*Discussiones Mathematicae*
*Graph Theory 40 (2020) 971–984*
*doi:10.7151/dmgt.2164*

# SUPER EDGE-CONNECTIVITY AND ZEROTH-ORDER RANDIĆ INDEX[1]

Zhihong He[2]

*School of Mathematics and Information Science*
*Yantai University, Yantai, 264005, China*
**e-mail:** zhihhe@126.com

and

Mei Lu

*Department of Mathematical Science*
*Tsinghua University, Beijing 100084, China*
**e-mail:** mlu@math.tsinghua.edu.cn

**Abstract**

Define the zeroth-order Randić index as $R^0(G) = \sum_{x\in V(G)} \frac{1}{\sqrt{d_G(x)}}$, where $d_G(x)$ denotes the degree of the vertex $x$. In this paper, we present two sufficient conditions for graphs and triangle-free graphs, respectively, to be super edge-connected in terms of the zeroth-order Randić index.

**Keywords:** zeroth-order Randić index, super edge-connected, degree, triangle-free graph, minimum degree.

**2010 Mathematics Subject Classification:** 05C20.

## 1. Introduction

Throughout this paper, we consider finite undirected simple connected graphs. Let $G$ be such a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Then the *order* and *size* of $G$ are $n = |V|$ and $m = |E|$, respectively. The *degree* of a vertex $u \in V$ is the number of edges incident with $u$ in $G$, denoted by

[1]Natural Science Funds of China (No.11501490, 61373019, 13071107) and by the Natural Science Foundation of Shandong Province (No. ZR2015AM006).

[2]Corresponding author.

$d(u) = d_G(u)$. The minimum of all the vertex degrees of $G$ is called the *minimum degree* of $G$, and denoted by $\delta = \delta(G)$. The *distance* between two vertices $u$ and $v$ of $G$ is the length of a shortest path connecting them in $G$. The maximum of distances over all pairs of vertices of $G$ is called the *diameter* of $G$, and denoted by $diam(G)$.

A *vertex-cut* in a graph $G$ is a set $X$ of vertices of $G$ such that $G - X$ is disconnected. The *vertex-connectivity* or simply the *connectivity* $\kappa = \kappa(G)$ of a graph $G$ is the minimum cardinality of a vertex-cut of $G$ if $G$ is not complete, and $\kappa(G) = n - 1$ if $G$ is the complete graph $K_n$ of order $n$. An *edge-cut* of a connected graph $G$ is a set of edges whose removal disconnects $G$. The *edge-connectivity* $\lambda(G)$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of $G$. An edge-cut $S$ is a *minimum edge-cut* or a $\lambda$-*cut*, if $|S| = \lambda(G)$, and an edge-cut $S$ is *trivial*, if $S$ consists of edges adjacent to a vertex of minimum degree. Notice that $\lambda(G) \leq \delta(G)$, and a graph $G$ with $\lambda(G) = \delta(G)$ is said to be *maximally edge-connected*, or $\lambda$-*optimal* for simplicity. Other terminology and notation needed will be introduced as it naturally occurs in the following and we use Bondy and Murty [3] for those not defined here.

The *zeroth-order Randić index* $R^0(G)$ was defined by in Kier and Hall in 1986 [12, 13] as

$$R^0(G) = \sum_{x \in V(G)} \frac{1}{\sqrt{d_G(x)}}.$$

Let $R(G) = \sum_{u \in V(G)} \frac{1}{d_G(u)}$, which is the known inverse degree of a graph.

Sufficient conditions for whether a graph is maximally edge-connected were given by several researchers.

**Theorem 1.** *Let $G$ be a connected graph of order $n$, minimum degree $\delta$ and edge-connectivity $\lambda$. Then $\lambda = \delta$ if*

(a) ([4]) $\delta \geq \lfloor \frac{n}{2} \rfloor$;

(b) ([14]) $d(u) + d(v) \geq n - 1$ *for all pairs $u$, $v$ of nonadjacent vertices;*

(c) ([6]) $R(G) < 2 + 2/\delta(\delta + 1) + (n - 2\delta)/(n - \delta - 2)(n - \delta - 1)$;

(d) ([6]) *$G$ is triangle-free and* $R(G) < 4 - 4(\delta - 1)(1/2\delta(2\delta + 2)) + 1/(n - 2\delta)(n - 2\delta + 2))$;

(e) ([5]) $R^0(G) < 2\delta^{-1/2} + \delta^{1/2} + (\delta - 1)(\delta + 1)^{-1/2} + (\delta - 1)(n - \delta - 1)^{-1/2} - (\delta - 2)(n - \delta - 2)^{-1/2}$;

(f) ([5]) *If $G$ is triangle-free and $R^0(G) < \min\{\gamma_1(-1/2, \delta), \gamma_2(-1/2, \delta)\}$, then $\lambda = \delta$, where* $\gamma_1(-1/2, \delta) = 3\delta^{1/2} + \delta^{-1/2} + (\delta - 1)(\delta + 1)^{-1/2} - \sqrt{2}(\delta - 1)(n - 2\delta)^{-1/2} + \sqrt{2}(\delta - 1)(n - 2\delta + 2)^{-1/2}$, $\gamma_2(-1/2, \delta) = 2\delta^{1/2} + \delta^{-1/2} + 2\delta(\delta + 1)^{-1/2} - \sqrt{2}(\delta - 2)(n - 2\delta - 2)^{-1/2} + \sqrt{2}(\delta - 2)(n - 2\delta)^{-1/2}$.

Other sufficient conditions, depending on paraments not directly related to the vertex degree, for graphs to be maximally edge-connected were given by several authors.

Bauer *et al.* [1] proposed the concept of super-connectedness. A graph $G$ is called *super-edge-connected* or *super-$\lambda$* if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super edge-connected graph is also maximally edge-connected. The study of super edge-connected graphs has a particular significance in the design of reliable networks [2]. Most of known sufficient conditions for a graph $G$ to be super-$\lambda$ are closely related to those in the preceding theorem.

**Theorem 2.** *Let $G$ be a connected graph. Then $G$ is super-$\lambda$ if*

(a) (Kelmans [11]) $n \leq 2\delta(G) - 1$;

(b) (Fiol [10]) $d(u) + d(v) \geq n$ *for all pairs* $u$ *and* $v$ *of nonadjacent vertices and* $G$ *is different from* $K_{n/2} \times K_2$;

(c) (Fiol [10]) $diam(G) = 2$ *and* $G$ *contains no* $K_\delta$ *with all its vertices of degree* $\delta$;

(d) (Fiol [10]) $G$ *is a bipartite graph with* $\delta \geq 3$ *and* $n \leq 4\delta - 3$;

(e) (Soneoka [17]) $n > \delta((\Delta - 1)^{diam(G)-1} - 1/(\Delta - 2) + 1) + (\Delta - 1)^{diam(G)-1}$;

(f) (Tian [19]) $R(G) < 2 + (n - 2\delta)/(n - \delta - 1)(n - \delta)$;

(g) (Tian [19]) $G$ *is triangle-free and* $R(G) < 2 + 1/\delta(\delta + 1) + (n - 2\delta - 1)/(n - \delta - 1)(n - \delta - 2)$.

In [6] Dankelmann *et al.* gave sufficient conditions for graphs to be maximally edge-connected in terms of the inverse degree, the minimum degree and the order of a graph. In [19] Tian *et al.* gave sufficient conditions for graphs to be super edge-connected in terms of the inverse degree, the minimum degree and the order of a graph.

Motivated by the results of Dankelmann *et al.* [6] and Tian *et al.* [19], in this paper we give sufficient conditions for arbitrary graphs and triangle-free graphs to be super edge-connected in terms of the zeroth-order general Randić index, minimum degree and the order.

## 2. Preliminary Lemmas

In this section, we will list or prove some lemmas which will be used in our later proofs.

**Lemma 3** (Lin *et al.* [15])**.** *Let* $x_1, x_2 \in N$ *and* $\alpha \in R$. *If* $x_1 - 2 \geq x_2 \geq 1$, *then*

(i) $(x_1 - 1)^\alpha + (x_2 + 1)^\alpha < x_1^\alpha + x_2^\alpha$ *if* $\alpha < 0$ *or* $\alpha > 1$;

(ii) $(x_1 - 1)^\alpha + (x_2 + 1)^\alpha > x_1^\alpha + x_2^\alpha$ *if* $0 < \alpha < 1$.

**Lemma 4** (Su [18])**.** *Let $x_1, \ldots, x_p$ and $A$ be positive reals with $\sum_{i=1}^{p} x_i \le A$. For any real number $\alpha < 0$, we have*

(i) $\sum_{i=1}^{p} x_i^{\alpha} \ge p^{1-\alpha} A^{\alpha}$;

(ii) *if, in addition $x_1, \ldots, x_p$, $A$ are positive integers, and $a$, $b$ are integers with $A = ap + b$ and $0 \le b < p$, then $\sum_{i=1}^{p} x_i^{\alpha} \ge (p-b)a^{\alpha} + b(a+1)^{\alpha}$.*

**Lemma 5** [18]**.** *Let $\Phi(x)$ be a continuous function on interval $[L, R]$ and $l + r = L + R$ for $l, r \in [L, R]$. Then*

(i) $\Phi(L) + \Phi(R) \ge \Phi(l) + \Phi(r)$ *if $\Phi(x)$ is convex;*

(ii) $\Phi(L) + \Phi(R) \le \Phi(l) + \Phi(r)$ *if $\Phi(x)$ is concave.*

We say that a graph is *triangle-free* if it does not contain a triangle as a subgraph.

**Lemma 6** (Dankelmann and Volkmann [7])**.** *Let $G$ be a triangle-free graph of order $n \le 4\delta - 1$. Then $\lambda = \delta$.*

A *complete $r$-partite graph $K_{n_1,n_2,\ldots,n_r}$* is a simple graph whose vertices can be partitioned into $r$ ($r \ge 2$) sets so that each pair of vertices is connected by an edge if and only if they belong to different sets of the partition.

The *Turán graph $T_{n,r}$* is the complete $r$-partite graph with $b$ partite sets of size $a+1$ and $r-b$ partite sets of size $a$, where $a = \left\lfloor \frac{n}{r} \right\rfloor$ and $b = n - ra$.

The following is a famous result due to Turán [20].

**Lemma 7** (Turán [20])**.**

(i) *Among all the $n$-vertex simple graphs with no $(r+1)$-clique, $T_{n,r}$ has the maximum number of edges.*

(ii) $|E(T_{n,r})| \le \left\lfloor \left(1 - \frac{1}{r}\right) \frac{n^2}{2} \right\rfloor$.

For two subsets $X$ and $Y$ of $V(G)$, let $[X, Y]$ be the set of edges with one endpoint in $X$ and the other one in $Y$, and $|[X, Y]|$ denotes the cardinality of $[X, Y]$.

The following lemma was proved by Dankelmann and Volkmann [8].

**Lemma 8** (Dankelmann and Volkmann [8])**.** *Let $G$ be a connected graph. If there exist two disjoint, nonempty sets $X, Y \subset V(G)$, $X \cup Y = V(G)$, and $|\,[X, Y]| < \delta$, then $|X| \ge \delta + 1$ and $|Y| \ge \delta + 1$.*

The result above also can be found in other literature, e.g. Dankelmann and Volkmann [7] and Plesník and Znám [16].

**Lemma 9.** *Let $x$ be a real number. Then $x^{-\frac{1}{2}} + \left(x + \frac{1}{2}\right)^{-\frac{1}{2}} \le (x+1)^{-\frac{1}{2}} + \left(x - \frac{1}{2}\right)^{-\frac{1}{2}}$.*

***Proof.*** Let $h(t) = t^{-\frac{1}{2}} - \left(t - \frac{1}{2}\right)^{-\frac{1}{2}}$. Clearly, $h(t)$ is increasing for $t \ge 1$, so we get $h(x+1) \ge h(x)$, i.e., $x^{-\frac{1}{2}} + \left(x + \frac{1}{2}\right)^{-\frac{1}{2}} \le (x+1)^{-\frac{1}{2}} + \left(x - \frac{1}{2}\right)^{-\frac{1}{2}}$, as desired. ■

## 3. Main Result

**Theorem 10.** *Let $G$ be a connected graph of order $n$, minimum degree $\delta$ and edge-connectivity $\lambda$. If*

$$R^0(G) < \min\left\{L_1\left(-\frac{1}{2},\delta\right), L_2\left(-\frac{1}{2},\delta\right)\right\}, \tag{1}$$

*then $G$ is super-$\lambda$, where $L_1\left(-\frac{1}{2},\delta\right) = \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}}$, $L_2\left(-\frac{1}{2},\delta\right) = (\delta+1)\delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} - (\delta-1)(n-\delta-2)^{-\frac{1}{2}}$.*

***Proof.*** In view of Theorem 2(a), we may assume $n \geq 2\delta$. Suppose to the contrary that $G$ is not *super*-$\lambda$. Let $S$ be a $\lambda$-cut such that each of two components of $G-S$ have at least two vertices, and let $W$ and $T$ denote the vertex sets of the two components of $G-S$. We claim that $\delta \leq |W|, |T| \leq n-\delta$. Assume $|W| \leq \delta - 1$. Then $\delta \geq 3$ and $\lambda(G) = |[W,\overline{W}]| \geq \delta|W| - |W|(|W|-1) \geq \delta|W| - (\delta-1)(|W|-1) = \delta - 1 + |W| \geq \delta + 1 > \delta$, a contradiction (because it is well known that $\lambda(G) \leq \delta(G)$). Similarly, we have $|T| \geq \delta$. Therefore, $\delta \leq |W|, |T| \leq n - \delta$.

If $\lambda(G) < \delta(G)$, then by Theorem 1(e), we have

$$\begin{aligned} R^0(G) &\geq 2\delta^{-\frac{1}{2}} + \delta^{\frac{1}{2}} + (\delta-1)(\delta+1)^{-\frac{1}{2}} + (\delta-1)(n-\delta-1)^{-\frac{1}{2}} \\ &- (\delta-2)(n-\delta-2)^{-\frac{1}{2}} \\ &= L_2\left(-\frac{1}{2},\delta\right) + \delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}} \\ &\geq L_2\left(-\frac{1}{2},\delta\right) \text{ (since } x^{-\frac{1}{2}} \text{ is a decreasing function for } x>0), \end{aligned}$$

a contradiction to equation (1). Thus we assume that $\lambda(G) = \delta(G)$ in the following argument.

Each vertex in $T$ is adjacent to at most $|T|-1$ vertices of $T$, and exactly $\delta$ edges join vertices of $T$ to vertices of $W$. Hence $\sum_{t\in T} d(t) \leq |T|(|T|-1) + \delta$. If $\delta < |T|$, then by Lemma 4

$$\begin{aligned} \sum_{t\in T} d^{-\frac{1}{2}}(t) &\geq (|T|-\delta)(|T|-1)^{-\frac{1}{2}} + \delta|T|^{-\frac{1}{2}} \\ &= [(|T|-1) + (1-\delta)](|T|-1)^{-\frac{1}{2}} + \delta|T|^{-\frac{1}{2}} \\ &= (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right]. \end{aligned}$$

If $|T| = \delta$, then $\sum_{t\in T} d(t) \leq |T|^2$, by Lemma 4, $\sum_{t\in T} d^{-\frac{1}{2}}(t) \geq |T|^{\frac{1}{2}}$. The same argument is also valid for $W$. Now we consider two cases.

*Case* 1. $|W| = \delta$ or $|T| = \delta$. Assume, without loss of generality, that $|W| = \delta$ and $|T| = n - \delta$. If $n = 2\delta$, then

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq 2\delta^{\frac{1}{2}} \\
&= \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2},\delta\right),
\end{aligned}$$

a contradiction to equation (1). Thus we suppose that $n > 2\delta$. By the above argument,

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq |W|^{\frac{1}{2}} + (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right] \\
&= \delta^{\frac{1}{2}} + (n-\delta-1)^{\frac{1}{2}} + (n-\delta-1)^{-\frac{1}{2}} - \delta\left[(n-\delta-1)^{-\frac{1}{2}} - (n-\delta)^{-\frac{1}{2}}\right] \\
&= \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2},\delta\right),
\end{aligned}$$

a contradiction to equation (1).

*Case* 2. $|W| \geq \delta+1$ and $|T| \geq \delta+1$. Since $G$ has a vertex $v$ of degree $\delta$, without loss of generality, assume $v \in W$. Then $\sum_{w\in W\setminus\{v\}} d(w) \leq (|W|-1)^2 + \delta$.

If $|W| = \delta+1$, then $\sum_{w\in W\setminus\{v\}} d(w) \leq |W|(|W|-1) = \delta(\delta+1)$, by Lemma 4, we have $\sum_{w\in W} d^{-\frac{1}{2}}(w) \geq \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}}$. Hence

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} \\
&\quad + (n-\delta-2)^{\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - \delta\left[(n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}\right] \\
&= \delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + (n-2\delta-1)(n-\delta-2)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} \\
&= L_2\left(-\frac{1}{2},\delta\right) + (n-\delta-2)^{\frac{1}{2}} - \delta^{\frac{1}{2}} \\
&\geq L_2\left(-\frac{1}{2},\delta\right) \text{ (since } x^{\frac{1}{2}} \text{ is an increasing function for } x>0\text{)},
\end{aligned}$$

a contradiction to equation (1).

If $|W| \geq \delta+2$, then by Lemma 4, we have

$$\begin{aligned}
\sum_{w\in W} d^{-\frac{1}{2}}(w) &\geq \delta^{-\frac{1}{2}} + (|W|-1-\delta)(|W|-1)^{-\frac{1}{2}} + \delta|W|^{-\frac{1}{2}} \\
&= \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} - \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t)\\
&\geq \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} - \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}}\right] + (|T|-1)^{\frac{1}{2}}\\
&+ (|T|-1)^{-\frac{1}{2}} - \delta\left[(|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right]\\
&= \delta^{-\frac{1}{2}} + (|W|-1)^{\frac{1}{2}} + (|T|-1)^{\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}}\\
&- \delta\left[(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\right].
\end{aligned}$$

To minimize the right-hand side of the last inequality, consider the function $h_1(x) = (x-1)^{-\frac{1}{2}} - x^{-\frac{1}{2}}$. It is easy to verify that $h_1''(x) > 0$ for $x > 1$, so $h_1(t)$ is convex. By $|W|, |T| \geq \delta + 1$, $|W| + |T| = n$, and Lemma 5, we have

$$\begin{aligned}
&(|W|-1)^{-\frac{1}{2}} - |W|^{-\frac{1}{2}} + (|T|-1)^{-\frac{1}{2}} - |T|^{-\frac{1}{2}}\\
&\leq \delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}.
\end{aligned}$$

Note that $h_2(x) = x^{-\frac{1}{2}}$ is a decreasing function and $h_3(x) = x^{\frac{1}{2}}$ is an increasing function for $x > 0$. We have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \geq \delta^{-\frac{1}{2}} + \delta^{\frac{1}{2}} + \delta^{\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}}\\
&- \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + (n-\delta-2)^{-\frac{1}{2}} - (n-\delta-1)^{-\frac{1}{2}}\right]\\
&= (\delta+1)\delta^{-\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} + \delta(n-\delta-1)^{-\frac{1}{2}} - (\delta-1)(n-\delta-2)^{-\frac{1}{2}}\\
&= Ł_2\left(-\frac{1}{2}, \delta\right),
\end{aligned}$$

a contradiction to equation (1). ■

We present a class of graphs to show that the condition in Theorem 10 cannot be improved.

**Example 11.** Let $n$ and $\delta$ be arbitrary integers with $n \geq 2\delta \geq 4$. Furthermore, let $G_1 \cong K_\delta$ with vertex set $V(G_1) = \{u_1, u_2, \ldots, u_\delta\}$ and let $H_2 \cong K_{n-\delta}$ with vertex set $V(G_2) = \{v_1, v_2, \ldots, v_{n-\delta}\}$. We define the graph $G$ as the union of $G_1$ and $G_2$ together with $\delta - 1$ edges $u_1v_1, u_2v_2, \ldots, u_\delta v_\delta$. Then $n(G) = n$, $\delta(G) = \delta$ and

$$R^0(G) = \delta^{\frac{1}{2}} + \delta(n-\delta)^{-\frac{1}{2}} + (n-2\delta)(n-\delta-1)^{-\frac{1}{2}} = L_1\left(-\frac{1}{2}, \delta\right).$$

But it is easy to see that $G$ is not super-$\lambda$.

Now we pay our attention to the maximally edge-connected triangle-free graphs. In the following we shall use the following four functions

$$H_1\left(-\frac{1}{2},t\right)=(2t-1)t^{-\frac{1}{2}}+\frac{n(n-2t+2)^{-\frac{1}{2}}}{\sqrt{2}}+\frac{(n-4t+2)(n-2t)^{-\frac{1}{2}}}{\sqrt{2}};$$
$$H_2\left(-\frac{1}{2},t\right)=(2t-1)t^{-\frac{1}{2}}+\sqrt{2}t(n-2t+3)^{-\frac{1}{2}}+\sqrt{2}(n-3t+1)(n-2t+1)^{-\frac{1}{2}};$$
$$H_3\left(-\frac{1}{2},t\right)=3t^{\frac{1}{2}}+t(t+1)^{-\frac{1}{2}}-\sqrt{2}t(n-2t)^{-\frac{1}{2}}+\sqrt{2}t(n-2t+2)^{-\frac{1}{2}};$$
$$H_4\left(-\frac{1}{2},t\right)=2t^{\frac{1}{2}}+2t(t+1)^{-\frac{1}{2}}-\sqrt{2}(t-1)(n-2t-2)^{-\frac{1}{2}}+\sqrt{2}t(n-2t)^{-\frac{1}{2}}.$$

**Theorem 12.** *Let $G$ be a connected triangle-free graph of order $n$, minimum degree $\delta$ and edge-connectivity $\lambda$. If*

$$(2)\qquad R^0(G)<\min\left\{H_1\left(-\frac{1}{2},\delta\right),H_2\left(-\frac{1}{2},\delta\right),H_3\left(-\frac{1}{2},\delta\right),H_4\left(-\frac{1}{2},\delta\right)\right\},$$

*then $G$ is super-$\lambda$.*

***Proof.*** Suppose to the contrary that $G$ is not super-$\lambda$. Let $F$ be a $\lambda$-cut such that each of two components of $G-F$ have at least two vertices, and let $W$ and $T$ denote the vertex sets of the two components. By the proof of Theorem 2.8 of [19] we get that $2\delta-1\le|W|,|T|\le n-2\delta+1$.

If $\lambda(G)<\delta(G)$, then $2\delta\le|W|,|T|\le n-2\delta$ (see, for example, [9]) and by Theorem 1(f)

$$R^0(G)\ge\min\left\{\gamma_1\left(-\frac{1}{2},\delta\right),\gamma_2\left(-\frac{1}{2},\delta\right)\right\},$$

where $\gamma_1\left(-\frac{1}{2},\delta\right)=H_3\left(-\frac{1}{2},\delta\right)+\delta^{-\frac{1}{2}}-(\delta+1)^{-\frac{1}{2}}+\sqrt{2}\Big[(n-2\delta)^{-\frac{1}{2}}-(n-2\delta+2)^{-\frac{1}{2}}\Big]\ge H_3\left(-\frac{1}{2},\delta\right)$, $\gamma_2\left(-\frac{1}{2},\delta\right)=H_4\left(-\frac{1}{2},\delta\right)+\sqrt{2}\left[(2\delta)^{-\frac{1}{2}}-(n-2\delta)^{-\frac{1}{2}}\right]+\sqrt{2}\left[(n-2\delta-2)^{-\frac{1}{2}}-(n-2\delta)^{-\frac{1}{2}}\right]\ge H_4\left(-\frac{1}{2},\delta\right)$. Therefore, we have

$$R^0(G)\ge\min\left\{H_3\left(-\frac{1}{2},\delta\right),H_4\left(-\frac{1}{2},\delta\right)\right\},$$

a contradiction to equation (2). Thus we assume that $\lambda(G)=\delta(G)$ in the following.

We know that $|\,[W,W]|\le\left\lfloor\frac{|W|^2}{4}\right\rfloor$ by Turán's Theorem, and thus

$$\sum_{w\in W}d(w)\le 2\left\lfloor\frac{|W|^2}{4}\right\rfloor+\delta.$$

If $|W|$ is even, then $\sum_{w\in W} d(w) \le |W|\frac{|W|}{2} + \delta$. By Lemma 4, we obtain

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge (|W|-\delta)\left(\frac{|W|}{2}\right)^{-\frac{1}{2}} + \delta\left(\frac{|W|}{2}+1\right)^{-\frac{1}{2}}. \tag{3}$$

If $|W|$ is odd, then $\sum_{w\in W} d(w) \le |W|\frac{|W|-1}{2} + \frac{|W|-1}{2} + \delta$. If $|W| > 2\delta - 1$, then by Lemma 4, we obtain

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge \left[\frac{|W|+1}{2}-\delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|-1}{2}+\delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}}. \tag{4}$$

If $|W| = 2\delta - 1$, then $\sum_{w\in W} d(w) \le |W|\frac{|W|+1}{2}$. By Lemma 4, we have

$$\sum_{w\in W} d^{-\frac{1}{2}}(w) \ge |W|\left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} = (2\delta-1)\delta^{-\frac{1}{2}}. \tag{5}$$

If $|W|, |T| = 2\delta - 1$, then by Lemma 4, we have

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \ge 2(2\delta-1)\delta^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + \frac{n(n-2\delta+2)^{-\frac{1}{2}}}{\sqrt{2}} + \frac{(n-4\delta+2)(n-2\delta)^{-\frac{1}{2}}}{\sqrt{2}} \\ &= H_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2). Thus, we assume in the following that one of $|W|$, $|T|$ is not equal to $2\delta - 1$, say $|T| > 2\delta - 1$ whenever $|W| = 2\delta - 1$. We consider five cases.

*Case* 1. $n$ is even and $|W| = 2\delta - 1$. As $n$ is even and $|W| = 2\delta - 1$, we know that $|T| = n - |W|$ is odd. Thus by inequalities (4) and (5),

$$\begin{aligned} R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \ge (2\delta-1)\delta^{-\frac{1}{2}} \\ &+ \left[\frac{|T|-1}{2}+\delta\right]\left[\frac{|T|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|T|+1}{2}-\delta\right]\left[\frac{|T|-1}{2}\right]^{-\frac{1}{2}} \\ &= (2\delta-1)\delta^{-\frac{1}{2}} + \left[\frac{n-2\delta}{2}+\delta\right]\left[\frac{n-2\delta+2}{2}\right]^{-\frac{1}{2}} \\ &+ \left[\frac{n-2\delta+2}{2}-\delta\right]\left[\frac{n-2\delta}{2}\right]^{-\frac{1}{2}} = (2\delta-1)\delta^{-\frac{1}{2}} \\ &+ \frac{n(n-2\delta+2)^{-\frac{1}{2}}}{\sqrt{2}} + \frac{(n-4\delta+2)(n-2\delta)^{-\frac{1}{2}}}{\sqrt{2}} = H_1\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

a contradiction to (2).

*Case* 2. $n$ is odd and $|W| = 2\delta - 1$. Since $n$ is odd and $|W| = 2\delta - 1$, we know that $|T| = n - |W|$ is even. Thus by inequalities (3) and (5),

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq (2\delta-1)\delta^{-\frac{1}{2}} + (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}} \\
&= (2\delta-1)\delta^{-\frac{1}{2}} + (n-3\delta+1)\left[\frac{n-2\delta+1}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{n-2\delta+3}{2}\right]^{-\frac{1}{2}} \\
&= (2\delta-1)\delta^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+3)^{-\frac{1}{2}} + \sqrt{2}(n-3\delta+1)(n-2\delta+1)^{-\frac{1}{2}} \\
&= H_2\left(-\frac{1}{2},\delta\right),
\end{aligned}$$

a contradiction to (2).

*Case* 3. $n$ and $|W|$ are both even, and $|W| > 2\delta - 1$. Immediately, we have that $|T| = n - |W|$ is also even. Thus by inequality (3),

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq (|W|-\delta)\left[\frac{|W|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}} \\
&= (|W|-\delta)\left[\frac{|W|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + (n-|W|-\delta)\left[\frac{n-|W|}{2}\right]^{-\frac{1}{2}} \\
&\quad + \delta\left[\frac{n-|W|}{2}+1\right]^{-\frac{1}{2}} = \sqrt{2}|W|^{\frac{1}{2}} + \sqrt{2}(n-|W|)^{\frac{1}{2}} - \delta h(|W|),
\end{aligned}$$

where $h(|W|) = \left[\frac{|W|}{2}\right]^{-\frac{1}{2}} - \left[\frac{|W|}{2}+1\right]^{-\frac{1}{2}} + \left[\frac{n-|W|}{2}\right]^{-\frac{1}{2}} - \left[\frac{n-|W|}{2}+1\right]^{-\frac{1}{2}}$.

Define a function $f_1(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}} - \left(\frac{x}{2}+1\right)^{-\frac{1}{2}}$. It is easy to verify that $f_1''(x) > 0$ for $x > 0$ , hence the function $f_1(x)$ is convex. By $|W|, |T| \in [2\delta, n-2\delta]$ and Lemma 5, we have $f_1(|W|)+f_1(n-|W|) \leq f_1(2\delta)+f_1(n-2\delta)$ and $f_2(x) = x^{\frac{1}{2}}$ is an increasing function for $x > 0$, thus

$$\begin{aligned}
R^0(G) &\geq \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} - \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + \left(\frac{n-2\delta}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-2\delta}{2}+1\right)^{-\frac{1}{2}}\right] \\
&= 3\delta^{\frac{1}{2}} + \delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+2)^{-\frac{1}{2}} = H_3\left(-\frac{1}{2},\delta\right),
\end{aligned}$$

a contradiction to (2).

*Case* 4. $n$ is even, $|W|$ is odd and $|W| > 2\delta + 1$. Then $|T| = n - |W|$ is odd. Since $|W|, |T| \geq 2\delta + 1$ and by (4),

$$\begin{aligned}
&R^0(G) = \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t)\\
&\geq \left[\frac{|W|-1}{2} + \delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|+1}{2} - \delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}}\\
&+ \left[\frac{n-|W|-1}{2} + \delta\right]\left[\frac{n-|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{n-|W|+1}{2} - \delta\right]\left[\frac{n-|W|-1}{2}\right]^{-\frac{1}{2}}\\
&= \frac{(|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&+ \frac{(n-|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(n-|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(n-|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&- \delta\left[\left(\frac{|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}}\right].
\end{aligned}$$

Define a function $f_3(x) = \left(\frac{x-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{x+1}{2}\right)^{-\frac{1}{2}}$. It is easy to verify that $f_3''(x) > 0$ for $x > 0$. Hence $f_3(x)$ is convex. Then we have

$$f_3(|W|) + f_3(n - |W|) \leq f_3(2\delta + 1) + f_3(n - 2\delta - 1).$$

Note that $f_2(x) = x^{\frac{1}{2}}$ is an increasing function and $f_4(x) = x^{-\frac{1}{2}}$ is a decreasing function for $x > 0$. Hence

$$\begin{aligned}
R^0(G) \geq\ & \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&+ \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\\
&- \delta\left[\delta^{-\frac{1}{2}} - (\delta+1)^{-\frac{1}{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} - \frac{(n-2\delta)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}}\right]\\
=\ & \delta^{\frac{1}{2}} + 3\delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}(\delta-2)(n-2\delta-2)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}}\\
=\ & H_4(-\frac{1}{2}, \delta) + \delta(\delta+1)^{-\frac{1}{2}} - \delta^{\frac{1}{2}} + \sqrt{2}(n-2\delta-2)^{-\frac{1}{2}} \geq H_4\left(-\frac{1}{2}, \delta\right),
\end{aligned}$$

a contradiction to (2).

*Case* 5. $n$ is odd and $|W| > 2\delta - 1$. Assume without loss of generality that $|W|$ is odd and $|T|$ is even. Then we have $|W| \geq 2\delta + 1$ and $|T| \geq 2\delta$. Thus by inequalities (3) and (4), we have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq \left[\frac{|W|-1}{2}+\delta\right]\left[\frac{|W|+1}{2}\right]^{-\frac{1}{2}} + \left[\frac{|W|+1}{2}-\delta\right]\left[\frac{|W|-1}{2}\right]^{-\frac{1}{2}} \\
&+ (|T|-\delta)\left[\frac{|T|}{2}\right]^{-\frac{1}{2}} + \delta\left[\frac{|T|}{2}+1\right]^{-\frac{1}{2}} \\
&= \frac{(|W|+1)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(|W|+1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(|W|-1)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(|W|-1)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(n-|W|)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&- \delta\left[\left(\frac{|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{|W|+1}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}}\right].
\end{aligned}$$

By Lemma 9, we obtain $\left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}} \leq \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}} + \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}}$, i.e., $\left(\frac{n-|W|}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|}{2}+1\right)^{-\frac{1}{2}} \leq \left(\frac{n-|W|-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-|W|+1}{2}\right)^{-\frac{1}{2}}$. Since $f_3(x) = \left(\frac{x-1}{2}\right)^{-\frac{1}{2}} - \left(\frac{x+1}{2}\right)^{-\frac{1}{2}}$ is convex and

$$f_3(|W|) + f_3(n-|W|) \leq f_3(2\delta+1) + f_3(n-2\delta-1).$$

Note that $f_2(x) = x^{\frac{1}{2}}$ is an increasing function and $f_4(x) = x^{-\frac{1}{2}}$ is a decreasing function $x > 0$. Therefore, we have

$$\begin{aligned}
R^0(G) &= \sum_{w\in W} d^{-\frac{1}{2}}(w) + \sum_{t\in T} d^{-\frac{1}{2}}(t) \\
&\geq \frac{(2\delta+2)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(2\delta+2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(n-2\delta-2)^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} + \frac{(2\delta)^{\frac{1}{2}}}{2^{-\frac{1}{2}}} \\
&- \delta\left[\left(\frac{2\delta}{2}\right)^{-\frac{1}{2}} - \left(\frac{2\delta+2}{2}\right)^{-\frac{1}{2}} + \left(\frac{n-2\delta-2}{2}\right)^{-\frac{1}{2}} - \left(\frac{n-2\delta}{2}\right)^{-\frac{1}{2}}\right] \\
&= 2\delta^{\frac{1}{2}} + 2\delta(\delta+1)^{-\frac{1}{2}} - \sqrt{2}(\delta-1)(n-2\delta-2)^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta)^{-\frac{1}{2}} \\
&= H_4\left(-\frac{1}{2}, \delta\right),
\end{aligned}$$

a contradiction to (2). ■

We present a class of graphs to show that the condition in Theorem 12 cannot be improved.

**Example 13.** For arbitrary integers $\delta$ and $n \geq 4\delta - 1$, let $G_1 \cong K_{\delta-1,\delta}$ with vertex set $V(K_{\delta-1,\delta}) = W \cup T$ (where $W = \{w_1, w_2, \ldots, w_{\delta-1}\}$ and $T = \{t_1, t_2, \ldots, t_\delta\}$). Let $G_2 \cong K_{\frac{n-2\delta+1}{2}, \frac{n-2\delta+1}{2}}$ with vertex set $V(G_2) = X \cup Y$ (where $X = \{x_1, x_2, \ldots, x_{\frac{n-2\delta+1}{2}}\}$ and $Y = \{y_1, y_2, \ldots, y_{\frac{n-2\delta+1}{2}}\}$). We define the graph $G$ as the union of $G_1$ and $G_2$ together with the $\delta$ edges $t_1x_1, t_2x_2, \ldots, t_\delta x_\delta$. Clearly, $G$ is triangle-free, has order $n$ and minimum degree $\delta$ and

$$\begin{aligned} R^0(G) &= (2\delta-1)\delta^{-\frac{1}{2}} + \sqrt{2}\delta(n-2\delta+3)^{-\frac{1}{2}} \\ &\quad + \sqrt{2}(n-3\delta+1)(n-2\delta+1)^{-\frac{1}{2}} = H_3\left(-\frac{1}{2}, \delta\right), \end{aligned}$$

but it is easy to see that $G$ is not super-$\lambda$.

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Received 12 April 2017
Revised 19 April 2018
Accepted 8 June 2018