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CONNECTED DOMINATION CRITICAL GRAPHS WITH CUT VERTICES

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Abstract

A graph G is said to be k-γ_c-critical if the connected domination number of G, $\gamma_c(G)$, is k and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. Let G be a k-γ_c-critical graph and $\zeta(G)$ the number of cut vertices of G. It was proved, in [1, 6], that, for $3 \leq k \leq 4$, every k-γ_c-critical graph satisfies $\zeta(G) \leq k - 2$. In this paper, we generalize that every k-γ_c-critical graph satisfies $\zeta(G) \leq k - 2$ for all $k \geq 5$. We also characterize all k-γ_c-critical graphs when $\zeta(G)$ is achieving the upper bound.

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1. Introduction

All graphs in this paper are finite, undirected and simple (no loops or multiple edges). For a graph G, let $V(G)$ denote the set of all vertices of G and let $E(G)$ denote the set of all edges of G. The *complement* $\overline{G}$ of G is the graph having the same set of vertices as G but the edge e is in $E(\overline{G})$ if and only if $e \notin E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The *open neighborhood* $N_G(v)$ of a vertex v in G is $\{u \in V(G) : uv \in E(G)\}$. Further, the *closed neighborhood* $N_G[v]$ of a vertex v in G is $N_G(v) \cup \{v\}$. For subsets X and Y of $V(G)$, $N_Y(X)$ is the set $\{y \in Y : yx \in E(G)$ for some $x \in X\}$. For a subgraph H of G, we use $N_Y(H)$ instead of $N_Y(V(H))$ and we use $N_H(X)$ instead of $N_{V(H)}(X)$. If $X = \{x\}$, we use $N_Y(x)$ instead of $N_Y(\{x\})$. The *degree* $deg(x)$ of a vertex x in G is $|N_G(x)|$. When no ambiguity occur, we write $N(x)$ and $N(X)$ instead of $N_G(x)$ and $N_G(X)$, respectively. An *end vertex* is a vertex of degree one and a *support vertex* is the vertex which is adjacent to an end vertex. A *star* $K_{1,n}$ is a graph of order $n+1$ containing one support vertex and n end vertices. The support vertex of a star is called the *center*. For a connected graph G, a vertex v of G is called a *cut vertex* if $G - v$ is not connected. The number of cut vertices of G is denoted by $\zeta(G)$. A *block* B of a graph G is a maximal connected subgraph such that B has no cut vertex. An *end block* of G is a block containing exactly one cut vertex of G. The *distance* $d(u, v)$ between vertices u and v of G is the length of a shortest (u, v)-path in G. The diameter of G $diam(G)$ is the maximum distance of any two vertices of G. For a connected graph G, a *bridge* xy of G is an edge such that $G - xy$ is not connected.

For a finite sequence of graphs $G_1, \ldots, G_l$ for $l \geq 2$, the *joins* $G_1 \vee \cdots \vee G_l$ is the graph consisting of the disjoint union of $G_1, \ldots, G_l$ and each vertex in G_i is joined to all vertices in G_{i+1} for $1 \leq i \leq l-1$ by edges. If $V(G_i) = \{x\}$, then we simply write $G_1 \vee \cdots \vee G_{i-1} \vee x \vee G_{i+1} \vee \cdots \vee G_l$. Moreover, for a subgraph H of G_2, the *join* $G_1 \vee_H G_2$ is the graph consisting of the disjoint union of G_1 and G_2 and edges that join each vertex in G_1 to each vertex in H.

For subsets D and X of $V(G)$, D *dominates* X if every vertex in X is either in D or adjacent to a vertex in D. If D dominates X, then we write $D \succ X$. We also write $a \succ X$ when $D = \{a\}$ and $D \succ x$ when $X = \{x\}$. Moreover, if $X = V(G)$, then D is a *dominating set* of G and we write $D \succ G$ instead of $D \succ V(G)$. A *connected dominating set* of a graph G is a dominating set D of G such that $G[D]$ is connected. If D is a connected dominating set of G, we then write $D \succ_c G$. A smallest connected dominating set is called a *γ_c-set*. The cardinality of a γ_c-set is called the *connected domination number* of G and is denoted by $\gamma_c(G)$. A graph G is said to be *k-γ_c-critical* if $\gamma_c(G) = k$ and $\gamma_c(G + uv) < k$ for any pair of non-adjacent vertices u and v of G.

For related results on k-γ_c-critical graphs, Chen *et al.* [3] completely characterized these graphs when $1 \leq k \leq 2$.

Theorem 1 [3]**.** *A graph G is 1-γ_c-critical if and only if G is a complete graph. Moreover, a graph G is 2-γ_c-critical if and only if $\overline{G} = \bigcup_{i=1}^{l} K_{1,n_i}$, where $l \geq 2$ and $n_i \geq 1$ for all $1 \leq i \leq l$.*

By Theorem 1, we observe that a k-γ_c-critical graph does not contain a cut vertex when $1 \leq k \leq 2$.

Observation 2. *Let G be a k-γ_c-critical graph with $1 \leq k \leq 2$. Then G has no cut vertex.*

For $k \geq 3$, there is no complete characterization of these graphs so far. However, there are some structural characterizations of k-γ_c-critical graphs when $3 \leq k \leq 4$ by focusing on the maximum number of cut vertices of the graphs. Ananchuen [1] proved that the number of cut vertices of a 3-γ_c-critical graph does not exceed one.

Theorem 3 [1]**.** *Let G be a 3-γ_c-critical graph. Then G contains at most one cut vertex.*

In our previous work in [6], we established the maximum number of cut vertices that 4-γ_c-critical graphs can have.

Theorem 4 [6]**.** *Let G be a 4-γ_c-critical graph. Then G contains at most two cut vertices.*

By these results, we naturally, ask for $k \geq 5$, whether every k-γ_c-critical graph contains at most $k - 2$ cut vertices. It turns out affirmatively as we shall see in the following theorem.

Theorem 5. *For $k \geq 5$, let G be a k-γ_c-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G) \leq k - 2$.*

The proof of this theorem is presented in Section 4. In this paper, we also characterize all k-γ_c-critical graphs when the number of cut vertices is achieving the upper bound.

For the outline of this paper, we provide related results and prove that there exists a forbidden subgraph of k-γ_c-critical graphs in Section 2. In Section 3, we characterize some end blocks of G. We then use the results from Sections 2 and 3 to establish the upper bound of the number of cut vertices of k-γ_c-critical graphs in Section 4. We also characterize all k-γ_c-critical graphs when $\zeta(G) = k - 2$ in Section 5. Finally, we discuss our result with some related result in another type of domination critical graphs in Section 6.

2. Related Results

In this section, we state a number of results that we make use of in establishing our theorems. At the end of this section, we also prove some crucial results which will be used to settle the maximum number of cut vertices of k-γ_c-critical graphs in Section 4. We begin with a result of Chartrand and Oellermann [2] which gives the relationship between the numbers of end blocks and cut vertices.

Lemma 6 (see [2], page 24)**.** *Let G be a connected graph with at least one cut vertex. Then G has at least two end blocks.*

In [3], Chen *et al.* established fundamental properties of k-γ_c-critical graphs.

Lemma 7 [3]**.** *Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G. Further, let D_{xy} be a γ_c-set of $G + xy$. Then*

(1) $k - 2 \leq |D_{xy}| \leq k - 1$,

(2) $D_{xy} \cap \{x, y\} \neq \emptyset$, *and*

(3) *if* $\{x\} = \{x, y\} \cap D_{xy}$, *then* $N_G(y) \cap D_{xy} = \emptyset$.

In [5], we further observed some structure of the subgraph of G (not $G + xy$) induced by D_{xy}. For completeness, we provide the proof.

Observation 8. *If $\{x, y\} \subseteq D_{xy}$, then $G[D_{xy}]$ consists of 2 components and each of which contains exactly one vertex of $\{x, y\}$.*

Proof. If $G[D_{xy}]$ is connected, then D_{xy} is a connected dominating set of G. It follows by Lemma 7(1) that $\gamma_c(G) \leq k - 1$, contradiction. Thus $G[D_{xy}]$ is not connected. As $(G + xy)[D_{xy}]$ is connected and xy is the only one edge which is added to G, it follows that xy is a bridge of $(G+xy)[D_{xy}]$. Therefore, $G[D_{xy}]$ has exactly 2 components and each of which contains exactly one vertex of $\{x, y\}$. This completes the proof. ■

When $k \geq 3$, Ananchuen [1] established structures of k-γ_c-critical graphs with a cut vertex.

Lemma 9 [1]**.** *For $k \geq 3$, let G be a k-γ_c-critical graph with a cut vertex c and let D be a connected dominating set. Then*

(1) *$G - c$ contains exactly two components,*

(2) *if C_1 and C_2 are the components of $G - c$, then $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete and*

(3) $c \in D$.

In our previous work in [6], we established the diameter of k-γ_c-critical graphs.

Lemma 10 [6]**.** *Let G be a k-γ_c-critical graph. Then $diam(G) \leq k$.*

We conclude this section by establishing a forbidden subgraph of k-γ_c-critical graphs when $k \geq 3$ in Lemma 12. We also need to prove the following lemma.

Lemma 11. *Let G be a k-γ_c-critical graph and let x and y be a pair of non-adjacent vertices of G such that $|D_{xy} \cap \{x, y\}| = 1$. Then, for a pair of vertices a and b in D_{xy}, we have that $N(a) \nsubseteq N[b]$.*

Proof. Suppose to the contrary that $N(a) \subseteq N[b]$ for some $a, b \in D_{xy}$.

Claim. $D_{xy} - \{a\} \succ_c a$.

Proof. As $|D_{xy} \cap \{x, y\}| = 1$, we must have $G[D_{xy}]$ is connected. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $G[D_{xy} - \{a\}]$ is connected. We next show that $D_{xy} - \{a\} \succ a$. As $G[D_{xy}]$ is connected, a must be adjacent to a vertex in D_{xy}. That is $D_{xy} - \{a\} \succ a$. Therefore $D_{xy} - \{a\} \succ_c a$. This settles the claim. □

Since $|D_{xy} \cap \{x, y\}| = 1$, we may assume without loss of generality that $\{x\} = D_{xy} \cap \{x, y\}$. We distinguish two cases.

Case 1. $a \neq x$. Because $N(a) \subseteq N[b]$ and $b \in D_{xy}$, it follows that $D_{xy} - \{a\} \succ V(G + xy) - \{a\}$. Thus, by the claim, we have $D_{xy} - \{a\} \succ_c G + xy$. This contradicts the minimality of D_{xy}. So Case 1 cannot occur.

Case 2. $a = x$. As $N(a) \subseteq N[b]$, we must have $D_{xy} - \{a\} \succ V(G+xy) - \{y, a\}$. By the claim, $D_{xy} - \{a\} \succ_c V(G) - \{y\}$. Because G is connected, it follows that $N(y) \neq \emptyset$. Let $z \in N(y)$. By Lemma 7(3), $z \notin D_{xy}$. As $D_{xy} \succ_c G + xy$, we must have that z is adjacent to a vertex in D_{xy}. If $za \notin E(G)$, then $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Lemma 7(1) implies that $|(D_{xy} - \{a\}) \cup \{z\}| \leq k - 1$ contradicting the minimality of $\gamma_c(G)$. Therefore, $za \in E(G)$. As $N(a) \subseteq N[b]$, we must have $zb \in E(G)$. Since $b \in D_{xy}$, it follows that $(D_{xy} - \{a\}) \cup \{z\} \succ_c G$. Similarly, $|(D_{xy} - \{a\}) \cup \{z\}| \leq k - 1$, a contradiction. So Case 2 cannot occur and this completes the proof. ■

We are ready to provide the construction of a forbidden subgraph of k-γ_c-critical graphs. For a connected graph G, let X, Y, X_1 and Y_1 be disjoint vertex subsets of $V(G)$. We, further, let $Z = X \cup X_1 \cup Y \cup Y_1$ and $\overline{Z} = V(G) - Z$. The induced subgraph $G[Z]$ is called a *bad subgraph* if

(i) $x_1 \succ X \cup X_1$ for any vertex $x_1 \in X_1$,

(ii) $N[x] \subseteq X \cup X_1$ for any vertex $x \in X$,

(iii) $y_1 \succ Y \cup Y_1$ for any vertex $y_1 \in Y_1$, and

(iv) $N[y] \subseteq Y \cup Y_1$ for any vertex $y \in Y$.

Figure 1 illustrates our set up.

Observe that $G[X_1]$ and $G[Y_1]$ are complete subgraphs. Further, if $\overline{Z} = \emptyset$, then there exists an edge x_1y_1 where $x_1 \in X_1$and $y_1 \in Y_1$ because G is connected. Thus $\{x_1, y_1\} \succ_c G$. This implies that $\gamma_c(G) \leq 2$. Therefore, if $\gamma_c(G) \geq 3$, then

$\overline{Z} \neq \emptyset$. The next lemma gives that every k-γ_c-critical graph has no bad subgraph as an induced subgraph.

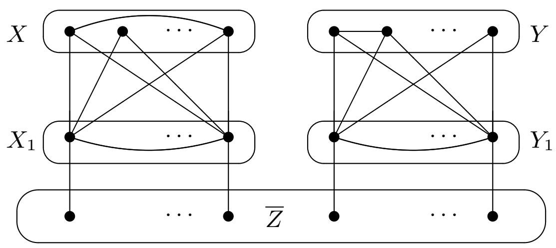

Figure 1. The induced subgraph $G[Z]$.

Lemma 12. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then G does not contain a bad subgraph as an induced subgraph.*

Proof. Suppose to the contrary that G contains $G[Z]$ as a bad subgraph. Let $x \in X$ and $y \in Y$. Consider $G + xy$. Lemma 7(2) implies that $D_{xy} \cap \{x, y\} \neq \emptyset$.

We first show that $\{x, y\} \subseteq D_{xy}$. Suppose to the contrary that $|D_{xy} \cap \{x, y\}| = 1$. Without loss of generality let $\{x\} = D_{xy} \cap \{x, y\}$. Since x is not adjacent to any vertex in Y_1, in order to dominate Y_1, $D_{xy} \cap (V(G) - X) \neq \emptyset$. Because $N[x] \subseteq X \cup X_1$, by the connectedness of $(G + xy)[D_{xy}]$, $D_{xy} \cap X_1 \neq \emptyset$. Let $x_1 \in D_{xy} \cap X_1$. Thus $N(x) \subseteq N[x_1]$ contradicting Lemma 11. Hence $\{x, y\} \subseteq D_{xy}$.

By Observation 8, $G[D_{xy}]$ has exactly two components H_1 and H_2 containing x and y, respectively. Let

$$U_1 = N(H_1) - V(H_1) \text{ and } U_2 = N(H_2) - V(H_2).$$

Thus $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$ because $D_{xy} = V(H_1) \cup V(H_2)$ and $D_{xy} \succ_c G + xy$. We next establish the following claim.

Claim. *For a vertex $u \in V(H_1) \cup U_1$, if $(V(H_1) \cup \{u\}) \cap X_1 \neq \emptyset$, then $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$. Similarly, for a vertex $v \in V(H_2) \cup U_2$, if $(V(H_2) \cup \{v\}) \cap Y_1 \neq \emptyset$, then $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$.*

Proof. Suppose that there exists $x_1 \in (V(H_1) \cup \{u\}) \cap X_1$. By Property (i) of bad subgraph, $x_1 \succ X \cup X_1$. Hence, $N[x] \subseteq N[x_1]$. Clearly, $G[V(H_1) \cup \{u\}]$ is connected. Since $x_1 \in V(H_1 - x) \cup \{u\}$, it follows that $G[V(H_1 - x) \cup \{u\}]$ is connected. As $N[x] \subseteq N[x_1]$, we must have $V(H_1 - x) \cup \{u\} \succ_c x$. Thus, it remains to show that $V(H_1 - x) \cup \{u\} \succ U_1$. Let $w \in U_1$. So, w is adjacent to a vertex of H_1. If $wx \notin E(G)$, then w is adjacent to a vertex of $H_1 - x$. But, if $wx \in E(G)$, then $wx_1 \in E(G)$. These imply that w is adjacent to a vertex in

$V(H_1-x)\cup\{u\}$. So $V(H_1-x)\cup\{u\} \succ U_1$. Therefore, $V(H_1-x)\cup\{u\} \succ_c U_1\cup\{x\}$. We can show that if $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$, then $V(H_2-y)\cup\{v\} \succ_c U_2\cup\{y\}$ by the similar arguments. This settles the claim. □

We note by the claim that u can be a vertex in H_1. Thus if $V(H_1)\cap X_1 \neq \emptyset$, then $V(H_1-x) \succ_c U_1\cup\{x\}$. Clearly $\overline{Z} \neq \emptyset$ because $k \geq 3$. To dominate $\overline{Z}$, we have $D_{xy}\cap(\overline{Z}\cup X_1\cup Y_1) \neq \emptyset$ because $N[x] \subseteq X\cup X_1$ and $N[y] \subseteq Y\cup Y_1$. Thus, by the connectedness of H_1 and H_2, $V(H_1)\cap X_1 \neq \emptyset$ or $V(H_2)\cap Y_1 \neq \emptyset$. Suppose without loss of generality that $V(H_1)\cap X_1 \neq \emptyset$. By applying the claim, we have that

$$V(H_1-x) \succ_c U_1\cup\{x\}. \tag{1}$$

Case 1. $U_1\cap U_2 \neq \emptyset$. Thus there is a vertex $v \in V(G)-(V(H_1)\cup V(H_2))$ such that v is adjacent to a vertex of H_1 and a vertex of H_2. That is $G[V(H_1)\cup\{v\}\cup V(H_2)]$ is connected.

We next show that $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. Suppose that $(V(H_2)\cup\{v\})\cap Y_1 = \emptyset$. By the connectedness of H_2, $V(H_2) \subseteq Y$ because $y \in Y$. Moreover, $v \in Y$ because v is adjacent to a vertex of H_2. So, Property (iv) implies that $N[v] \subseteq Y\cup Y_1$. As v is adjacent to a vertex of H_1, we must have that $V(H_1)\cap(Y\cup Y_1) \neq \emptyset$. By the connectedness of H_1, $V(H_1)\cap Y_1 \neq \emptyset$. Property (iii) yields that there exists a vertex of H_1 adjacent to a vertex of H_2. So H_1 and H_2 are the same component, a contradiction. Hence $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. By the claim, we have that

$$V(H_2-y)\cup\{v\} \succ_c U_2\cup\{y\}. \tag{2}$$

Since $V(G) = U_1\cup U_2\cup V(H_1)\cup V(H_2)$, by (1) and (2), $V(H_1-x)\cup V(H_2-y)\cup\{v\} \succ_c G$. Hence

$$(D_{xy}-\{x,y\})\cup\{v\} = V(H_1-x)\cup V(H_2-y)\cup\{v\} \succ_c G.$$

Lemma 7(1) yields that $|(D_{xy}-\{x,y\})\cup\{v\}| \leq k-1$ contradicting $\gamma_c(G)=k$. So Case 1 cannot occur.

Case 2. $U_1\cap U_2 = \emptyset$. Since G is connected, there exist vertices u and v in $V(G)-(V(H_1)\cup V(H_2))$ such that $u \in U_1, v \in U_2$ and $uv \in E(G)$. Therefore $G[V(H_1)\cup\{u,v\}\cup V(H_2)]$ is connected.

We will show that $(V(H_1)\cup\{u\})\cap X_1 \neq \emptyset$ and $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$. Suppose to the contrary that $(V(H_1)\cup\{u\})\cap X_1 = \emptyset$. So $V(H_1)\cup\{u\} \subseteq X$ by the connectedness of $G[V(H_1)\cup\{u\}]$. Since H_1 and H_2 are different components, by Property (i), $V(H_2)\cap X_1 = \emptyset$. Thus $v \in X_1$ because $uv \in E(G)$ and $N[u] \subseteq X\cup X_1$. This implies by Property (i) that $v \succ H_1$, in particular $v \in U_1$. Thus $v \in U_1\cap U_2$. This contradicts $U_1\cap U_2 = \emptyset$. Hence, $(V(H_1)\cup\{u\})\cap X_1 \neq \emptyset$. By the same arguments, we have $(V(H_2)\cup\{v\})\cap Y_1 \neq \emptyset$.

Hence, by the claim, we have that $V(H_1 - x) \cup \{u\} \succ_c U_1 \cup \{x\}$ and $V(H_2 - y) \cup \{v\} \succ_c U_2 \cup \{y\}$. As $V(G) = U_1 \cup U_2 \cup V(H_1) \cup V(H_2)$, we must have that $(D_{xy} - \{x, y\}) \cup \{u, v\} \succ_c G$. Lemma 7(1) gives that $|(D_{xy} - \{x, y\}) \cup \{u, v\}| \leq k-1$ contradicting $\gamma_c(G) = k$. So Case 2 cannot occur. Therefore G does not contain a bad subgraph as an induced subgraph. This completes the proof. ■

By applying Lemma 12, we easily establish the maximum number of end vertices of k-γ_c-critical graphs.

Corollary 13 [8]**.** *For $k \geq 3$, every k-γ_c-critical graph has at most one end vertex.*

Proof. Suppose to the contrary that G has x and y as two end vertices. Let x_1 and y_1 be the support vertices adjacent to x and y, respectively. Thus x_1 and y_1 are cut vertices. Since $\gamma_c(G) \geq 3$, $V(G) - \{x, x_1, y, y_1\} \neq \emptyset$. Thus, Lemma 9(1) implies that $x_1 \neq y_1$. Choose $X_1 = \{x_1\}, Y_1 = \{y_1\}, X = \{x\}$ and $Y = \{y\}$. Clearly $G[X_1 \cup Y_1 \cup X \cup Y]$ is a bad subgraph contradicting Lemma 12. Hence, G has at most one end vertex and this completes the proof. ■

It is worth noting that very recently Taylor and van der Merwe [8] proved Corollary 13 as well. They proved the corollary with contrapositive but did not apply the concept of a bad subgraph in their proof.

3. The Characterizations of Some End Blocks

In this section, we provide characterizations of some blocks of k-γ_c-critical graphs. For a connected graph G, we let $\mathcal{A}(G)$ be the set of all cut vertices of G.

We first show that for a connected graph G and a pair of non-adjacent vertices x and y of G, $\mathcal{A}(G) = \mathcal{A}(G + xy)$ if x and y are in the same block of G.

Lemma 14. *For a connected graph G, let B be a block of G and $x, y \in V(B)$ such that $xy \notin E(G)$. Then $\mathcal{A}(G) = \mathcal{A}(G + xy)$.*

Proof. Since G is a subgraph of $G+xy$, $\mathcal{A}(G+xy) \subseteq \mathcal{A}(G)$. Suppose there exists c such that $c \in \mathcal{A}(G)$ but $c \notin \mathcal{A}(G+xy)$. Thus $(G+xy)-c$ is connected. Let C be the component of $G-c$ containing vertices of $V(B)-\{c\}$ and C' be a component of $G-c$ which is not C. Further, let $a \in N_{C'}(c)$ and $b \in N_C(c)$. Since c is a cut vertex of G, there is only one path a, c, b from a to b. But c is not a cut vertex in $G + xy$. This implies that $G - c$ has a path $P = p_1, p_2, \ldots, x, y, \ldots, p_r$ from b to a where $b = p_1, a = p_r, x = p_i$ and $y = p_{i+1}$ for some $1 \leq i \leq r-1$ and $r \geq 2$. We see that P must contain an edge xy and $c \notin \{p_1, p_2, \ldots, p_r\}$. Since C and C' are the two different components of $G - c$, by the connectedness of the path P, $\{p_1, p_2, \ldots, p_i\} \subseteq V(C)$ and $\{p_{i+1}, \ldots, p_r\} \subseteq V(C')$. So $x \in V(C)$ and $y \in V(C')$

contradicting x and y are in the same block. Therefore $\mathcal{A}(G) \subseteq \mathcal{A}(G+xy)$ and thus, $\mathcal{A}(G) = \mathcal{A}(G+xy)$. This completes the proof. ■

For a k-γ_c-critical graph G with a cut vertex, let B be an end block of G containing non-adjacent vertices x and y. Clearly, $V(B+xy) = V(B)$.

Lemma 15. *For an integer $k \geq 3$, let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. For all $x, y \in V(B)$ such that $xy \notin E(G)$, $|D_{xy} \cap V(B+xy)| < |D \cap V(B)|$.*

Proof. Let c be the cut vertex of G such that $\mathcal{A}(G) \cap V(B) = \{c\}$. Note that $D - V(B)$ and $D \cap V(B)$ are disjoint as well as $D_{xy} - V(B+xy)$ and $D_{xy} \cap V(B+xy)$. We first establish the following claim.

Claim. $|D_{xy} - V(B+xy)| \geq |D - V(B)|$.

Proof. Suppose to the contrary that $|D_{xy} - V(B+xy)| < |D - V(B)|$. Clearly, $|D_{xy} - V(B+xy)| = |D_{xy} - V(B)|$. Thus $|D_{xy} - V(B)| < |D - V(B)|$. We will show that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G$. Firstly, we show that $G[(D_{xy} - V(B)) \cup (D \cap V(B))]$ is connected. As D_{xy} is a γ_c-set of $G+xy$, we must have that $(G+xy)[D_{xy}]$ is connected. Since $xy \in E(B+xy)$, we have that $G[(D_{xy} - V(B+xy)) \cup \{c\}]$ is connected. Hence, $G[(D_{xy} - V(B)) \cup \{c\}]$ is connected. Clearly, $G[D \cap V(B)]$ is connected. Moreover, $c \in D \cap V(B)$ by Lemma 9(3). Thus $G[(D_{xy} - V(B)) \cup (D \cap V(B))]$ is connected.

We next show that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ G$. Because $D_{xy} \succ_c G+xy$ and $xy \in E(B+xy)$, it follows that $(D_{xy} - V(B)) \cup \{c\} \succ V(G) - V(B)$. It is easy to see that $D \cap V(B) \succ V(B)$. So $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ G$. This implies that $(D_{xy} - V(B)) \cup (D \cap V(B)) \succ_c G$. But

$$\begin{aligned}
|(D_{xy} - V(B)) \cup (D \cap V(B))| &\leq |(D_{xy} - V(B))| + |(D \cap V(B))| \\
&< |(D - V(B))| + |(D \cap V(B))| \\
&= |(D - V(B)) \cup (D \cap V(B))| = |D|,
\end{aligned}$$

contradicting the minimality of D. Therefore $|D_{xy} - V(B+xy)| \geq |D - V(B)|$ and this settles the claim. □

We are now ready to prove this lemma. Suppose to the contrary that $|D_{xy} \cap V(B+xy)| \geq |D \cap V(B)|$. Thus

$$\begin{aligned}
|D_{xy}| &= |(D_{xy} - V(B+xy)) \cup (D_{xy} \cap V(B+xy))| \\
&= |(D_{xy} - V(B+xy))| + |(D_{xy} \cap V(B+xy))| \\
&\geq |(D - V(B))| + |(D_{xy} \cap V(B+xy))| \quad \text{(by the claim)} \\
&\geq |(D - V(B))| + |(D \cap V(B))| \\
&= |(D - V(B)) \cup (D \cap V(B))| = |D|,
\end{aligned}$$

contradicting Lemma 7(1). Therefore, $|D_{xy} \cap V(B + xy)| < |D \cap V(B)|$ and this completes the proof. ■

We now introduce four classes of graphs such that some graph in these classes is an end block of a k-γ_c-critical graph. For vertices c, z_1 and z_2, we let

$$\mathcal{B}_0 = \{c \vee K_{t_1} : \text{for an integer } t_1 \geq 1\},$$
$$\mathcal{B}_1 = \{c \vee K_{t_2} \vee z_1 : \text{for an integer } t_2 \geq 1\}, \text{ and}$$
$$\mathcal{B}_{2,1} = \{c \vee K_{t_3} \vee K_{t_4} \vee z_2 : \text{for integers } t_3, t_4 \geq 1\}.$$

Before we construct the next class, it is worth to introduce a graph T which occurs in the characterization of k-γ_c-critical graphs with a maximum number of cut vertices. For positive integers $l \geq 2, r$ and n_i, we let $\mathcal{S} = \bigcup_{i=1}^{l} K_{1,n_i}$ and

$$T = \mathcal{S} \text{ or}$$
$$T = \mathcal{S} \cup \overline{K_r}.$$

Then, for $1 \leq i \leq l$, we let $s_0^i, s_1^i, s_2^i, \ldots, s_{n_i}^i$ be the vertices of a star K_{1,n_i} centered at s_0^i. We, further, let $S = \bigcup_{i=1}^{l} \{s_1^i, s_2^i, \ldots, s_{n_i}^i\}$ and $S' = \bigcup_{i=1}^{l} \{s_0^i\}$, moreover, let $S'' = V(\overline{K_r})$ if $T = \mathcal{S} \cup \overline{K_r}$ and $S'' = \emptyset$ if $T = \mathcal{S}$. We note that

$$\overline{T} = \overline{\mathcal{S}} \text{ or}$$
$$\overline{T} = \overline{\mathcal{S}} \vee K_r.$$

That is, $\overline{T}$ can be obtained by removing the edges in the stars of $\mathcal{S}$ from a complete graph on $S \cup S' \cup S''$. Throughout this paper, we are, in fact, using the complement of T. We are ready to define the next class. Recall that, for graphs G_1 and G_2 such that G_2 has H as a subgraph, the join $G_1 \vee_H G_2$ is the graph constructed from the disjoint union of G_1 and G_2 by joining each vertex in G_1 to each vertex in H with an edge.

$$\mathcal{B}_{2,2} = \left\{c \vee_{\overline{T}[S]} \overline{T} : \text{for positive integers } l \geq 2, r \text{ and } n_i\right\}.$$

We note by the construction that, in $\overline{T}$, every vertex in S is adjacent to exactly $|S' \cup S''| - 1$ vertices in $S' \cup S''$. A graph in this class is illustrated in Figure 2. According to the figure, an *oval* denotes a complete subgraph, *double lines* between subgraphs denote joining every vertex of one subgraph to every vertex of the other subgraph and a *dash line* denotes a removed edge.

It is worth noting that, for an end block B of a k-γ_c-critical graph having D as a γ_c-set, the number of vertices in $D \cap V(B)$ can be as large as k. We will give an example by using the graph $\overline{T}$. For an integer $k \geq 5$, let $K_{n_1}, \ldots, K_{n_{k-3}}$ be $k-3$ copies of complete graphs with $n_1, \ldots, n_{k-3} \geq 2$ and let a_1 and a_2 be two isolated vertices. It is not difficult to see that the graph

$$a_1 \vee a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$$

is a k-γ_c-critical graph having $R = a_2 \vee K_{n_1} \vee \cdots \vee K_{n_{k-3}} \vee_{\overline{T}[S]} \overline{T}$ as an end block. Clearly, $|D \cap V(R)| = k$.

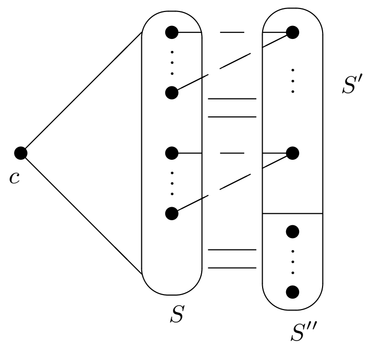

Figure 2. A graph G in the class $\mathcal{B}_{2,2}$.

In the following, we characterize an end block B such that $|D \cap V(B)| \leq 3$. Let c be the cut vertex of G in B and H be the component of $G - c$ such that $G[V(H) \cup \{c\}] = B$. We further let

$W = N_H(c)$,

$W' = \{w' \in V(H) - W : w'w \in E(G)$ for some $w \in W\}$ and

$W'' = V(H) - (W \cup W')$.

Note that W' or W'' can be empty. Since $c \in V(B)$, we have that $|D \cap V(H)| = i$ if and only if $|D \cap V(B)| = i + 1$ for all $i \geq 0$. Thus, $|D \cap V(B)| \geq 1$.

Lemma 16. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 1$, then $B \in \mathcal{B}_0$.*

Proof. In view of Lemma 9(2), $G[W]$ is complete. Lemma 9(3) gives, further, that $D \cap V(B) = \{c\}$. Since $D \succ B$ and $|(D \cap V(B)) - \{c\}| = 0$, it follows that $W' \cup W'' = \emptyset$ and $c \succ W$. So $B \in \mathcal{B}_0$. This completes the proof. ■

Lemma 17. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. If $|D \cap V(B)| = 2$, then $B \in \mathcal{B}_1$.*

Proof. Let $\{y\} = (D \cap V(B)) - \{c\}$. By the connectedness of $G[D]$, $y \in W$. Thus $W'' = \emptyset$ and $V(H) = W \cup W'$. Suppose that there exist $u, v \in V(H)$ such that $uv \notin V(G)$. Consider $G + uv$. Lemma 7(2) gives that $D_{uv} \cap \{u, v\} \neq \emptyset$. Lemma 14 gives also that $c \in D_{uv}$. Hence, $|D_{uv} \cap V(B + uv)| \geq 2$ contradicting Lemma 15. Thus $G[W \cup W']$ is complete. Let $z_1 \in W'$. Consider $G + cz_1$. Since $|D \cap V(B)| = 2$, by Lemma 15, $|D_{cz_1} \cap V(B + cz_1)| \leq 1$. Lemmas 9(3) and 14 yield that $c \in D_{cz_1}$. So $|D_{cz_1} \cap V(H)| = 0$. This implies that $c \succ B + cz_1$. Since $\{z_1\} = N_{G+cz_1}(c) \cap W'$, $W' = \{z_1\}$. So $B \in \mathcal{B}_1$ and this completes the proof. ■

Lemma 18. *Let G be a k-γ_c-critical graph with a γ_c-set D and let B be an end block of G. Suppose that $|D \cap V(B)| = 3$. Then $B \in \mathcal{B}_{2,1}$ if $W'' \neq \emptyset$ and $B \in \mathcal{B}_{2,2}$ if $W'' = \emptyset$. Consequently, $B \in \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2}$.*

Proof. Suppose that $|D \cap V(B)| = 3$. Lemma 9(2) implies that $G[W]$ is complete. We first establish the following claim.

Claim. *For any non-adjacent vertices $u, v \in W \cup W' \cup W''$, we have $c \in D_{uv} \cap V(B + uv)$ and $|D_{uv} \cap W \cap \{u, v\}| = 1$.*

Proof. Lemma 15 implies that $|D_{uv} \cap V(B + uv)| \leq 2$. In view of Lemmas 9(3) and 14, $c \in D_{uv} \cap V(B + uv)$. Thus $|D_{uv} \cap \{u, v\}| \leq 1$. Lemma 7(2) then gives that $|D_{uv} \cap \{u, v\}| = 1$. So $|D_{uv} \cap W \cap \{u, v\}| = 1$ because $(G + uv)[D_{uv}]$ is connected. This settles the claim. □

Suppose there exist $u, v \in W' \cup W''$ such that $uv \notin E(G)$. Consider $G + uv$. By the claim $|D_{uv} \cap W \cap \{u, v\}| = 1$ contradicting $W \cap \{u, v\} = \emptyset$. Thus $G[W' \cup W'']$ is complete.

We first consider the case when $W'' \neq \emptyset$. Let $w \in W$ and $z_2 \in W''$. Consider $G + wz_2$. By the claim, $D_{wz_2} \cap V(B + wz_2) = \{c, w\}$. Since $\{z_2\} = W'' \cap N_{G+wz_2}(w)$, it follows that $W'' = \{z_2\}$. Suppose there exists $w' \in W'$ such that $ww' \notin E(G)$. Consider $G + ww'$. By the claim, $D_{ww'} \cap V(B + ww') = \{c, w\}$. Thus $D_{ww'}$ does not dominate z_2, a contradiction. Therefore $G[W \cup W']$ is complete and $B \in \mathcal{B}_{2,1}$.

We finally consider the case when $W'' = \emptyset$. We will show that, for all $w \in W$, $|N_{W'}(w)| = |W'| - 1$. If $w \succ W'$, then $(D - V(H)) \cup \{w\} \succ_c G$. But $|(D - V(H)) \cup \{w\}| = k - 1$ contradicting $\gamma_c(G) = k$. Thus $|N_{W'}(w)| \leq |W'| - 1$. If w is not adjacent to x, y in W', then consider $G + wx$. By the claim, $D_{wx} \cap V(B + wx) = \{c, w\}$. Clearly D_{wx} does not dominate y, a contradiction. Hence, $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$. We now have that $G[W \cup W']$ is the complement of disjoint union of isolated vertices in W' and stars whose centers are in W' and all of end vertices are in W. It remains to show that there are at least two stars in $\overline{G}[W \cup W']$. Suppose to the contrary that, in $\overline{G}[W \cup W']$, there is exactly one star centered at w'. Because $|N_{W'}(w)| = |W'| - 1$ for all $w \in W$, w' is not adjacent to any vertex in W. So $w' \in W''$ contradicting $W'' = \emptyset$. Hence, there are at least two stars in $\overline{G}[W \cup W']$. This completes the proof. ■

4. The Upper Bound of the Number of Cut Vertices

In this section, we establish the maximum number of cut vertices of k-γ_c-critical graphs. In view of Observation 2, it suffices to restrict our attention to the case $k \geq 3$. We begin this section by showing that G does not have two end blocks in $\mathcal{B}_0 \cup \mathcal{B}_1$.

Lemma 19. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then G contains at most one end block B such that $B \in \mathcal{B}_0 \cup \mathcal{B}_1$.*

Proof. Suppose to the contrary that there exist two different end blocks U and R which are, respectively, in the classes $\mathcal{B}_i$ and $\mathcal{B}_j$ where $\{i, j\} \subseteq \{0, 1\}$. Let u be the cut vertex of G in U. If $U \in \mathcal{B}_0$, then $U = u \vee K_{t_1}$ for some integer $t_1 \geq 1$. If $U \in \mathcal{B}_1$, then there exist an integer $t_2 \geq 1$ and a vertex z_1 of U such that $U = u \vee K_{t_2} \vee z_1$. Then, we choose

$$X_1 = \begin{cases} \{u\} & \text{if } U \in \mathcal{B}_0, \\ V(K_{t_2}) & \text{if } U \in \mathcal{B}_1, \end{cases}$$

and we choose

$$X = \begin{cases} V(K_{t_1}) & \text{if } U \in \mathcal{B}_0, \\ \{z_1\} & \text{if } U \in \mathcal{B}_1. \end{cases}$$

Clearly, U contains X and X_1 which satisfy the Properties (i) and (ii), respectively.

We now consider R. Let r be the cut vertex of G in R. If $R \in \mathcal{B}_0$, then $R = r \vee K_{t'_1}$ for some integer $t'_1 \geq 1$. But, if $R \in \mathcal{B}_1$, then there exist an integer $t'_2 \geq 1$ and a vertex w_1 of R such that $R = r \vee K_{t'_2} \vee w_1$. Then, we choose

$$Y_1 = \begin{cases} \{r\} & \text{if } R \in \mathcal{B}_0, \\ V(K_{t'_2}) & \text{if } R \in \mathcal{B}_1, \end{cases}$$

and we choose

$$Y = \begin{cases} V(K_{t'_1}) & \text{if } R \in \mathcal{B}_0, \\ \{w_1\} & \text{if } R \in \mathcal{B}_1. \end{cases}$$

Clearly, R contains Y and Y_1 which satisfy the Properties (i) and (ii), respectively.

We observe that X, Y and Y_1 are pairwise disjoint because U and R are different blocks. Suppose that $Y_1 \cap X_1 \neq \emptyset$. By the choice of X_1 and Y_1, if $X_1 = V(K_{t_2})$ or $Y_1 = V(K_{t'_2})$, then $Y_1 \cap X_1 = \emptyset$ because U and R are different end blocks, contradicting the assumption that $Y_1 \cap X_1 \neq \emptyset$. Hence, $X_1 = \{u\}$ and $Y_1 = \{r\}$. This implies that $u = r$, moreover, U and R are both in $\mathcal{B}_0$. Thus $u \succ U$ and $u \succ R$. Lemma 9(1) yields that $G - u$ has $U - u$ and $R - u$ as the two components. We have that $G = K_{t_1} \vee u \vee K_{t'_1}$. Clearly, $u \succ_c G$ contradicting $\gamma_c(G) \geq 3$. Hence, $Y_1 \cap X_1 = \emptyset$. So, G contains a bad subgraph contradicting Lemma 12. This completes the proof. ■

In the following, for a block B of G, we let

$$\mathcal{A}(B) = V(B) \cap \mathcal{A}(G).$$

We also let

$$\zeta(G) = |\mathcal{A}(G)|, \zeta(B) = |\mathcal{A}(B)| \text{ and}$$
$$\zeta_0(G) = \max\{\zeta(B) : B \text{ is a block of } G\}.$$

When no ambiguity can occur, we abbreviate $\zeta_0(G)$ to ζ_0. Clearly, $\zeta_0 \le \zeta(G)$. In the following lemma, we establish the existence of ζ_0 end blocks.

Lemma 20. *For any k-γ_c-critical graph G, let B_0 be a block of G containing ζ_0 cut vertices $c_1, c_2, \ldots, c_{\zeta_0}$. Then there exist mutually disjoint end blocks $B_1, B_2, \ldots, B_{\zeta_0}$.*

Proof. In view of Lemma 9(1), $G - c_i$ has only two components for $1 \le i \le \zeta_0$. Let C_i be the component of $G - c_i$ that does not contain any vertex of B_0. If graph $G[\{c_i\} \cup V(C_i)]$ does not contain any cut vertex, then $G[\{c_i\} \cup V(C_i)]$ is an end block and we let $B_i = G[\{c_i\} \cup V(C_i)]$. If graph $G[\{c_i\} \cup V(C_i)]$ contains a cut vertex, then, by Lemma 6, $G[\{c_i\} \cup V(C_i)]$ has at least two end blocks. Therefore, at least one end block of $G[\{c_i\} \cup V(C_i)]$ does not contain c_i and we let B_i be this end block. In both cases of the choice, B_i is an end block of G. Obviously, $B_1, B_2, \ldots, B_{\zeta_0}$ are mutually disjoint and this completes the proof. ■

Lemma 21. *For $k \ge 3$, let G be a k-γ_c-critical graph with a γ_c-set D and let $B_1, B_2, \ldots, B_{\zeta_0}$ be the end blocks of G from Lemma 20. Moreover, for $1 \le i \le \zeta_0$, we let $x_i \in \mathcal{A}(G) \cap V(B_i)$. Then at least $\zeta_0 - 1$ of the end blocks $B_1, B_2, \ldots, B_{\zeta_0}$ satisfy $|(D \cap V(B_i)) - \{x_i\}| \ge 2$.*

Proof. Lemma 19 gives that at least $\zeta_0 - 1$ blocks of $\{B_i | 1 \le i \le \zeta_0\}$ are not in $\mathcal{B}_j$ where $j \in \{0, 1\}$. Without loss of generality let $B_1, B_2, \ldots, B_{\zeta_0 - 1}$ be such blocks. Hence

$$|(D \cap V(B_i)) - \{x_i\}| \ge 2$$

for $1 \le i \le \zeta_0 - 1$ and this completes the proof. ■

We next let $\overline{\mathcal{A}} = \mathcal{A}(G) - \mathcal{A}(B_0)$ and $\overline{\zeta} = |\overline{\mathcal{A}}|$. That is, $\overline{\mathcal{A}}$ is the set of cut vertices which are not in B_0. Clearly,

$$\zeta(G) = \overline{\zeta} + \zeta_0. \tag{3}$$

Recall that, for $1 \le i \le \zeta_0$, C_i is the component of $G - c_i$ which does not contain any vertex of B_0. We also let

$$j_0 = \min\left\{|D \cap V(C_i)| : \text{for all } 1 \le i \le \zeta_0\right\}.$$

The following theorem gives the relationship of $\zeta_0, \overline{\zeta}, j_0$ and k.

Theorem 22. *For $k \ge 3$, let G be a k-γ_c-critical graph. Then $3\zeta_0 - 2 + \overline{\zeta} + j_0 \le k$.*

Proof. Lemma 9(3) yields that $\mathcal{A}(G) \subseteq D$. For each end block B_i of G which is a consequent of Lemma 20, $1 \leq i \leq \zeta_0$, let $x_i \in \mathcal{A}(G) \cap V(B_i)$. Clearly $(D \cap V(B_1)) - \{x_1\}, (D \cap V(B_2)) - \{x_2\}, \ldots, (D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}$ and $\mathcal{A}(G)$ are pairwise disjoint. These imply that

$$\sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \leq k. \tag{4}$$

In view of Lemma 21, at least $\zeta_0 - 1$ end blocks of $B_1, B_2, \ldots, B_{\zeta_0}$ are not in $\mathcal{B}_0 \cup \mathcal{B}_1$. Without loss of generality let $B_1, B_2, \ldots, B_{\zeta_0 - 1}$ be such blocks. So $2 \leq |(D \cap V(B_i)) - \{x_i\}|$ for $1 \leq i \leq \zeta_0 - 1$. Therefore

$$2(\zeta_0 - 1) \leq \sum_{i=1}^{\zeta_0 - 1} \big|(D \cap V(B_i)) - \{x_i\}\big|. \tag{5}$$

By the minimality of j_0,

$$0 \leq j_0 \leq \big|(D \cap V(B_{\zeta_0})) - \{x_{\zeta_0}\}\big|. \tag{6}$$

Therefore

$$\begin{aligned} 3\zeta_0 - 2 + j_0 + \overline{\zeta} &= 2(\zeta_0 - 1) + j_0 + \overline{\zeta} + \zeta_0 \\ &\leq \sum_{i=1}^{\zeta_0 - 1} |(D \cap V(B_i)) - \{x_i\}| + j_0 + \zeta(G) \quad \text{(by (3) and (5))} \\ &\leq \sum_{i=1}^{\zeta_0} |(D \cap V(B_i)) - \{x_i\}| + \zeta(G) \quad \text{(by (6))} \\ &\leq k \quad \text{(by (4))}, \end{aligned}$$

as required. ■

Theorem 22 implies the following corollary.

Corollary 23. *For $k \geq 3$, let G be a k-γ_c-critical graph. Then $\zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor$.*

Proof. Theorem 22 implies that $3\zeta_0 \leq k + 2 - \overline{\zeta} - j_0$. As $\overline{\zeta}, j_0 \geq 0$, we must have that

$$\zeta_0 \leq \left\lfloor \frac{k+2}{3} \right\rfloor$$

and this completes the proof. ■

Note that Theorem 22 together with $\zeta(G) = \overline{\zeta} + \zeta_0$ give

$$2\zeta_0 \le k - \zeta(G) - j_0 + 2. \tag{7}$$

We are now ready to establish Theorem 5. For completeness, we recall the statement of this theorem.

Theorem 5. *For $k \ge 3$, let G be a k-γ_c-critical graph with $\zeta(G)$ cut vertices. Then $\zeta(G) \le k - 2$.*

Proof. Suppose to the contrary that $|\mathcal{A}(G)| > k - 2$. Lemma 9(3) gives that $|\mathcal{A}(G)| \le k$. Thus either $|\mathcal{A}(G)| = k$ or $|\mathcal{A}(G)| = k-1$, in particular, $k-\zeta(G) \le 1$. This implies by (7) that

$$2\zeta_0 \le k - \zeta(G) - j_0 + 2 \le 3.$$

Therefore

$$\zeta_0 \le 1.$$

If $\zeta(G) \ge 2$, then we always have a block containing more than one cut vertex. Thus $\zeta_0 \ge 2$, a contradiction. Therefore $\zeta(G) \le 1$. As $k - \zeta(G) \le 1$, we must have that

$$k \le 2,$$

contradicting $k \ge 3$. Hence $\zeta(G) \le k - 2$ and this completes the proof. ■

5. Characterizations

In this section, we characterize all k-γ_c-critical graphs G when $\zeta(G) = k - 2$. We first give the construction of a k-γ_c-critical graph with $k - 2$ cut vertices.

The class $\mathcal{F}(k)$

Let B be a graph in the class $\mathcal{B}_{2,2}$ containing c, S, S' and S'' which are defined in $\mathcal{B}_{2,2}$. We, further, let $P_{k-1} = z_0, z_1, \ldots, z_{k-2}$ be a path of order $k-1$. A graph G in the class $\mathcal{F}(k)$ is constructed from the graphs B and P_{k-1} by identifying z_{k-2} with c. A graph G in the class $\mathcal{F}(k)$ is illustrated in Figure 3.

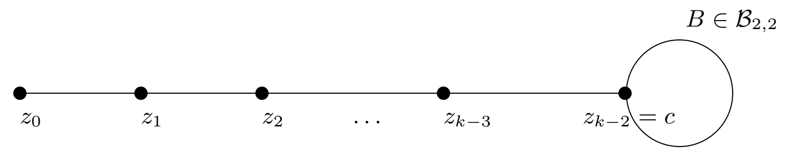

Figure 3. A graph G in the class $\mathcal{F}(k)$.

Lemma 24. *Let $G \in \mathcal{F}(k)$. Then G is a k-γ_c-critical graph with $k-2$ cut vertices.*

Proof. Clearly, $z_1, z_2, \ldots, z_{k-2}$ are the $k-2$ cut vertices of G. We observe that $\{z_1, z_2, \ldots, z_{k-2}, s_1^1, s_0^2\} \succ_c G$. Therefore $\gamma_c(G) \leq k$.

We next show that $\gamma_c(G) \geq k$. Let D be a γ_c-set of G. Since z_1 is a cut vertex, by the connectedness of $G[D]$, $z_1 \in D$. We first suppose that $D \cap S'' \neq \emptyset$. As $z_1 \in D$, by the connectedness of $G[D]$, we must have $\{z_2, z_3, \ldots, z_{k-2}, y\} \subseteq D$ where $y \in D \cap S$. Thus $\gamma_c(G) = |D| \geq k$ and $\gamma_c(G) = k$. We now suppose that $D \cap S'' = \emptyset$. To dominate B, $|D \cap (S \cup S')| \geq 2$. Similarly, by the connectedness of $G[D]$, we have $\{z_2, z_3, \ldots, z_{k-2}\} \subseteq D$ and thus $\gamma_c(G) \geq k$. Therefore $\gamma_c(G) = k$.

We next establish the criticality. Let u and v be two non-adjacent vertices of G and $S_1 = S \cup S' \cup S''$. We first consider the case when $\{u, v\} \subseteq S_1$. Thus $\{u, v\} = \{s_j^i, s_0^i\}$ for some $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, n_i\}$. Clearly $\{z_1, z_2, \ldots, z_{k-2}, s_j^i\} \succ_c G + uv$ and $\gamma_c(G + uv) \leq k - 1$.

We now consider the case when $|\{u, v\} \cap S_1| = 1$. Without loss of generality let $\{v\} = \{u, v\} \cap S_1$. If $u = z_{k-2}$, then $v \notin S$. So $\{z_{k-2}, v\} \succ S_1$. Thus $\{z_1, z_2, \ldots, z_{k-2}, v\} \succ_c G + uv$. Therefore $\gamma_c(G + uv) \leq k - 1$. Since $l \geq 2$, there exists $v' \in S - \{v\}$ such that $\{v, v'\} \succ_c S_1$. Then, if $u \in \{z_1, z_2, \ldots, z_{k-3}\}$, we have $\{z_1, z_2, \ldots, u, \ldots, z_{k-3}, v, v'\} \succ_c G + uv$. Hence $\gamma_c(G + uv) \leq k - 1$. If $u = z_0$, then $\{z_2, z_3, \ldots, z_{k-2}, v, v'\} \succ_c G + uv$ and thus, $\gamma_c(G + uv) \leq k - 1$.

We finally consider the case when $|\{u, v\} \cap S_1| = 0$. Therefore $\{u, v\} \subseteq \{z_0, z_1, \ldots, z_{k-2}\}$. Thus $u = z_i$ and $v = z_j$ for some $i \neq j \in \{0, 1, 2, \ldots, k-2\}$. Without loss of generality let $i < j$. Clearly $i + 2 \leq j$. Hence

$$\big(\{z_1, \ldots, z_{k-2}\} - \{z_{i+1}\}\big) \cup \{s_1^1, s_0^2\} \succ_c G + uv.$$

So $\gamma_c(G + uv) \leq k - 1$. Thus G is a k-γ_c-critical graph and this completes the proof. ■

Let $\mathcal{Z}(k, \zeta)$ be the class of k-γ_c-critical graphs containing ζ cut vertices. As the graphs in these class have been characterized in [1] and [6] when $3 \leq k \leq 4$, we turn attention to the case when $k \geq 5$.

Lemma 25. *For $k \geq 5$, let $G \in \mathcal{Z}(k, \zeta)$ where $\zeta \in \{k-3, k-2\}$. Then G has only two end blocks and the remaining blocks contain two cut vertices.*

Proof. Clearly $\zeta(G) \geq k - 3$. We have by (7) that

$$2\zeta_0 \leq k - \zeta(G) - j_0 + 2 \leq k - (k-3) - j_0 + 2 \leq 5.$$

That is $\zeta_0 \leq 2$. Lemma 9(1) implies that G has only two end blocks and the other blocks contain two cut vertices. This completes the proof. ■

In view of Lemma 25, hereafter, G has exactly two end blocks, R_1, R_{k-1} say, and the other blocks $R_2, R_3, \ldots, R_{k-2}$ which contain two cut vertices. Without loss of generality let $z_1 \in V(R_1), z_{k-2} \in V(R_{k-1})$ and $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$ (see Figure 4).

Lemma 26. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and R_1, R_{k-1} be two end blocks. Then $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$.*

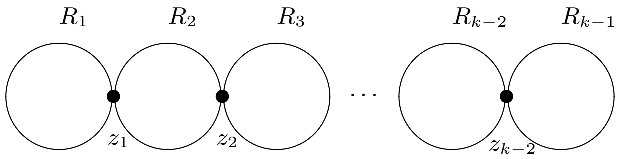

Figure 4. The structure of $G \in \mathcal{Z}(k, \zeta)$ where $\zeta \in \{k-3, k-2\}$.

Proof. Lemma 9(3) yields that $\mathcal{A}(G) \subseteq D$. As $\zeta(G) = k-2$, we must have $|D - \mathcal{A}(G)| = 2$. Clearly $(D - \mathcal{A}(G)) \cap (V(R_1) \cup V(R_{k-1})) \neq \emptyset$, otherwise $R_1, R_{k-1} \in \mathcal{B}_0$ contradicting Lemma 19.

Without loss of generality let $|(D \cap V(R_1)) - \{z_1\}| \leq |(D \cap V(R_{k-1})) - \{z_{k-2}\}|$. Suppose to the contrary that $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 1$. Thus $R_1, R_{k-1} \in \mathcal{B}_0 \cup \mathcal{B}_1$ contradicting Lemma 19. Hence $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$ and this completes the proof. ■

Lemma 27. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and $R_2, R_3, \ldots, R_{k-2}$ be blocks which contain two cut vertices such that $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$. Then*

$$\{z_{i-1}, z_i\} \succ_c R_i \quad \textit{for} \quad 2 \leq i \leq k-1, \textit{ in particular,} \quad z_{i-1}z_i \in E(G).$$

Proof. As $\zeta(G) = k-2$, by Lemma 26, we must have $D \cap V(R_i) = \{z_{i-1}, z_i\}$ for $2 \leq i \leq k-2$. Therefore

$$\{z_{i-1}, z_i\} \succ_c R_i.$$

Clearly, $z_{i-1}z_i \in E(G)$ and this completes the proof. ■

Lemma 28. *For $k \geq 5$, let $G \in \mathcal{Z}(k, k-2)$ and R_i be a block of G containing two cut vertices z_{i-1} and z_i for $2 \leq i \leq k-2$. Then $V(R_i) = \{z_{i-1}, z_i\}$.*

Proof. By Lemma 26, $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Without loss of generality let $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Since $|D \cap \mathcal{A}(G)| = k-2$, $|(D \cap V(R_1)) - \{z_1\}| = 0$ and thus, Lemma 16 gives that $R_1 \in \mathcal{B}_0$.

We consider the case when $i = 2$. Let $z \in V(R_1) - \{z_1\}$. Suppose there exists $u \in V(R_2) - \{z_1, z_2\}$. Consider $G + uz$. We see that z_2 is a cut vertex of $G + uz$.

Lemma 9(2) implies that $z_2 \in D_{uz}$. If $|D_{uz} - V(R_1)| \leq k-2$, then, by Lemma 27, $(D_{uz} - V(R_1)) \cup \{z_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Hence, by Lemma 7(1), $|D_{uz} - V(R_1)| = k-1$. Since $u \notin V(R_1)$, by Lemma 7(2), $\{u\} = D_{uz} \cap \{u, z\}$. Lemma 27 implies that $(D_{uz} - \{u\}) \cup \{z_1\} \succ_c G$. But $|(D_{uz} - \{u\}) \cup \{z_1\}| = k-1$ contradicting $\gamma_c(G) = k$. Hence, $V(R_2) = \{z_1, z_2\}$.

We consider the case when $3 \leq i \leq k-2$. Suppose to the contrary that $R_i' = V(R_i) - \{z_{i-1}, z_i\} \neq \emptyset$. Lemma 27 gives that $\{z_{i-1}, z_i\} \succ_c R_i'$ and $z_{i-1}z_i \in E(G)$. If there exists a vertex $b' \in R_i'$ which is not adjacent to z_j for some $j \in \{i, i-1\}$, then $b'z_{2i-1-j} \in E(G)$. Note that $b', z_j \in N_{R_i}(z_{2i-1-j})$. Thus $G[N_{R_i}(z_{2i-1-j})]$ is not a complete graph contradicting Lemma 9(2). Therefore, $z_i \succ R_i'$ and $z_{i-1} \succ R_i'$. We now have that $z_i \succ V(R_i), z_{i-1} \succ V(R_i)$ and $N[b'] \subseteq R_i' \cup \{z_{i-1}, z_i\}$ for all $b' \in R_i'$. Moreover, we have that $z_1 \succ R_1$ and $N[b] \subseteq V(R_1)$ for all $b \in V(R_1)$. Choose

$$X_1 = \{z_1\}, X = V(R_1) - \{z_1\}, Y = R_i' \text{ and } Y_1 = \{z_i, z_{i-1}\}.$$

Clearly X, X_1, Y and Y_1 form a bad subgraph. This contradicts Lemma 12. Hence, $R_i' = \emptyset$ for all $2 \leq i \leq k-3$. This completes the proof. ■

The following theorem gives the characterization of the graphs in the class $\mathcal{Z}(k, k-2)$.

Theorem 29. *For $k \geq 5$, we have that $\mathcal{Z}(k, k-2) = \mathcal{F}(k)$.*

Proof. Lemma 24 implies that $\mathcal{F}(k) \subseteq \mathcal{Z}(k, k-2)$. It suffices to show that a k-γ_c-critical graph with $k-2$ cut vertices is in $\mathcal{F}(k)$. Let G be a k-γ_c-critical graph with $k-2$ cut vertices. Lemma 25 implies that G has only two end blocks, R_1, R_{k-1} say, and the other blocks $R_2, R_3, \ldots, R_{k-2}$ which contain two cut vertices. Let $z_1 \in V(R_1), z_{k-2} \in V(R_{k-1})$ and $z_{i-1}, z_i \in V(R_i)$ for $2 \leq i \leq k-2$. Thus $\mathcal{A}(G) = \{z_1, z_2, \ldots, z_{k-2}\}$. Lemma 26 implies that $|(D \cap V(R_1)) - \{z_1\}| = 2$ or $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$. Without loss of generality let

$$|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2.$$

Thus $|(D \cap V(R_1)) - \{z_1\}| = 0$. By Lemma 16, $R_1 \in \mathcal{B}_0$. Clearly, $z_1 \succ R_1$. As $\zeta(G) = k-2$, we must have $D \cap V(R_i) = \{z_{i-1}, z_i\}$ for $2 \leq i \leq k-2$ and $D \cap V(R_1) = \{z_1\}$. By Lemma 27,

$$\{z_{i-1}, z_i\} \succ_c R_i.$$

Let $z_0 \in V(R_1) - \{z_1\}$. Clearly $d(z_1, z_0) = 1$. The following claim characterizes R_{k-1}.

Claim. $R_{k-1} \in \mathcal{B}_{2,2}$.

Proof. Since $|(D \cap V(R_{k-1})) - \{z_{k-2}\}| = 2$, there exists $w \in V(R_{k-1}) - \{z_{k-2}\}$ such that $d(w, z_{k-2}) \geq 2$. Thus

$$d(z_0, w) \geq d(z_0, z_1) + d(z_1, z_2) + \cdots + d(z_{k-2}, w) \geq k.$$

Lemma 10 gives that $d(z_0, w) = k$. Hence $d(z_{k-2}, w') \leq 2$ for all $w' \in V(R_{k-1}) - \{z_{k-2}\}$. So $R_{k-1} \notin \mathcal{B}_{2,1}$. By Lemma 18, $R_{k-2} \in \mathcal{B}_{2,2}$ and thus establishing the claim. □

Lemma 28 implies that, for all $i \in \{2, 3, \ldots, k-2\}$, $V(R_i) = \{z_{i-1}, z_i\}$. So far, it remains to show that $V(R_1) = \{z_1, z_0\}$. Consider $G + z_2z_0$. Since z_2 is a cut vertex of $G + z_2z_0$, $z_2 \in D_{z_2z_0}$ by the connectedness of $(G + z_2z_0)[D_{z_2z_0}]$. We note by Lemma 27 that $z_1z_2 \in E(G)$. Then, if $|D_{z_2z_0} - V(R_1)| \leq k-2$, we have that $(D_{z_2z_0} - V(R_1)) \cup \{z_1\} \succ_c G$ contradicting $\gamma_c(G) = k$. Therefore, by Lemma 7(1), $|D_{z_2z_0} - V(R_1)| = k-1$. Thus $\{z_2\} = \{z_2, z_0\} \cap D_{z_2z_0}$ and this implies that $z_2 \succ R_1$ in $G + z_2z_0$. Since $V(R_1) \cap N_{G+z_2z_0}(z_2) = \{z_0\}$, $V(R_1) = \{z_1, z_0\}$ and this completes the proof. ■

6. Discussion

In this section, we discuss the related result on an another type of domination critical graphs. For a graph G, a vertex subset D of G is a *total dominating set* of G if every vertex of G is adjacent to a vertex in D. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. A graph G is said to be *k-γ_t-critical* if $\gamma_t(G) = k$ and $\gamma_t(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. For $k = 3$, it was pointed out by Ananchuen in [1] that a graph G is 3-γ_t-critical if and only if G is 3-γ_c-critical. In [7], the authors established the similar result when $k = 4$. Therefore we have the following result.

Theorem 30 ([1] and [7])**.** *For $k \in \{3, 4\}$, a connected graph G is k-γ_t-critical if and only if G is k-γ_c-critical.*

For related results on k-γ_t-critical graphs, Hattingh *et al.* [4] established the upper bound of the number of end vertices of k-γ_t-critical graphs. They proved the following.

Theorem 31 [4]**.** *For $k \geq 5$, every k-γ_t-critical graph has at most $k-2$ end vertices.*

They, further, established the existence of k-γ_t-critical graphs with prescribe end vertices according to the bound from Theorem 31.

Theorem 32 [4]**.** *For integers $k \geq 3$ and $0 \leq h \leq k-2$ except only the case when $k = 4$ and $h = 2$, there exists a k-γ_t-critical graph with h end vertices.*

Hence, by Corollary 13 and Theorem 30, we can conclude that there is no 4-γ_t-critical graph with two end vertices. This fulfills Theorem 32 in the following way.

Corollary 33. *For integers $k \geq 3$ and $0 \leq h \leq k-2$, there exists a k-γ_t-critical graph with h end vertices if and only if $k \neq 4$ or $h \neq 2$.*

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