Discussiones Mathematicae
Graph Theory 40 (2020) 1025–1033
doi:10.7151/dmgt.2159

LOW 5-STARS AT 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO VERTICES OF DEGREE FROM 7 TO 9 [1]

Oleg V. Borodin, Mikhail A. Bykov

and

Anna O. Ivanova

Sobolev Institute of Mathematics
Novosibirsk, 630090, Russia

e-mail: brdnoleg@math.nsc.ru
131093@mail.ru
shmgnanna@mail.ru

Abstract

In 1940, Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P_5}$ of 3-polytopes with minimum degree 5.

Given a 3-polytope P, by $h_5(P)$ we denote the minimum of the maximum degrees (height) of the neighborhoods of 5-vertices (minor 5-stars) in P.

Recently, Borodin, Ivanova and Jensen showed that if a polytope P in $\mathbf{P_5}$ is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-vertex, then $h_5(P)$ can be arbitrarily large. Therefore, we consider the subclass $\mathbf{P_5^*}$ of 3-polytopes in $\mathbf{P_5}$ that avoid $(5, 5, 6, 6, \infty)$-vertices.

For each P^* in $\mathbf{P_5^*}$ without vertices of degree from 7 to 9, it follows from Lebesgue's Theorem that $h_5(P^*) \le 17$. Recently, this bound was lowered by Borodin, Ivanova, and Kazak to the sharp bound $h_5(P^*) \le 15$ assuming the absence of vertices of degree from 7 to 11 in P^*.

In this note, we extend the bound $h_5(P^*) \le 15$ to all P^*s without vertices of degree from 7 to 9.

Keywords: planar map, planar graph, 3-polytope, structural properties, 5-star, weight, height.

2010 Mathematics Subject Classification: 05C75.

[1] The work was funded by the Russian Science Foundation, grant 16-11-10054.

1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a 3-*polytope*) is denoted by $d(x)$. As proved by Steinitz [14], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs. A k-*vertex* is a vertex v with $d(v) = k$. A k^+-*vertex* (k^--*vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. The set of 3-polytopes with minimum degree 5 is denoted by $\mathbf{P_5}$, and its elements are P_5s. We will drop the argument whenever it is clear from context.

The *height* of a subgraph S of a 3-polytope is the maximum degree of the vertices of S in the 3-polytope. A k-star, a star with k rays, is *minor* if its center v has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $h_k(P_5)$ we denote the minimum height of minor k-stars in a given 3-polytope P_5.

In 1904, Wernicke [15] proved that every P_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 6^--neighbors. So $h_1 \le h_2 \le 6$ in $\mathbf{P_5}$, where both bounds are sharp.

In 1940, in attempts to solve the Four Color Problem, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in P_5s.

In particular, this description implies the results in [11, 15] and shows that there is a 5-vertex with three 7^--neighbors. Thus $h_3 \le 7$, which is sharp due to Borodin [1]. Jendrol' and Madaras [12] gave a precise description of minor 3-stars in P_5s.

Lebesgue [13] also proved $h_4(P_5) \le 11$, which was strengthened by Borodin and Woodall [10] to the tight bound $h_4(P_5) \le 10$. Recently, Borodin and Ivanova [2] obtained a precise description of 4-stars in P_5s.

The more general problem of describing 5-stars at 5-vertices in $\mathbf{P_5}$ remains widely open.

Recently, precise upper bounds have been obtained for the minimum height $h_5(P_5)$ of minor 5-stars in several natural subclasses of $\mathbf{P_5}$.

Note that Borodin, Ivanova and Jensen [5] showed that if a polytope P_5 is allowed to have a 5-vertex adjacent to two 5-vertices and two more vertices of degree at most 6, called a $(5, 5, 6, 6, \infty)$-*vertex*, then $h_5(P_5)$ can be arbitrarily large. (In fact, every 5-vertex in the construction in [5] has two 5-neighbors and two 6-neighbors.) Therefore, from now on we restrict ourselves to the subclass $\mathbf{P_5^*}$ of the 3-polytopes in $\mathbf{P_5}$ avoiding $(5, 5, 6, 6, \infty)$-vertices.

For each P_5^* in $\mathbf{P_5^*}$, it follows from Lebesgue's Theorem that $h_5(P_5^*) \le 41$. This bound was lowered to $h_5(P_5^*) \le 28$ by Borodin, Ivanova, and Jensen [5] and then to $h_5(P_5^*) \le 23$ in Borodin-Ivanova [4]. On the other hand, it was shown in [5] that the upper bound for $h_5(P_5^*)$ cannot go down below 20. We conjecture that $h_5(P_5^*) \le 20$ whenever $P_5^* \in \mathbf{P_5^*}$.

Back in 1996, Jendrol' and Madaras [12] showed that if a polytope P_5^{**} has a 5-vertex adjacent to four 5-vertices, then $h_5(P_5^{**})$ can be arbitrarily large. Therefore, considering subclasses of $\mathbf{P_5^*}$ without vertices of degree from 6 to a certain k_6 with $k_6 > 6$, we should deal only with 3-polytopes P_5^{**}s having no 5-vertices with four 5-neighbors.

For every P_5^{**} in $\mathbf{P_5^*}$ with $k_6 = 9$, Lebesgues' bound $h_5(P_5^{**}) \le 14$ was improved by Borodin and Ivanova [3] to the sharp bound $h_5(P_5^{**}) \le 12$. Later on, Borodin, Ivanova and Nikiforov [9] proved the same bound assuming the absence only of vertices of degree from 6 to 8, improving Lebesgues' bound $h_5(P_5^{**}) \le 17$.

For each P_5^{**} with no vertices of degree 6 or 7, it follows from Lebesgue's Theorem that $h_5(P_5) \le 23$, and Borodin, Ivanova, Kazak and Vasil'eva [7] have obtained the best possible bound $h_5(P_5^{**}) \le 14$.

For each P_5^{**} with no 6-vertices, Lebegues' bound $h_5(P_5^{**}) \le 41$ was improved by Borodin, Ivanova and Nikiforov [8] to the sharp bound $h_5(P_5^{**}) \le 17$. We note that the sharpness was confirmed in [8] by a construction on almost 3000 vertices.

Another natural direction of research towards a tight version of Lebesgue's Theorem is considering subclasses of $\mathbf{P_5^*}$ with no vertices of degree from 7 to a certain integer k_7 with $k_7 > 7$.

For $k_7 = 11$, Lebesgue's bound $h_5(P^*) \le 17$ was lowered by Borodin, Ivanova, and Kazak [6] to the sharp bound $h_5(P^*) \le 15$. The purpose of this note is to extend this bound to all P^*s such that $k_7 = 9$.

Theorem 1. *Every 3-polytope P^* with minimum degree 5 and neither $(5,5,6,6,\infty)$-vertices nor vertices of degree from 7 to 9 satisfies $h_5(P^*) \le 15$, which bound is best possible.*

Problem 2. Is it true that every 3-polytope P^* with minimum degree 5 and no $(5,5,6,6,\infty)$-vertices satisfies $h_5(P^*) \le 15$ provided that

(a) P^* has no vertices of degree 7 and 8?
(b) only 7-vertices are forbidden in P^*?

2. Proof of Theorem 1

The sharpness of the bound 15 in Theorem 1 follows from a construction in [6].

Now suppose a 3-polytope P_5' is a counterexample to the main statement of Theorem 1. In particular, each minor 5-star in P_5' contains a 16^+-vertex along with either another 10^+-vertex or at least three 6-vertices.

Let P_5 be a counterexample on the same vertices as P_5' with the maximum possible number of edges. For brevity, a vertex v with $d(v) \neq 6$ is a *non-6-vertex*.

Remark 3. P_5 has no two non-6-vertices being nonconsecutive along the boundary of a 4^+-face. Indeed, otherwise adding a diagonal between these vertices would result in a counterexample with greater edges than P_5.

Corollary 4. *In P_5, each 4^+-face has at most two non-6-vertices, and if it has two such vertices, then they are adjacent to each other.*

Discharging.

Let V, E, and F be the sets of vertices, edges, and faces of P_5. Euler's formula $|V| - |E| + |F| = 2$ for P_5 implies

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of P_5 as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12.

The *final charge* $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R9 below (see Figure 1).

For a vertex v, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to v in a fixed cyclic order. If f is a face, then $v_1, \ldots, v_{d(f)}$ are the vertices incident with f in the same cyclic order.

A vertex is *simplicial* if it is completely surrounded by 3-faces.

R1. Every 4^+-face gives 1 to every incident non-6-vertex.

R2. Suppose $f = uvw$ is a 3-face with $d(u) = 5$ and $d(v) \geq 10$.

(a) If $d(w) \geq 6$, then u receives from v either $\frac{2}{5}$ if $d(v) \leq 15$ or $\frac{2}{3}$ otherwise.

(b) If $d(w) = 5$, then u (as well as w) receives from v either $\frac{1}{5}$ if $d(v) \leq 15$ or $\frac{1}{3}$ otherwise.

R3. A non-simplicial 5-vertex v such that there are 3-faces v_1vv_2 and v_2vv_3 with $d(v_2) \geq 16$ gives $\frac{2}{3}$ to v_2.

R4. A simplicial 5-vertex v with $d(v_2) \geq 16$ and $d(v_1) \geq 10$ gives $\frac{1}{3}$ to v_2.

R5. A simplicial 5-vertex v with $d(v_2) \geq 16$ and $d(v_1) = d(v_3) = 6$ gives $\frac{1}{3}$ to v_2.

R6. A simplicial 5-vertex v with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = 5$, and $d(v_4) \geq 10$ gives $\frac{2}{5}$ to v_2.

R7. A simplicial 5-vertex v with $d(v_2) \geq 16$, $d(v_1) = 6$, $d(v_3) = d(v_4) = 5$ (hence $d(v_5) \geq 10$) gives $\frac{1}{2}$ to v_2.

Remark 5. Note that a simplicial 5-vertex v with $d(v_2) \geq 16$, $d(v_1) = d(v_4) = 6$, and $d(v_3) = 5$ gives nothing to v_2.

R8. A simplicial 5-vertex v with $d(v_2) \geq 16$, $d(v_1) = d(v_3) = d(v_4) = 5$, and $d(v_5) \geq 10$ gives $\frac{1}{15}$ to v_2.

R9. A simplicial 5-vertex v with $d(v_2) \geq 16$, $d(v_1) = d(v_3) = 5$, $d(v_4) \geq 6$, and $d(v_5) \geq 10$ gives $\frac{4}{15}$ to v_2.

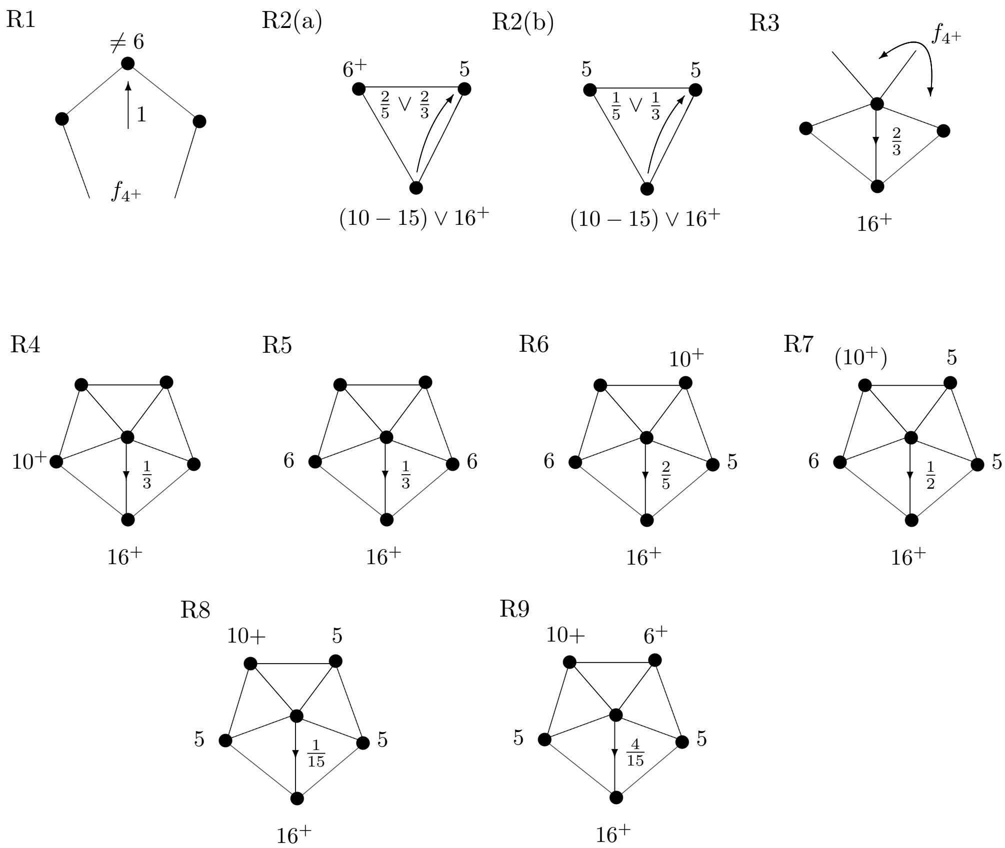

Figure 1. Rules of discharging.

Checking $\boldsymbol{\mu'(x) \geq 0}$ whenever $\boldsymbol{x \in V \cup F}$.

First consider a face f in P_5. If $d(f) = 3$, then f does not participate in discharging, and so $\mu'(v) = \mu(f) = 2 \times 3 - 6 = 0$. Note that every 4^+-face is incident with at most two non-6-vertices due to Corollary 4, which implies that $\mu'(v) = 2d(f) - 6 - 2 \times 1 \geq 0$ by R1.

Now suppose $v \in V$.

Case 1. $d(v) \geq 18$. Since v sends at most $\frac{2}{3}$ to its 5-neighbors through each 3-face by R2, we have $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3} \geq 0$.

Case 2. $16 \leq d(v) \leq 17$. If v is not simplicial, then it sends at most $\frac{2}{3}$ through each of at most $d(v) - 1$ faces, so $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} = \frac{d(v)-16}{3} \geq 0$, as desired. From now on, suppose v is simplicial.

If v has two consecutive 6^+-neighbors, then again $\mu'(v) \geq d(v) - 6 - (d(v) - 1) \times \frac{2}{3} \geq 0$. So we can assume from now on that each 3-face incident with v is incident with a 5-vertex.

If v has at least one non-simplicial 5-neighbor v_2, then v receives $\frac{2}{3}$ from v_2 by R3, which implies $\mu'(v) \geq d(v) - 6 + \frac{2}{3} - d(v) \times \frac{2}{3} = \frac{d(v)-16}{3} \geq 0$. Thus suppose all 5-vertices adjacent to v are simplicial.

If v has a 10^+-neighbor v_2, then v receives $\frac{1}{3} + \frac{1}{3}$ from the 5-vertices v_1 and v_3 by R4, which again implies $\mu'(v) \geq 0$.

Summarizing, from now on our v is simplicial, has no 10^+-neighbors, no two consecutive 6-neighbors, and no non-simplicial 5-neighbors.

Suppose $S_k = v_0, \ldots, v_k$ is a sequence of neighbors of v with $d(v_0) = 6$, $d(v_k) = 6$, while $d(v_i) = 5$ whenever $1 \leq i \leq k - 1$ and $k \geq 2$. (It is not excluded that $S_k = S_{d(v)}$, which happens when v has precisely one 6-neighbor.) Let w_i, $1 \leq i \leq k - 1$, $k \geq 2$, be the common neighbor of v_{i-1} and v_i different from v.

Since $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3}$, we can say that v has the *deficiency* equal to $\frac{1}{3}$ if $d(v) = 17$ or $\frac{2}{3}$ if $d(v) = 16$.

Our next goal is to estimate the total return to v from its 5-neighbors by R4–R9 and show that it is not less than the deficiency of v.

Remark 6. As we remember, our v has no S_1s. Note that v_1 in S_2 returns $\frac{1}{3}$ to v by R5. As for S_3, it can happen that neither v_1 nor v_2 returns anything to v, which is the case only when v_1 and v_2 have a common 6-neighbor (see Remark 5).

Lemma 7. *The total return from (the three 5-vertices of) an S_4 is at least $\frac{2}{3}$.*

Proof. If $d(w_2) \geq 10$ or $d(w_2) = 5$, then v receives at least $\frac{2}{5}$ from its 5-neighbor v_1 by R6 or R7, respectively. The same is true for v_3. So, if $d(w_2) \neq 6$ and $d(w_3) \neq 6$, our v returns at least $\frac{4}{5}$, which is more than enough. Thus we can assume by symmetry that $d(w_2) = 6$. Note that in this case $d(w_3) \geq 10$, for v_2 is not a $(5, 5, 6, 6, \infty)$-vertex. Since v_2 gives $\frac{4}{15}$ to v by R9, while v_3 gives $\frac{2}{5}$ by R6, we have the desired return of $\frac{2}{3}$. ■

Lemma 8. *The total return from the three extreme 5-vertices v_1, v_2, and v_3 of an S_k with $k \geq 5$ is at least $\frac{1}{3}$.*

Proof. We have nothing to prove unless $d(w_2) = 6$, which implies that $d(w_3) \geq 10$. Now v_2 still gives $\frac{4}{15}$ to v by R9, while v_3 gives al least $\frac{1}{15}$ by R8 or R9, which returns sum up to the desired $\frac{1}{3}$. ■

By symmetry, we deduce the following fact from Lemma 8.

Corollary 9. *The total return from an S_k is at least $\frac{1}{3}$ if $5 \le k \le 6$ and at least $\frac{2}{3}$ if $k \ge 7$.*

If v is completely surrounded by 5-vertices (which means that no S_k is defined), then the total return to v is at least $16 \times \frac{1}{15} > \frac{2}{3}$, and hence we can assume from now on that the neighborhood of v is partitioned into S_ks.

If $d(v) = 17$, then to pay off the deficiency of $\frac{1}{3}$ it suffices to note that every S_k with $k \ne 3$ returns at least $\frac{1}{3}$ to v, while 3 does not divide 17 (which implies that v cannot be surrounded only by S_3s).

Finally, suppose that $d(v) = 16$. As follows from Lemma 7 combined with Corollary 9, we are able to cover the deficiency of $\frac{2}{3}$ unless the neighborhood of v consists of several S_3 and at most one S_k such that $k \in \{2, 5, 6\}$. However, the residue of 16 modulo 3 is neither 0 nor 2, a contradiction.

Case 3. $10 \le d(v) \le 15$. Now R2 implies that $\mu'(v) \ge d(v) - 6 - d(v) \times \frac{2}{5} = \frac{3(d(v)-10)}{5} \ge 0$ since v sends either nothing or $\frac{2}{5}$ through each incident face.

Case 4. $d(v) = 6$. Since v does not participate in discharging, we have $\mu'(v) = \mu(v) = 6 - 6 = 0$.

Case 5. $d(v) = 5$. If v is incident with a 4^+-face, then $\mu'(v) \ge 5 - 6 + 1 = 0$ due to R1 combined with the fact that each 16^+-neighbor v_2 gives more to v by R2 than v returns to v_2 by R3. Therefore, in what follows we can assume that v is simplicial.

Remark 10. Each 16^+-neighbor v_2 gives v through the faces v_1vv_2, v_2vv_3 by R2 and returns from v along edge vv_2 by R4–R9:

(a) $\frac{4}{3}$ versus $\frac{1}{3}$ if $d(v_1) \ge 6$ and $d(v_3) \ge 6$,

(b) 1 versus at most $\frac{1}{2}$ if $d(v_1) = 5$ and $d(v_3) \ge 6$, or

(c) $\frac{2}{3}$ versus at most $\frac{4}{15}$ if $d(v_1) = 5$ and $d(v_3) = 5$.

Remark 10 combined with examining R4–R9 more carefully implies the following observation.

Remark 11. The donation of a 16^+-neighbor v_2 to v exceeds the return from v to v_2 by less than $\frac{1}{2}$ only when v obeys R9, in which case we have $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$.

Subcase 5.1. v participates in R9. Thus suppose $d(v_1) = d(v_3) = 5$, $d(v_2) \ge 16$, $d(v_4) \ge 6$, and $d(v_5) \ge 10$. Note that v acquires $\frac{2}{3} - \frac{4}{15} = \frac{2}{5}$ from v_2 by R2 combined with R9.

If $d(v_5) \ge 16$, then v_5 gives 1 to v by R2, and v returns to v_5 either $\frac{1}{3}$ by R4 if $d(v_4) \ge 10$ or $\frac{2}{5}$ by R6 if $d(v_4) = 6$. Thus the total acquisition of v from v_5 is at least $\frac{3}{5}$, and we are done.

If $d(v_5) \le 15$, then v_5 gives $\frac{3}{5}$ to v by R2, and we are done again.

Subcase 5.2. v does not participates in R9. In view of Remark 11, we already have nothing to prove if v has at least two 16^+-neighbors. So suppose v_2 is the only 16^+-neighbor of v.

If $d(v_1) \ge 10$, then v_1 gives v at least $\frac{3}{5}$ by R2, while v_2's resulting donation to v is $1 - \frac{1}{3}$ by R2 and R4. This implies $\mu'(v) > 0$.

By symmetry, suppose $d(v_1) \le d(v_3) \le 6$. If $d(v_1) = d(v_3) = 6$, then v_1 gives $\frac{4}{3}$ to v by R2 and takes back $\frac{1}{3}$ from v by R5, which implies $\mu'(v) \ge 0$.

Subcase 5.2.1. $d(v_1) = 5$ and $d(v_3) = 6$. Now v_2 gives 1 to v by R2. If $d(v_5) > 6$, which means that in fact $10 \le d(v_4) \le 15$, then we have $\mu'(v) \ge -1 + 1 - \frac{2}{5} + \frac{2}{5} = 0$ by R2 and R6.

If $d(v_5) = 6$, then we have $d(v_4) = 6$ or $d(v_4) \ge 10$ due to the absence of a $(5, 5, 6, 6, \infty)$-vertex. In both cases, $\mu'(v) \ge -1 + 1 = 0$ by R2 since v returns nothing to v_2.

Finally, $d(v_5) = 5$. Now $d(v_4) \ge 10$ due to the absence of $(5, 5, 6, 6, \infty)$-vertex, and we have $\mu'(v) \ge -1 + 1 - \frac{1}{2} + \frac{3}{5} > 0$ by R2 and R7.

Subcase 5.2.2. $d(v_1) = d(v_3) = 5$. Here v_2 gives $\frac{2}{3}$ to v by R2. Since v is not a $(5, 5, 6, 6, \infty)$-vertex, we can assume that $10 \le d(v_4) \le 15$. Furthermore, R9 is not applicable to v by an above assumption, so $d(v_5) = 5$. This means that v obeys R8, and we have $\mu'(v) = -1 + \frac{2}{3} - \frac{1}{15} + \frac{2}{5} = 0$, as desired.

Thus we have proved $\mu'(x) \ge 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

References

[1] O.V. Borodin, *Structural properties of planar maps with the minimal degree* 5, Math. Nachr. **158** (1992) 109–117.
doi:10.1002/mana.19921580108

[2] O.V. Borodin and A.O. Ivanova, *Describing* 4*-stars at* 5*-vertices in normal plane maps with minimum degree* 5, Discrete Math. **313** (2013) 1710–1714.
doi:10.1016/j.disc.2013.04.025

[3] O.V. Borodin and A.O. Ivanova, *Light neighborhoods of* 5*-vertices in* 3*-polytopes with minimum degree* 5, Sib. Èlektron. Mat. Izv. **13** (2016) 584–591.
doi:10.17377/semi.2016.13.045

[4] O.V. Borodin and A.O. Ivanova, *Light and low* 5*-stars in normal plane maps with minimum degree* 5, Sib. Math. J. **57** (2016) 470–475.
doi:10.1134/S0037446616030071

[5] O.V. Borodin, A.O. Ivanova and T.R. Jensen, 5*-stars of low weight in normal plane maps with minimum degree* 5, Discuss. Math. Graph Theory **34** (2014) 539–546.
doi:10.7151/dmgt.1748

[6] O.V. Borodin, A.O. Ivanova and O.N. Kazak, *Describing neighborhoods of* 5*-vertices in* 3*-polytopes with minimum degree* 5 *and without vertices of degrees from* 7 *to* 11, Discuss. Math. Graph Theory **38** (2018) 615–625.
doi:10.7151/dmgt.2024

[7] O.V. Borodin, A.O. Ivanova, O.N. Kazak and E.I. Vasil'eva, *Heights of minor* 5*-stars in* 3*-polytopes with minimum degree* 5 *and no vertices of degree* 6 *and* 7, Discrete Math. **341** (2018) 825–829.
doi:10.1016/j.disc.2017.11.021

[8] O.V. Borodin, A.O. Ivanova and D.V. Nikiforov, *Low minor* 5*-stars in* 3*-polytopes with minimum degree* 5 *and no* 6*-vertices*, Discrete Math. **340** (2017) 1612–1616.
doi:10.1016/j.disc.2017.03.002

[9] O.V. Borodin, A.O. Ivanova and D.V. Nikiforov, *Low and light* 5*-stars in* 3*-polytopes with minimum degree* 5 *and restrictions on the degrees of major vertices*, Sib. Math. J. **58** (2017) 600–605.
doi:10.1134/S003744661704005X

[10] O.V. Borodin and D.R. Woodall, *Short cycles of low weight in normal plane maps with minimum degree* 5, Discuss. Math. Graph Theory **18** (1998) 159–164.
doi:10.7151/dmgt.1071

[11] P. Franklin, *The four color problem*, Amer. J. Math. **44** (1922) 225–236.
doi:10.2307/2370527

[12] S. Jendrol' and T. Madaras, *On light subgraphs in plane graphs of minimum degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.
doi:10.7151/dmgt.1035

[13] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (9) (1940) 27–43.

[14] E. Steinitz, *Polyeder und Raumeinteilungen*, in: Enzykl. Math. Wiss. (Geometrie), **3** (1922) 1–139.

[15] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.
doi:10.1007/BF01444968

Received 18 December 2017
Revised 25 June 2018
Accepted 25 June 2018