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SUPER EDGE-CONNECTIVITY AND ZEROTH-ORDER RANDIĆ INDEX[1]

Zhihong He[2]

School of Mathematics and Information Science
Yantai University, Yantai, 264005, China
e-mail: zhihhe@126.com

and

Mei Lu

Department of Mathematical Science
Tsinghua University, Beijing 100084, China
e-mail: mlu@math.tsinghua.edu.cn

Abstract

Define the zeroth-order Randić index as $R^0(G) = \sum_{x \in V(G)} \frac{1}{\sqrt{d_G(x)}}$, where $d_G(x)$ denotes the degree of the vertex x. In this paper, we present two sufficient conditions for graphs and triangle-free graphs, respectively, to be super edge-connected in terms of the zeroth-order Randić index.

Keywords: zeroth-order Randić index, super edge-connected, degree, triangle-free graph, minimum degree.

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1. Introduction

Throughout this paper, we consider finite undirected simple connected graphs. Let G be such a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Then the *order* and *size* of G are $n = |V|$ and $m = |E|$, respectively. The *degree* of a vertex $u \in V$ is the number of edges incident with u in G, denoted by

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[2]Corresponding author.

$d(u) = d_G(u)$. The minimum of all the vertex degrees of G is called the *minimum degree* of G, and denoted by $\delta = \delta(G)$. The *distance* between two vertices u and v of G is the length of a shortest path connecting them in G. The maximum of distances over all pairs of vertices of G is called the *diameter* of G, and denoted by $diam(G)$.

A *vertex-cut* in a graph G is a set X of vertices of G such that $G - X$ is disconnected. The *vertex-connectivity* or simply the *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if G is the complete graph K_n of order n. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The *edge-connectivity* $\lambda(G)$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. An edge-cut S is a *minimum edge-cut* or a λ-*cut*, if $|S| = \lambda(G)$, and an edge-cut S is *trivial*, if S consists of edges adjacent to a vertex of minimum degree. Notice that $\lambda(G) \leq \delta(G)$, and a graph G with $\lambda(G) = \delta(G)$ is said to be *maximally edge-connected*, or λ-*optimal* for simplicity. Other terminology and notation needed will be introduced as it naturally occurs in the following and we use Bondy and Murty [3] for those not defined here.

The *zeroth-order Randić index* $R^0(G)$ was defined by in Kier and Hall in 1986 [12, 13] as

$$R^0(G) = \sum_{x \in V(G)} \frac{1}{\sqrt{d_G(x)}}.$$

Let $R(G) = \sum_{u \in V(G)} \frac{1}{d_G(u)}$, which is the known inverse degree of a graph.

Sufficient conditions for whether a graph is maximally edge-connected were given by several researchers.

Theorem 1. *Let G be a connected graph of order n, minimum degree δ and edge-connectivity λ. Then $\lambda = \delta$ if*

(a) ([4]) $\delta \geq \lfloor \frac{n}{2} \rfloor$;

(b) ([14]) $d(u) + d(v) \geq n - 1$ *for all pairs u, v of nonadjacent vertices;*

(c) ([6]) $R(G) < 2 + 2/\delta(\delta+1) + (n-2\delta)/(n-\delta-2)(n-\delta-1)$;

(d) ([6]) *G is triangle-free and* $R(G) < 4 - 4(\delta-1)(1/2\delta(2\delta+2)) + 1/(n-2\delta)(n-2\delta+2))$;

(e) ([5]) $R^0(G) < 2\delta^{-1/2} + \delta^{1/2} + (\delta-1)(\delta+1)^{-1/2} + (\delta-1)(n-\delta-1)^{-1/2} - (\delta-2)(n-\delta-2)^{-1/2}$;

(f) ([5]) *If G is triangle-free and $R^0(G) < \min\{\gamma_1(-1/2,\delta), \gamma_2(-1/2,\delta)\}$, then $\lambda = \delta$, where* $\gamma_1(-1/2,\delta) = 3\delta^{1/2} + \delta^{-1/2} + (\delta-1)(\delta+1)^{-1/2} - \sqrt{2}(\delta-1)(n-2\delta)^{-1/2} + \sqrt{2}(\delta-1)(n-2\delta+2)^{-1/2}$, $\gamma_2(-1/2,\delta) = 2\delta^{1/2} + \delta^{-1/2} + 2\delta(\delta+1)^{-1/2} - \sqrt{2}(\delta-2)(n-2\delta-2)^{-1/2} + \sqrt{2}(\delta-2)(n-2\delta)^{-1/2}$.

Other sufficient conditions, depending on paraments not directly related to the vertex degree, for graphs to be maximally edge-connected were given by several authors.

Bauer *et al.* [1] proposed the concept of super-connectedness. A graph G is called *super-edge-connected* or *super-λ* if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super edge-connected graph is also maximally edge-connected. The study of super edge-connected graphs has a particular significance in the design of reliable networks [2]. Most of known sufficient conditions for a graph G to be super-λ are closely related to those in the preceding theorem.

Theorem 2. *Let G be a connected graph. Then G is super-λ if*

(a) (Kelmans [11]) $n \le 2\delta(G) - 1$;

(b) (Fiol [10]) $d(u) + d(v) \ge n$ *for all pairs u and v of nonadjacent vertices and G is different from* $K_{n/2} \times K_2$;

(c) (Fiol [10]) $diam(G) = 2$ *and G contains no K_δ with all its vertices of degree δ*;

(d) (Fiol [10]) *G is a bipartite graph with $\delta \ge 3$ and $n \le 4\delta - 3$*;

(e) (Soneoka [17]) $n > \delta((\Delta-1)^{diam(G)-1} - 1/(\Delta-2)+1) + (\Delta-1)^{diam(G)-1}$;

(f) (Tian [19]) $R(G) < 2 + (n-2\delta)/(n-\delta-1)(n-\delta)$;

(g) (Tian [19]) *G is triangle-free and* $R(G) < 2 + 1/\delta(\delta+1) + (n-2\delta-1)/(n-\delta-1)(n-\delta-2)$.

In [6] Dankelmann *et al.* gave sufficient conditions for graphs to be maximally edge-connected in terms of the inverse degree, the minimum degree and the order of a graph. In [19] Tian *et al.* gave sufficient conditions for graphs to be super edge-connected in terms of the inverse degree, the minimum degree and the order of a graph.

Motivated by the results of Dankelmann *et al.* [6] and Tian *et al.* [19], in this paper we give sufficient conditions for arbitrary graphs and triangle-free graphs to be super edge-connected in terms of the zeroth-order general Randić index, minimum degree and the order.

2. Preliminary Lemmas

In this section, we will list or prove some lemmas which will be used in our later proofs.

Lemma 3 (Lin *et al.* [15]). *Let $x_1, x_2 \in N$ and $\alpha \in R$. If $x_1 - 2 \ge x_2 \ge 1$, then*

(i) $(x_1-1)^\alpha + (x_2+1)^\alpha < x_1^\alpha + x_2^\alpha$ *if* $\alpha < 0$ *or* $\alpha > 1$;

(ii) $(x_1-1)^\alpha + (x_2+1)^\alpha > x_1^\alpha + x_2^\alpha$ *if* $0 < \alpha < 1$.

Lemma 4 (Su [18])**.** *Let $x_1, \ldots, x_p$ and A be positive reals with $\sum_{i=1}^{p} x_i \leq A$. For any real number $\alpha < 0$, we have*

(i) $\sum_{i=1}^{p} x_i^{\alpha} \geq p^{1-\alpha} A^{\alpha}$;

(ii) *if, in addition $x_1, \ldots, x_p$, A are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then $\sum_{i=1}^{p} x_i^{\alpha} \geq (p-b)a^{\alpha} + b(a+1)^{\alpha}$.*

Lemma 5 [18]**.** *Let $\Phi(x)$ be a continuous function on interval $[L, R]$ and $l + r = L + R$ for $l, r \in [L, R]$. Then*

(i) $\Phi(L) + \Phi(R) \geq \Phi(l) + \Phi(r)$ *if $\Phi(x)$ is convex;*

(ii) $\Phi(L) + \Phi(R) \leq \Phi(l) + \Phi(r)$ *if $\Phi(x)$ is concave.*

We say that a graph is *triangle-free* if it does not contain a triangle as a subgraph.

Lemma 6 (Dankelmann and Volkmann [7])**.** *Let G be a triangle-free graph of order $n \leq 4\delta - 1$. Then $\lambda = \delta$.*

A *complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$* is a simple graph whose vertices can be partitioned into r ($r \geq 2$) sets so that each pair of vertices is connected by an edge if and only if they belong to different sets of the partition.

The *Turán graph $T_{n,r}$* is the complete r-partite graph with b partite sets of size $a+1$ and $r-b$ partite sets of size a, where $a = \left\lfloor \frac{n}{r} \right\rfloor$ and $b = n - ra$.

The following is a famous result due to Turán [20].

Lemma 7 (Turán [20])**.**

(i) *Among all the n-vertex simple graphs with no $(r+1)$-clique, $T_{n,r}$ has the maximum number of edges.*

(ii) $|E(T_{n,r})| \leq \left\lfloor \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \right\rfloor$.

For two subsets X and Y of $V(G)$, let $[X, Y]$ be the set of edges with one endpoint in X and the other one in Y, and $|[X, Y]|$ denotes the cardinality of $[X, Y]$.

The following lemma was proved by Dankelmann and Volkmann [8].

Lemma 8 (Dankelmann and Volkmann [8])**.** *Let G be a connected graph. If there exist two disjoint, nonempty sets $X, Y \subset V(G)$, $X \cup Y = V(G)$, and $|[X, Y]| < \delta$, then $|X| \geq \delta + 1$ and $|Y| \geq \delta + 1$.*

The result above also can be found in other literature, e.g. Dankelmann and Volkmann [7] and Plesník and Znám [16].

Lemma 9. *Let x be a real number. Then $x^{-\frac{1}{2}} + \left(x + \frac{1}{2}\right)^{-\frac{1}{2}} \leq (x+1)^{-\frac{1}{2}} + \left(x - \frac{1}{2}\right)^{-\frac{1}{2}}$.*

Proof. Let $h(t) = t^{-\frac{1}{2}} - \left(t - \frac{1}{2}\right)^{-\frac{1}{2}}$. Clearly, $h(t)$ is increasing for $t \geq 1$, so we get $h(x+1) \geq h(x)$, i.e., $x^{-\frac{1}{2}} + \left(x + \frac{1}{2}\right)^{-\frac{1}{2}} \leq (x+1)^{-\frac{1}{2}} + \left(x - \frac{1}{2}\right)^{-\frac{1}{2}}$, as desired. ■

3. Main Result

Theorem 10. *Let G be a connected graph of order n, minimum degree δ and edge-connectivity λ. If*

$$R^0(G) < \min\left\{L_1\left(-\frac{1}{2},\delta\right), L_2\left(-\frac{1}{2},\delta\right)\right\}, \tag{1}$$

then G is super-λ, where $L_1\left(-\frac{1}{2},\delta\right) = \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}}$, $L_2\left(-\frac{1}{2},\delta\right) = (\delta+1)\delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} - (\delta-1)(n-\delta-2)^{-\frac{1}{2}}$.

Proof. In view of Theorem 2(a), we may assume $n \geq 2\delta$. Suppose to the contrary that G is not *super*-λ. Let S be a λ-cut such that each of two components of $G-S$ have at least two vertices, and let W and T denote the vertex sets of the two components of $G-S$. We claim that $\delta \leq |W|, |T| \leq n-\delta$. Assume $|W| \leq \delta - 1$. Then $\delta \geq 3$ and $\lambda(G) = |[W,\overline{W}]| \geq \delta|W| - |W|(|W|-1) \geq \delta|W| - (\delta-1)(|W|-1) = \delta - 1 + |W| \geq \delta + 1 > \delta$, a contradiction (because it is well known that $\lambda(G) \leq \delta(G)$). Similarly, we have $|T| \geq \delta$. Therefore, $\delta \leq |W|, |T| \leq n - \delta$.

If $\lambda(G) < \delta(G)$, then by Theorem 1(e), we have

$$\begin{aligned} R^0(G) &\geq 2\delta^{-\frac{1}{2}} + \delta^{\frac{1}{2}} + (\delta-1)(\delta+1)^{-\frac{1}{2}} + (\delta-1)(n-\delta-1)^{-\frac{1}{2}} \\ &- (\delta-2)(n-\delta-2)^{-\frac{1}{2}} \\ &= L_2\left(-\frac{1}{2},\delta\right) + \delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}} \\ &\geq L_2\left(-\frac{1}{2},\delta\right) \text{ (since } x^{-\frac{1}{2}} \text{ is a decreasing function for } x > 0), \end{aligned}$$

a contradiction to equation (1). Thus we assume that $\lambda(G) = \delta(G)$ in the following argument.

Each vertex in T is adjacent to at most $|T|-1$ vertices of T, and exactly δ edges join vertices of T to vertices of W. Hence $\sum_{t\in T} d(t) \leq |T|(|T|-1) + \delta$. If $\delta < |T|$, then by Lemma 4

$$\begin{aligned} \sum_{t\in T} d^{-\frac{1}{2}}(t) &\geq (|T|-\delta)(|T|-1)^{-\frac{1}{2}} + \delta|T|^{-\frac{1}{2}} \\ &= [(|T|-1) + (1-\delta)](|T|-1)^{-\frac{1}{2}} + \delta|T|^{-\frac{1}{2}} \\ &= (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right]. \end{aligned}$$

If $|T| = \delta$, then $\sum_{t\in T} d(t) \leq |T|^2$, by Lemma 4, $\sum_{t\in T} d^{-\frac{1}{2}}(t) \geq |T|^{\frac{1}{2}}$. The same argument is also valid for W. Now we consider two cases.

Case 1. $|W| = \delta$ or $|T| = \delta$. Assume, without loss of generality, that $|W| = \delta$ and $|T| = n - \delta$. If $n = 2\delta$, then

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq 2\delta^{\frac{1}{2}} \\ &= \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to equation (1). Thus we suppose that $n > 2\delta$. By the above argument,

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\ &\geq |W|^{\frac{1}{2}} + (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right] \\ &= \delta^{\frac{1}{2}} + (n-\delta-1)^{\frac{1}{2}} + (n-\delta-1)^{-\frac{1}{2}} - \delta\left[(n-\delta-1)^{-\frac{1}{2}} - (n-\delta)^{-\frac{1}{2}}\right] \\ &= \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to equation (1).

Case 2. $|W| \geq \delta + 1$ and $|T| \geq \delta + 1$. Since G has a vertex v of degree δ, without loss of generality, assume $v \in W$. Then $\sum_{w\in W\setminus\{v\}} d(w) \leq (|W|-1)^2 + \delta$.

If $|W| = \delta+1$, then $\sum_{w\in W\setminus\{v\}} d(w) \leq |W|(|W|-1) = \delta(\delta+1)$, by Lemma 4, we have $\sum_{w\in W} d^{-\frac{1}{2}}(w) \geq \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}}$. Hence

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} \\ &\quad + (n-\delta-2)^{\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - \delta\left[(n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}\right] \\ &= \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + (n-2\delta-1)(n-\delta-2)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} \\ &= L_2\left(-\frac{1}{2}, \delta\right) + (n-\delta-2)^{\frac{1}{2}} - \delta^{\frac{1}{2}} \\ &\geq L_2\left(-\frac{1}{2}, \delta\right) \text{ (since } x^{\frac{1}{2}} \text{ is an increasing function for } x > 0\text{)}, \end{aligned}$$

a contradiction to equation (1).

If $|W| \geq \delta + 2$, then by Lemma 4, we have

$$\begin{aligned} \sum_{w\in W} d^{-\frac{1}{2}}(w) &\geq \delta^{-\frac{1}{2}} + (|W|-1-\delta)(|W|-1)^{-\frac{1}{2}} + \delta|W|^{-\frac{1}{2}} \\ &= \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} - \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}}\right]. \end{aligned}$$

Hence

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} - \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}}\right] + (|T|-1)^{\frac{1}{2}} \\
&+ (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right] \\
&= \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} + (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} \\
&- \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right].
\end{aligned}$$

To minimize the right-hand side of the last inequality, consider the function $h_1(x) = (x-1)^{-\frac{1}{2}} - x^{-\frac{1}{2}}$. It is easy to verify that $h_1''(x) > 0$ for $x > 1$, so $h_1(t)$ is convex. By $|W|, |T| \geq \delta + 1$, $|W| + |T| = n$, and Lemma 5, we have

$$\begin{aligned}
&(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}} \\
&\leq \delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}.
\end{aligned}$$

Note that $h_2(x) = x^{-\frac{1}{2}}$ is a decreasing function and $h_3(x) = x^{\frac{1}{2}}$ is an increasing function for $x > 0$. We have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq \delta^{-\frac{1}{2}} + \delta^{\frac{1}{2}} + \delta^{\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} \\
&- \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}\right] \\
&= (\delta+1)\delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} - (\delta-1)(n-\delta-2)^{-\frac{1}{2}} \\
&= Ł_2\left(-\frac{1}{2}, \delta\right),
\end{aligned}$$

a contradiction to equation (1). ■

We present a class of graphs to show that the condition in Theorem 10 cannot be improved.

Example 11. Let n and δ be arbitrary integers with $n \geq 2\delta \geq 4$. Furthermore, let $G_1 \cong K_\delta$ with vertex set $V(G_1) = \{u_1, u_2, \ldots, u_\delta\}$ and let $H_2 \cong K_{n-\delta}$ with vertex set $V(G_2) = \{v_1, v_2, \ldots, v_{n-\delta}\}$. We define the graph G as the union of G_1 and G_2 together with $\delta - 1$ edges $u_1v_1, u_2v_2, \ldots, u_\delta v_\delta$. Then $n(G) = n$, $\delta(G) = \delta$ and

$$R^0(G) = \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2}, \delta\right).$$

But it is easy to see that G is not super-λ.

Now we pay our attention to the maximally edge-connected triangle-free graphs. In the following we shall use the following four functions

$$H_1\left(-\frac{1}{2},t\right)=(2t-1)t^{-\frac{1}{2}}+\frac{n(n-2t+2)^{-\frac{1}{2}}}{\sqrt{2}}+\frac{(n-4t+2)(n-2t)^{-\frac{1}{2}}}{\sqrt{2}};$$
$$H_2\left(-\frac{1}{2},t\right)=(2t-1)t^{-\frac{1}{2}}+\sqrt{2}t(n-2t+3)^{-\frac{1}{2}}+\sqrt{2}(n-3t+1)(n-2t+1)^{-\frac{1}{2}};$$
$$H_3\left(-\frac{1}{2},t\right)=3t^{\frac{1}{2}}+t(t+1)^{-\frac{1}{2}}-\sqrt{2}t(n-2t)^{-\frac{1}{2}}+\sqrt{2}t(n-2t+2)^{-\frac{1}{2}};$$
$$H_4\left(-\frac{1}{2},t\right)=2t^{\frac{1}{2}}+2t(t+1)^{-\frac{1}{2}}-\sqrt{2}(t-1)(n-2t-2)^{-\frac{1}{2}}+\sqrt{2}t(n-2t)^{-\frac{1}{2}}.$$

Theorem 12. *Let G be a connected triangle-free graph of order n, minimum degree δ and edge-connectivity λ. If*

$$\text{(2)} \qquad R^0(G)<\min\left\{H_1\left(-\frac{1}{2},\delta\right),H_2\left(-\frac{1}{2},\delta\right),H_3\left(-\frac{1}{2},\delta\right),H_4\left(-\frac{1}{2},\delta\right)\right\},$$

then G is super-λ.

Proof. Suppose to the contrary that G is not super-λ. Let F be a λ-cut such that each of two components of $G-F$ have at least two vertices, and let W and T denote the vertex sets of the two components. By the proof of Theorem 2.8 of [19] we get that $2\delta-1\le|W|,|T|\le n-2\delta+1$.

If $\lambda(G)<\delta(G)$, then $2\delta\le|W|,|T|\le n-2\delta$ (see, for example, [9]) and by Theorem 1(f)

$$R^0(G)\ge\min\left\{\gamma_1\left(-\frac{1}{2},\delta\right),\gamma_2\left(-\frac{1}{2},\delta\right)\right\},$$

where $\gamma_1\left(-\frac{1}{2},\delta\right)=H_3\left(-\frac{1}{2},\delta\right)+\delta^{-\frac{1}{2}}-(\delta+1)^{-\frac{1}{2}}+\sqrt{2}\Big[(n-2\delta)^{-\frac{1}{2}}-(n-2\delta+2)^{-\frac{1}{2}}\Big]\ge H_3\left(-\frac{1}{2},\delta\right)$, $\gamma_2\left(-\frac{1}{2},\delta\right)=H_4\left(-\frac{1}{2},\delta\right)+\sqrt{2}\left[(2\delta)^{-\frac{1}{2}}-(n-2\delta)^{-\frac{1}{2}}\right]+\sqrt{2}\left[(n-2\delta-2)^{-\frac{1}{2}}-(n-2\delta)^{-\frac{1}{2}}\right]\ge H_4\left(-\frac{1}{2},\delta\right)$. Therefore, we have

$$R^0(G)\ge\min\left\{H_3\left(-\frac{1}{2},\delta\right),H_4\left(-\frac{1}{2},\delta\right)\right\},$$

a contradiction to equation (2). Thus we assume that $\lambda(G)=\delta(G)$ in the following.

We know that $|\,[W,W]|\le\left\lfloor\frac{|W|^2}{4}\right\rfloor$ by Turán's Theorem, and thus

$$\sum_{w\in W}d(w)\le 2\left\lfloor\frac{|W|^2}{4}\right\rfloor+\delta.$$

If $|W|$ is even, then $\sum_{w\in W} d(w) \le |W|\frac{|W|}{2} + \delta$. By Lemma 4, we obtain

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge (|W|-\delta)\left(\frac{|W|}{2}\right)^{-\frac{1}{2}} + \delta\left(\frac{|W|}{2}+1\right)^{-\frac{1}{2}}. \tag{3}$$

If $|W|$ is odd, then $\sum_{w\in W} d(w) \le |W|\frac{|W|-1}{2} + \frac{|W|-1}{2} + \delta$. If $|W| > 2\delta - 1$, then by Lemma 4, we obtain

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge \left[\frac{|W|+1}{2} - \delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|-1}{2} + \delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}}. \tag{4}$$

If $|W| = 2\delta - 1$, then $\sum_{w\in W} d(w) \le |W|\frac{|W|+1}{2}$. By Lemma 4, we have

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge |W|\left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} = (2\delta-1)\delta^{-\frac{1}{2}}. \tag{5}$$

If $|W|, |T| = 2\delta - 1$, then by Lemma 4, we have

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \ge 2(2\delta-1)\delta^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + \frac{n(n-2\delta+2)^{-\frac{1}{2}}}{\sqrt{2}} + \frac{(n-4\delta+2)(n-2\delta)^{-\frac{1}{2}}}{\sqrt{2}} \\ &= H_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2). Thus, we assume in the following that one of $|W|$, $|T|$ is not equal to $2\delta - 1$, say $|T| > 2\delta - 1$ whenever $|W| = 2\delta - 1$. We consider five cases.

Case 1. n is even and $|W| = 2\delta - 1$. As n is even and $|W| = 2\delta - 1$, we know that $|T| = n - |W|$ is odd. Thus by inequalities (4) and (5),

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \ge (2\delta-1)\delta^{-\frac{1}{2}} \\ &+ \left[\frac{|T|-1}{2} + \delta\right]\left[\frac{|T|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|T|+1}{2} - \delta\right]\left[\frac{|T|-1}{2}\right]^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + \left[\frac{n-2\delta}{2} + \delta\right]\left[\frac{n-2\delta+2}{2}\right]^{-\frac{1}{2}} \\ &+ \left[\frac{n-2\delta+2}{2} - \delta\right]\left[\frac{n-2\delta}{2}\right]^{-\frac{1}{2}} = (2\delta-1)\delta^{-\frac{1}{2}} \\ &+ \frac{n(n-2\delta+2)^{-\frac{1}{2}}}{\sqrt{2}} + \frac{(n-4\delta+2)(n-2\delta)^{-\frac{1}{2}}}{\sqrt{2}} = H_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2).

Case 2. n is odd and $|W| = 2\delta - 1$. Since n is odd and $|W| = 2\delta - 1$, we know that $|T| = n - |W|$ is even. Thus by inequalities (3) and (5),

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\ &\geq (2\delta-1)\delta^{-\frac{1}{2}} + (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + (n-3\delta+1)\left[\frac{n-2\delta+1}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{n-2\delta+3}{2}\right]^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+3)^{-\frac{1}{2}} + \sqrt{2}(n-3\delta+1)(n-2\delta+1)^{-\frac{1}{2}} \\ &= H_2\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2).

Case 3. n and $|W|$ are both even, and $|W| > 2\delta - 1$. Immediately, we have that $|T| = n - |W|$ is also even. Thus by inequality (3),

$$\begin{aligned} R^0(G) = &\sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\ \geq &(|W|-\delta)\left[\frac{|W|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}} \\ = &(|W|-\delta)\left[\frac{|W|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + (n-|W|-\delta)\left[\frac{n-|W|}{2}\right]^{-\frac{1}{2}} \\ &+ \delta\left[\frac{n-|W|}{2}+1\right]^{-\frac{1}{2}} = \sqrt{2}|W|^{\frac{1}{2}} + \sqrt{2}(n-|W|)^{\frac{1}{2}} - \delta h(|W|), \end{aligned}$$

where $h(|W|) = \left[\frac{|W|}{2}\right]^{-\frac{1}{2}} - \left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + \left[\frac{n-|W|}{2}\right]^{-\frac{1}{2}} - \left[\frac{n-|W|}{2}+1\right]^{-\frac{1}{2}}$.

Define a function $f_1(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}} - \left(\frac{x}{2}+1\right)^{-\frac{1}{2}}$. It is easy to verify that $f_1''(x) > 0$ for $x > 0$, hence the function $f_1(x)$ is convex. By $|W|, |T| \in [2\delta, n-2\delta]$ and Lemma 5, we have $f_1(|W|)+f_1(n-|W|) \leq f_1(2\delta)+f_1(n-2\delta)$ and $f_2(x) = x^{\frac{1}{2}}$ is an increasing function for $x > 0$, thus

$$\begin{aligned} R^0(G) &\geq \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} - \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + \left(\frac{n-2\delta}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-2\delta}{2}+1\right)^{-\frac{1}{2}}\right] \\ &= 3\delta^{\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+2)^{-\frac{1}{2}} = H_3\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2).

Case 4. n is even, $|W|$ is odd and $|W| > 2\delta + 1$. Then $|T| = n - |W|$ is odd. Since $|W|, |T| \geq 2\delta + 1$ and by (4),

$$
\begin{aligned}
&R^0(G) = \sum_{w \in W} d^{-\frac{1}{2}}(w) + \sum_{t \in T} d^{-\frac{1}{2}}(t) \\
&\geq \left[\frac{|W|-1}{2} + \delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|+1}{2} - \delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}} \\
&+ \left[\frac{n-|W|-1}{2} + \delta\right]\left[\frac{n-|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{n-|W|+1}{2} - \delta\right]\left[\frac{n-|W|-1}{2}\right]^{-\frac{1}{2}} \\
&= \frac{(|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&+ \frac{(n-|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(n-|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(n-|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&- \delta\left[\left(\frac{|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}}\right].
\end{aligned}
$$

Define a function $f_3(x) = \left(\frac{x-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{x+1}{2}\right)^{-\frac{1}{2}}$. It is easy to verify that $f_3''(x) > 0$ for $x > 0$. Hence $f_3(x)$ is convex. Then we have

$$f_3(|W|) + f_3(n - |W|) \leq f_3(2\delta + 1) + f_3(n - 2\delta - 1).$$

Note that $f_2(x) = x^{\frac{1}{2}}$ is an increasing function and $f_4(x) = x^{-\frac{1}{2}}$ is a decreasing function for $x > 0$. Hence

$$
\begin{aligned}
R^0(G) \geq\ & \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&+ \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&- \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} - \frac{(n-2\delta)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\right] \\
=\ & \delta^{\frac{1}{2}} + 3\delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}(\delta-2)(n-2\delta-2)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}} \\
=\ & H_4(-\frac{1}{2}, \delta) + \delta(\delta+1)^{-\frac{1}{2}} - \delta^{\frac{1}{2}} + \sqrt{2}(n-2\delta-2)^{-\frac{1}{2}} \geq H_4\left(-\frac{1}{2}, \delta\right),
\end{aligned}
$$

a contradiction to (2).

Case 5. n is odd and $|W| > 2\delta - 1$. Assume without loss of generality that $|W|$ is odd and $|T|$ is even. Then we have $|W| \geq 2\delta + 1$ and $|T| \geq 2\delta$. Thus by inequalities (3) and (4), we have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t)\\
&\geq \left[\frac{|W|-1}{2}+\delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|+1}{2}-\delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}}\\
&+ (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}}\\
&= \frac{(|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(n-|W|)^{\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&- \delta\left[\left(\frac{|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}}\right].
\end{aligned}$$

By Lemma 9, we obtain $\left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}} \leq \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}} + \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}}$, i.e., $\left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}} \leq \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}}$. Since $f_3(x) = \left(\frac{x-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{x+1}{2}\right)^{-\frac{1}{2}}$ is convex and

$$f_3(|W|) + f_3(n-|W|) \leq f_3(2\delta+1) + f_3(n-2\delta-1).$$

Note that $f_2(x) = x^{\frac{1}{2}}$ is an increasing function and $f_4(x) = x^{-\frac{1}{2}}$ is a decreasing function $x > 0$. Therefore, we have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t)\\
&\geq \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&- \delta\left[\left(\frac{2\delta}{2}\right)^{-\frac{1}{2}} - \left(\frac{2\delta+2}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-2\delta-2}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-2\delta}{2}\right)^{-\frac{1}{2}}\right]\\
&= 2\delta^{\frac{1}{2}} + 2\delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}(\delta-1)(n-2\delta-2)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}}\\
&= H_4\left(-\frac{1}{2},\delta\right),
\end{aligned}$$

a contradiction to (2). ■

We present a class of graphs to show that the condition in Theorem 12 cannot be improved.

Example 13. For arbitrary integers δ and $n \geq 4\delta-1$, let $G_1 \cong K_{\delta-1,\delta}$ with vertex set $V(K_{\delta-1,\delta}) = W \cup T$ (where $W = \{w_1, w_2, \ldots, w_{\delta-1}\}$ and $T = \{t_1, t_2, \ldots, t_\delta\}$). Let $G_2 \cong K_{\frac{n-2\delta+1}{2},\frac{n-2\delta+1}{2}}$ with vertex set $V(G_2) = X \cup Y$ (where $X = \{x_1, x_2, \ldots, x_{\frac{n-2\delta+1}{2}}\}$ and $Y = \{y_1, y_2, \ldots, y_{\frac{n-2\delta+1}{2}}\}$). We define the graph G as the union of G_1 and G_2 together with the δ edges $t_1x_1, t_2x_2, \ldots, t_\delta x_\delta$. Clearly, G is triangle-free, has order n and minimum degree δ and

$$\begin{aligned} R^0(G) &= (2\delta-1)\delta^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+3)^{-\frac{1}{2}} \\ &\quad + \sqrt{2}(n-3\delta+1)(n-2\delta+1)^{-\frac{1}{2}} = H_3\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

but it is easy to see that G is not super-λ.

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