size of the smallest (respectively, the largest) maximal independent set in G. It is well known that an independent set is maximal if and only if it is also dominating. Hence, we can say that the domination, which is defined even for non-independent sets, is the property which makes an independent set maximal. Furthermore, every set which is both independent and dominating is a minimal dominating set of G. This leads to the well known inequality chain

$$\gamma(G) \leq i(G) \leq \beta(G).$$

Fink and Jacobson [7, 8] generalized the concepts of independent and dominating sets. Let k be a positive integer. A set S of vertices in a graph G is *k-independent* if the maximum degree of the subgraph induced by S is at most $k-1$. The maximum cardinality of a k-independent set of G is the *k-independence number* of G and is denoted $\beta_k(G)$. A k-independent set of G with maximum cardinality is called a $\beta_k(G)$-set. The subset S is *k-dominating* if every vertex of $V \setminus S$ has at least k neighbors in S. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G.

Relationships between two parameters $\gamma_k(G)$ and $\beta_k(G)$ have been studied by several authors. Favaron [5] proved that for any graph G and positive integer k, $\gamma_k(G) \leq \beta_k(G)$. Also, Favaron [6] proved that for every graph G and positive integer $k \leq \Delta$, $\beta_k(G) + \gamma_{\Delta-k+1}(G) \geq n$. Jacobson, Peters and Rall [12] showed that for every graph G and positive integer $k \leq \delta$, $\beta_k(G) + \gamma_{\delta-k+1}(G) \leq n$. Hansberg, Meierling and Volkmann [9] showed that if G is a connected r-partite graph and k is an integer such that $\Delta \geq k$, then $\gamma_k(G) \leq \frac{\beta(G)}{r}(r(r-1)+k-1)$. For more information on k-independence number and k-domination see [2].

The relation between 2-independent set and some domination parameters have been studied by several authors (see for example [1, 3, 4, 13]).

Motivated by the aforementioned works, we consider the difference of $\beta_2(T) - \gamma(T)$ for trees and prove that for any tree T of order $n \geq 2$, $\frac{4\beta_2(T)}{3} - \gamma(T) \geq i(T)$ and characterize all extreme trees. Also we prove that for every T of order $n \geq 2$, $i(T) \leq \frac{3\beta_2(T)}{4}$, and we classify all trees attaining this inequality.

2. A Lower Bound on the Difference $\frac{4\beta_2(T)}{3} - \gamma(T)$

In this section we show that for every tree T of order $n \geq 2$, $\frac{4\beta_2(T)}{3} - \gamma(T) \geq i(T)$ and we characterize all extreme trees. We proceed with some definitions and lemmas.

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G once. The subdivision star $S(K_{1,t})$ for $t \geq 1$, is called a *healthy spider* S_t. A *wounded spider* $S_{t,q}$ $(0 \leq q \leq t-1)$ is the tree obtained from $K_{1,t}$ $(t \geq 1)$ by subdividing q edges of $K_{1,t}$. Note that stars are wounded spiders. A *spider* is a healthy or a wounded spider.

Lemma 1. *Let T' be a tree and $v \in V(T')$. If T is the tree obtained from T' by adding a path $P_4 = u_1u_2u_3u_4$ and joining v to u_2, then $\gamma(T) + i(T) \leq \gamma(T') + i(T') + 4$ and $\beta_2(T) = \beta_2(T') + 3$.*

Proof. Clearly, any (independent) dominating set of T' can be extended to a (independent) dominating set of T by adding u_1, u_3 and this implies that $\gamma(T) +$

$i(T) \le \gamma(T') + i(T') + 4$.

Also, obviously any $\beta_2(T')$-set can be extended to an 2-independent set of T by adding u_1, u_3, u_4 yielding $\beta_2(T) \ge \beta_2(T') + 3$. On the other hand, if S is a $\beta_2(T)$-set then clearly $|S \cap \{u_1, u_2, u_3, u_4\}| \le 3$ and so $S \cap V(T')$ is a 2-independent set of T' of size at least $\beta_2(T) - 3$ implying that $\beta_2(T) \le \beta_2(T') + 3$. Thus $\beta_2(T) = \beta_2(T') + 3$. ■

Lemma 2. *Let T' be a tree and $v \in V(T')$. If T is the tree obtained from T' by adding a path $P_3 = u_1u_2u_3$ and joining v to u_1, then $\gamma(T) \le \gamma(T') + 1$, $i(T) \le i(T') + 1$ and $\beta_2(T) = \beta_2(T') + 2$.*

Proof. Clearly, any (independent) dominating set of T' can be extended to a (independent) dominating set of T by adding u_2 and this implies that $\gamma(T) \le \gamma(T') + 1$ and $i(T) \le i(T') + 1$.

Also, obviously any $\beta_2(T')$-set can be extended to an 2-independent set of T by adding u_2, u_3 yielding $\beta_2(T) \ge \beta_2(T') + 2$. On the other hand, if S is a $\beta_2(T)$-set then clearly $|S \cap \{u_1, u_2, u_3\}| \le 2$ and hence $S \cap V(T')$ is a 2-independent set of T' of size at least $\beta_2(T) - 2$ implying that $\beta_2(T) \le \beta_2(T') + 2$. Thus $\beta_2(T) = \beta_2(T') + 2$. ■

Lemma 3. *If T is a spider of order $n \ge 2$, then $\gamma(T) + i(T) \le \frac{4\beta_2(T)}{3}$ with equality if and only if $T = P_4$.*

Proof. If $T = S_t$ is a healthy spider for some $t \ge 1$, then obviously $\gamma(T) + i(T) = 2t$ because $\gamma(T) = t$ and $i(T) = t$. Also $\beta_2(T) = 2t$. Hence $\gamma(T) + i(T) = \beta_2(T) < \frac{4\beta_2(T)}{3}$. Now let $T = S_{t,q}$ be a wounded spider. If $q = 0$, then T is a star and we have $\gamma(T) + i(T) = 2 \le t = \beta_2(T) < \frac{4\beta_2(T)}{3}$. Suppose $q > 0$. If $t = 2$, then $T = P_4$ and clearly $\gamma(T) + i(T) = \frac{4\beta_2(T)}{3}$. If $t \ge 3$, then clearly $\gamma(T) + i(T) = 2q + 2$ and $\beta_2(T) = t + q$ and so $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$. ■

Next we introduce a family $\mathcal{T}$ of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = P_4$, and if $k \ge 2$, then T_{i+1} can be obtained recursively from T_i by the operation $\mathcal{T}_1$ for $1 \le i \le k - 1$.

Operation $\mathcal{T}_1$. If $v \in T_i$ is a support vertex, then $\mathcal{T}_1$ adds a path $P_4 = u_1u_2u_3u_4$ and joins v to u_2.

Observation 4. *Let $T \in \mathcal{T}$. Then the following conditions are satisfied.*

1. *Every support vertex is adjacent to exactly one leaf.*
2. *Every vertex of T is a leaf or support vertex.*
3. *Both of $L(T)$ and $V(T) - L(T)$ are $\gamma(T)$-set.*
4. *$L(T)$ is a $i(T)$-set.*

5. $L(T) \subset \beta_2(T)$-*set.*
6. $\beta_2(T) = |L(T)| + |V(T) - L(T)|/2 = 3\gamma(T)/2$.

Theorem 5. *If T is a tree of order $n \geq 2$, then*

$$\gamma(T) + i(T) \leq \frac{4\beta_2(T)}{3} \tag{1}$$

with equality if and only if $T \in \mathcal{T}$.

Proof. The proof is by induction on n. The results are trivial for trees of order $n = 2, 3, 4$. Let $n \geq 5$ and suppose that for every non-trivial tree T of order less than n the results are true. Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star and clearly $\gamma(T) + i(T) = 2 < \frac{4\beta_2(T)}{3}$ by Lemma 3. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{r,s}$. Since $r + s \geq 3$, if we suppose $r \geq s$, then we have $r \geq 2$. If $r \geq s \geq 2$, then $\gamma(T) + i(T) = s + 3 < \frac{4(r+s)}{3} = \frac{4\beta_2(T)}{3}$. If $s = 1$, then $\gamma(T) + i(T) = 4 < \frac{4(r+2)}{3} = \frac{4\beta_2(T)}{3}$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$.

Let $v_1 v_2 \cdots v_D$ be a diametrical path in T such that $\deg(v_2)$ is as large as possible. Root T at v_D. Consider the following cases.

Case 1. $\deg_T(v_2) \geq 4$. Suppose $T' = T - \{v_1\}$. Clearly, any $\gamma(T)$-set and any $\gamma(T')$-set contains v_2 and this implies that $\gamma(T) = \gamma(T')$. Let S be a $i(T')$-set. If $v_2 \in S$, then S is an independent dominating set of T and if $v_2 \notin S$, then $S \cup \{v_1\}$ is an independent dominating set of T yielding $i(T) \leq i(T') + 1$. On the other hand, if S is a $\beta_2(T')$-set such that $|S \cap L(T')|$ is as large as possible, then clearly $v_2 \notin S$ and $S \cup \{v_1\}$ is a 2-independent set of T implying that $\beta_2(T) \geq |S| + 1 = \beta_2(T') + 1$. By the induction hypothesis, we have

$$\gamma(T) + i(T) \leq \gamma(T') + i(T') + 1 \leq \frac{4\beta_2(T')}{3} + 1 \leq \frac{4\beta_2(T) - 1}{3} < \frac{4\beta_2(T)}{3}.$$

Case 2. $\deg_T(v_2) = 3$. Assume that $L_{v_2} = \{v_1, z\}$. First let $\deg(v_3) = 2$. Suppose $T' = T - T_{v_3}$. As Case 1, we have $\gamma(T) = \gamma(T') + 1$ and $i(T) \leq i(T') + 1$. On the other hand, if S is a $\beta_2(T')$-set, then $S \cup \{v_1, v_2\}$ is a 2-independent set of T yielding $\beta_2(T) \geq |S| + 2 = \beta_2(T') + 2$. By the induction hypothesis, we have

$$\gamma(T) + i(T) \leq \gamma(T') + i(T') + 2 \leq \frac{4\beta_2(T')}{3} + 2 \leq \frac{4\beta_2(T) - 2}{3} < \frac{4\beta_2(T)}{3}.$$

Now let $\deg(v_3) \geq 3$. Let $L_{v_3} = \{x_1, \ldots, x_l\}$. If $L_{v_3} \neq \emptyset$, then let $C_2 = \{y_1, \ldots, y_k\}$ be the children of v_3 with depth 1 and degree 2, if any, and redlet $z_1, \ldots, z_t$ be the children of v_3 with depth 1 and degree 3 where $z_1 = v_2$. Let $T' = T - T_{v_3}$. Clearly, any $\gamma(T')$-set can be extended to a dominating set of T by adding v_3 and its children of depth 1 and this yields $\gamma(T) \leq \gamma(T') + |C_2| + t + 1$. Also, any $i(T')$-set can be extended to an independent dominating set of T by adding all children of v_3 implying that $i(T) \le i(T') + |C_2| + t + |L_{v_3}|$. On the other hand, any $\beta_2(T')$-set, can be extended to a 2-independent set of T by adding L_{v_3}, $y_1, \dots, y_k$ and their children, if any, and the children of $z_1, \dots, z_t$ yielding $\beta_2(T) \ge \beta_2(T') + |L_{v_3}| + 2t + 2|C_2|$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma(T) + i(T) &\le \gamma(T') + i(T') + 2|C_2| + 2t + |L_{v_3}| + 1\\ &\le \frac{4\beta_2(T')}{3} + 2|C_2| + 2t + |L_{v_3}| + 1\\ &\le \frac{4\beta_2(T) - 8|C_2| - 8t - 4|L_{v_3}|}{3} + 2|C_2| + 2t + |L_{v_3}| + 1\\ &\le \frac{4\beta_2(T) - 2|C_2| - 2t - |L_{v_3}| + 3}{3} \le \frac{4\beta_2(T)}{3}.\end{aligned}$$

We claim that the equality does not hold. Suppose, to the contrary, that $\gamma(T) + i(T) = \frac{4\beta_2(T)}{3}$. Then all inequalities occurring the above chain must be equalities and this holds if and only if $\gamma(T') + i(T') = \frac{4\beta_2(T')}{3}$, $|C_2| = 0$, $t = 1$ and $|L_{v_3}| = 1$. Thus $\deg_T(v_3) = 3$ and v_3 is adjacent with a leaf w. By the induction hypothesis, we have $T' \in \mathcal{T}$. It follows from Observation 4 that v_4 is either a leaf or is a weak support vertex. We distinguish the following subcases.

Subcase 2.1. $\deg_T(v_4) = 2$. If $\operatorname{diam}(T) = 4$, then clearly $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$ which is a contradiction. Let $\operatorname{diam}(T) \ge 5$. Let $T' = T - T_{v_4}$. Clearly, any $\gamma(T')$-set can be extended to a dominating set of T by adding v_3, v_2, any $i(T')$-set can be extended to a dominating set of T by adding v_3, v_1, z, and any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_3, w, v_1, z. By the induction hypothesis, we obtain $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$ a contradiction.

Subcase 2.2. v_4 is a support vertex. Let $T' = T - T_{v_2}$. Clearly, any $\gamma(T')$-set can be extended to a dominating set of T by adding v_2 and any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding v_1, z. Let S' be a $i(T')$-set. If $v_3 \notin S'$, then let $S = S' \cup \{v_2\}$ and if $v_3 \in S'$, then let $S = (S' \setminus \{v_3\}) \cup \{w, v_2\}$. Obviously, S is an independent dominating set of T yielding $i(T) \le i(T') + 1$. By the induction hypothesis, we obtain $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$, a contradiction. This proved our claim.

Case 3. $\deg_T(v_2) = 2$. If $\deg_T(v_3) = 2$, then let $T' = T - T_{v_3}$. By Lemma 2 and the induction hypothesis, we have $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$. Let $\deg_T(v_3) \ge 3$. By the choice of diametrical path we may assume that all children of v_3 with depth 1, have degree 2. First we suppose that there is a pendant path $v_3 z_2 z_1$. Let $T' = T - T_{v_2}$. Clearly, any $\gamma(T')$-set and any $i(T')$-set can be extended to a dominating set of T by adding v_1 yielding $\gamma(T) \le \gamma(T') + 1$ and $i(T) \le i(T') + 1$. Let S' be a $\beta_2(T')$-set. If $v_3 \notin S'$, then let $S = S' \cup \{v_1, v_2\}$ and if $v_3 \in S'$, then let $S = (S' \setminus \{v_3\}) \cup \{v_1, v_2, z_1, z_2\}$. Obviously, S is a 2-independent set of T yielding

$\beta_2(T) \geq \beta_2(T')+2$. By the induction hypothesis, we obtain $\gamma(T)+i(T) < \frac{4\beta_2(T)}{3}$. Now let all children of v_3 with exception v_2 are leaves. If $\deg_T(v_3) \geq 4$, then as above we can see that $\gamma(T) + i(T) < \frac{4\beta_2(T)}{3}$. Henceforth, we assume that $\deg_T(v_3) = 3$. Let w be the leaf adjacent to v_3. Suppose $T' = T - T_{v_3}$. By the induction hypothesis and Lemma 1 we have

$$\gamma(T) + i(T) = \gamma(T') + i(T') + 4 \leq \frac{4\beta_2(T')}{3} + 4 \leq \frac{4\beta_2(T) - 3}{3} + 4 = \frac{4\beta_2(T)}{3}.$$

If the equality holds, then we must have $\gamma(T') + i(T') = \frac{4\beta_2(T')}{3}$ and it follows from the induction hypothesis that we have $T' \in \mathcal{T}$. Thus each vertex of T' is either a leaf or a support vertex. We claim that v_4 is not a leaf in T'. Suppose, to the contrary, that v_4 is a leaf in T'. If $\mathrm{diam}(T) = 4$, then T is a wounded spider and by Lemma 3 we have $\gamma(T)+i(T) < \frac{4\beta_2(T)}{3}$, a contradiction. Let $\mathrm{diam}(T) \geq 5$. Since v_5 is not a strong support vertex, we observe that v_6 is a support vertex too. We consider two subcases.

Subcase 3.1. $\deg(v_5) = 2$. Let $T'' = T - T_{v_4}$ and let w, v_5 be two leaves adjacent to v_6 in T''. It follows from the induction hypothesis that $T'' \notin \mathcal{T}$ and so $\gamma(T'') + i(T'') < \frac{4\beta_2(T'')}{3}$. As above cases, we can see that $\gamma(T) \leq \gamma(T'') + 2$, $i(T) \leq i(T') + 2$ and $\beta_2(T) \geq \beta_2(T'') + 3$. This implies that

$$\gamma(T) + i(T) \leq \gamma(T'') + i(T'') + 4 < \frac{4\beta_2(T'')}{3} + 4 = \frac{4\beta_2(T)}{3},$$

which is a contradiction.

Subcase 3.2. $\deg(v_5) \geq 3$. Since $T' \in \mathcal{T}$ and v_4 is a leaf, every vertex $z \in N_T(v_5) \setminus \{v_4\}$ is a support vertex. Let $T'' = T - T_{v_4}$ and let u be a leaf adjacent to v_6 in T''. As above, we have $\gamma(T) \leq \gamma(T'') + 2$ and $i(T) \leq i(T') + 2$. Let S' be a $\beta_2(T'')$-set. If $v_5 \notin S'$ or $v_5 \in S$ and $z \notin S'$ for each $z \in N_T(v_5) \setminus \{v_4\}$, then $S = S' \cup \{v_4, w, v_2, v_1\}$ is a 2-independent set of T yielding $\beta_2(T) \geq \beta_2(T'')+4$ and by the induction hypothesis we obtain

$$\gamma(T) + i(T) \leq \gamma(T'') + i(T'') + 4 \leq \frac{4\beta_2(T'')}{3} + 4 < \frac{4\beta_2(T)}{3},$$

a contradiction again. Assume that $v_5 \in S'$ and $z \in S'$ for some $z \in N_T(v_5) \setminus \{v_4\}$ We may assume, without loss of generality, that $z = v_6$. Then $u \notin S'$ and the set $S = (S' \setminus \{v_5\}) \cup \{u, v_4, w, v_2, v_1\}$ is a 2-independent set of T yielding $\beta_2(T) \geq \beta_2(T'') + 4$ and as above we get a contradiction.

Consequently, v_4 is a support vertex of T'. Now T can be obtained from T' by operation $\mathcal{T}_1$ and so $T \in \mathcal{T}$. This completes the proof. ■

The next result is an immediate consequence of Theorem 5.

Corollary 6. *If T is a tree of order $n \geq 2$, then $\gamma(T) \leq \frac{2\beta_2(T)}{3}$.*

3. Independent Domination and 2-Independence of Trees

In this section we show that for any T of order $n \geq 2$, $i(T) \leq \frac{3\beta_2(T)}{4}$ and we characterize all extreme trees. First we introduce a family $\mathcal{F}$ of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = DS_{2,2}$, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by the operation $\mathcal{O}$ for $1 \leq i \leq k-1$.

Operation $\mathcal{O}$. If $v \in V(T_i)$ is a strong support vertex with $|L_v| = 2$, then operation $\mathcal{O}$ adds a double star $DS_{2,2}$ and joins a support vertex of $DS_{2,2}$ to v.

Observation 7. *If $T \in \mathcal{F}$, then*

1. *$L(T)$ is a $\beta_2(T)$-set of T and so $\beta_2(T) = \frac{2n(T)}{3}$,*
2. *every strong support vertex is adjacent with exactly two leaves,*
3. *$|L(T)| = 2|V(T) - L(T)|$,*
4. *$i(T) = \frac{n(T)}{2}$,*
5. *$i(T) = \frac{3\beta_2(T)}{4}$.*

Theorem 8. *If T is a tree of order $n \geq 2$, then*

$$i(T) \leq \frac{3\beta_2(T)}{4}, \tag{2}$$

with equality if and only if $T \in \mathcal{F}$.

Proof. The proof is by induction on n. The statements clearly hold for all trees of order $n = 2, 3, 4$. Let $n \geq 5$, and suppose that for every nontrivial tree T of order less than n the results are true. Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star and clearly $i(T) = 1 < \frac{3\beta_2(T)}{4}$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{r,s}$ for some $r \geq s \geq 1$. If $r \geq s \geq 2$, then

$$i(T) = s + 1 \leq \frac{3(r+s)}{4} = \frac{3\beta_2(T)}{4},$$

with equality if and only if $r = s = 2$ and this if and only if $T \in \mathcal{F}$. If $s = 1$, then $i(T) = 2 < \frac{3(r+2)}{4} = \frac{3\beta_2(T)}{4}$. Hence we may assume that $\mathrm{diam}(T) \geq 4$. Let $v_1 v_2 \cdots v_D$ be a diametrical path in T such that $t = \deg(v_2)$ is as large as possible. Let $L_{v_2} = \{z_1 = v_1, z_2, \ldots, z_{t-1}\}$. Let k_1 be the number of children of v_3 with depth 0, k_2 be the number of children of v_3 with depth 1 and degree at most three and k_3 be the number of children of v_3 with depth 1 and degree at least four. First let $2k_2 + 5k_3 > k_1$. Assume that $T' = T - T_{v_3}$. Clearly any $i(T')$-set can be extended by adding all children of v_3 to an independent dominating set of T and so $i(T) \leq i(T') + k_1 + k_2 + k_3$. On the other hand, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all leaves in

L_{v_3}, all children of v_3 with degree at most three and one of their children, and all leaves adjacent to the children of v_3 with degree at least four implying that $\beta_2(T) \geq \beta_2(T') + k_1 + 2k_2 + 3k_3$. By the induction hypothesis, we obtain

$$\begin{aligned} i(T) &\leq i(T') + k_1 + k_2 + k_3 \leq \frac{3\beta_2(T')}{4} + k_1 + k_2 + k_3 \\ &\leq \frac{3\beta_2(T) - 3k_1 - 6k_2 - 9k_3}{4} + k_1 + k_2 + k_3 \\ &\leq \frac{3\beta_2(T)}{4} + \frac{k_1 - 2k_2 - 5k_3}{4} < \frac{3\beta_2(T)}{4}. \end{aligned}$$

Henceforth, we assume that $2k_2 + 5k_3 \leq k_1$. This implies that v_3 is a strong support vertex, that is $k_1 \geq 2$. Consider the following cases.

Case 1. $t \geq 4$. Let $w_1, w_2 \in L_{v_3}$ and let $T' = T - \{z_1, z_2, w_1, w_2\}$. If S' is a $\beta_2(T')$-set, then the set $S = (S' \setminus \{v_2, v_3\}) \cup L_{v_2} \cup L_{v_3}$ if $|S' \cap \{v_2, v_2\}| = 2$, and $S = (S' \setminus \{v_2, v_3\}) \cup \{z_1, z_2, w_1, w_2\}$ if $|S' \cap \{v_2, v_3\}| \leq 1$, is a 2-independent set of T yielding $\beta_2(T) \geq \beta_2(T') + 3$. Now we show that $i(T) \leq i(T') + 2$. Let D' be a $i(T')$-set. Since D' is independent, we have $|D' \cap \{v_3, v_2\}| \leq 1$. If $|D' \cap \{v_3, v_2\}| = 0$, then $(D' - L_{v_2}) \cup \{v_2\}$ is a $i(T')$-set. Hence we may assume that $|D' \cap \{v_3, v_2\}| = 1$. Let $D = D' \cup \{z_1, z_2\}$ if $v_3 \in D'$, and $D = D' \cup \{w_1, w_2\}$ if $v_2 \in D'$. Clearly, D is an independent dominating set of T and so $i(T) \leq i(T') + 2$. By the induction hypothesis, we obtain

$$i(T) \leq i(T') + 2 \leq \frac{3\beta_2(T')}{4} + 2 < \frac{3\beta_2(T)}{4}.$$

Case 2. $t = 3$ and $k_1 \geq 3$. Let $w_1, w_2, w_3 \in L_{v_3}$ and $T' = T - \{z_1, z_2, w_1, w_2\}$. If S' is a $\beta_2(T')$-set, then the set $S = (S' \setminus \{v_2, v_3\}) \cup L_{v_2} \cup L_{v_3}$ if $|S' \cap \{v_1, v_2\}| = 2$, and $S = (S' \setminus \{v_2, v_3\}) \cup \{z_1, z_2, w_1, w_2\}$ if $|S' \cap \{v_2, v_3\}| \leq 1$, is a 2-independent set of T yielding $\beta_2(T) \geq \beta_2(T') + 3$. As above, we can see that $i(T) \leq i(T') + 2$ and by the induction hypothesis, we have $i(T) \leq i(T') + 2 \leq \frac{3\beta_2(T')}{4} + 2 < \frac{3\beta_2(T)}{4}$.

Case 3. $t = 3$ and $k_1 = 2$. We deduce from $2k_2 + 5k_3 \leq k_1$ that $k_2 \leq 1$ and $k_3 = 0$. Since $t = \deg(v_2) = 3$, then $k_2 = 1$. This yields $\deg_T(v_3) = 4$ and $T_{v_3} = DS_{2,2}$. Let $L_{v_3} = \{w_1, w_2\}$ and $T' = T - T_{v_3}$. Clearly, every $i(T')$-set can be extended to an independent dominating set of T by adding v_2, w_1, w_2 yielding $i(T) \leq i(T') + 3$. On the other hand, any $\beta_2(T')$ can be extended to a 2-independent set by adding z_1, z_2, w_1, w_2 and so $\beta_2(T) \geq \beta_2(T') + 4$. By the induction hypothesis we have

$$i(T) \leq i(T') + 3 \leq \frac{3\beta_2(T')}{4} + 3 \leq \frac{3(\beta_2(T) - 4)}{4} + 3 \leq \frac{3\beta_2(T)}{4}.$$

If the equality holds, then we must have $i(T') = \frac{3\beta_2(T')}{4}$ and this if and only if $T' \in \mathcal{F}$. Hence each vertex of T' is either a leaf or a strong support vertex. Now we show that v_4 is a support vertex of T'. Assume that v_4 is not a support vertex of T'. Then v_4 is a leaf of T' and v_5 is its support vertex in T'. Let $T'' = T - T_{v_4}$. Then clearly $T'' \notin \mathcal{F}$ and so $i(T'') < \frac{3\beta_2(T'')}{4}$. Obviously, every $i(T')$-set can be extended to an independent dominating set of T by adding v_2, z_1, z_2 yielding $i(T) \le i(T') + 3$, and any $\beta_2(T')$ can be extended to a 2-independent set by adding z_1, z_2, w_1, w_2 and so $\beta_2(T) \ge \beta_2(T') + 4$. Therefore

$$i(T) \le i(T'') + 3 < \frac{3\beta_2(T')}{4} + 3 \le \frac{3(\beta_2(T) - 4)}{4} + 3 \le \frac{3\beta_2(T)}{4},$$

which is a contradiction. Thus v_4 is a support vertex. Now T can be obtained from T' be operation $\mathcal{O}$ and so $T \in \mathcal{F}$.

Case 4. $t = 2$. Let $w_1, w_2 \in L_{v_3}$ and $T' = T - \{v_1, v_2\}$. Clearly, any $i(T')$-set can be extended to an independent dominating set of T by adding v_1, and this implies that $i(T) \le i(T') + 1$. On the other hand, for any $\beta_2(T')$-set S', the set $S = (S' \setminus \{v_3\}) \cup L_{v_3} \cup \{v_1, v_2\}$ is a 2-independent set of T yielding $\beta_2(T) \ge \beta_2(T') + 2$. It follows from the induction hypothesis that

$$i(T) \le i(T') + 1 \le \frac{3\beta_2(T')}{4} + 1 < \frac{3\beta_2(T)}{4},$$

and the proof is complete. ■

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