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A NOTE ON THE FAIR DOMINATION NUMBER IN OUTERPLANAR GRAPHS

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Abstract

For $k \geq 1$, a k-fair dominating set (or just kFD-set), in a graph G is a dominating set S such that $|N(v) \cap S| = k$ for every vertex $v \in V - S$. The k-fair domination number of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A fair dominating set, abbreviated FD-set, is a kFD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of G that is not the empty graph, is the minimum cardinality of an FD-set in G. In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

Keywords: fair domination, outerplanar graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [13]. Specifically, let G be a simple graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, then we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. A graph G of order at least three is 2-*connected* if the deletion of any vertex does not disconnect the graph. A *cut-vertex* in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a *block*. A graph G is *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph G is *Hamiltonian* if there is a spanning cycle in G. For a subset S of vertices of G, we denote by $G[S]$ the subgraph of G induced by S.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A vertex v is said to be *dominated* by a set S if $N[v] \cap S \neq \emptyset$.

Caro *et al.* [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The *k-fair domination number* of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$*-set*. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$*-set*. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro *et al.* [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph G of order n, and among open problems posed by Caro *et al.* [1], one asks to find $fd(G)$ for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block K in an outerplanar graph G a *strong-block* if K contains at least three vertices. We call a vertex w in a strong-block K of an outerplanar graph G a *special cut-vertex* if w belongs to a shortest path from K to a strong-block $K' \neq K$. We call a strong-block K in an outerplanar graph G a *leaf-block* if K contains exactly one special cut-vertex. We denote by $r(G)$ the number of strong-blocks of a graph G. The following is straightforward.

Observation 1. *Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.*

We make use of the following.

Observation 2 (Caro *et al.* [1]). *Every* $1FD$*-set in a graph contains all its strong support vertices.*

Theorem 3 (Leydolda *et al.* [14]). *An outerplanar graph G is Hamiltonian if and only if it is* 2*-connected.*

Theorem 4 (Hajian *et al.* [9]). *If G is a unicyclic graph of order n, then* $fd_1(G) \leq (n+1)/2$.

2. Main Result

Theorem 5. *If G is an outerplanar graph of order n and size m with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m-3n+3)/2 - r$. This bound is sharp.*

Proof. Let G be an outerplanar graph of order n and size m with $r \geq 1$ strong-blocks. We prove that $fd_1(G) \leq (4m-3n+3)/2 - r$. The result follows from Theorem 4 if G is a unicyclic graph. Thus assume that G is not a unicyclic graph. Suppose to the contrary that $fd_1(G) > (4m-3n+3)/2 - r$. Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is as minimum as possible. Let $K_1, K_2, \ldots, K_r$ be the r strong-blocks of G. By Theorem 3, K_j is Hamiltonian, for $1 \leq j \leq r$. Let $C^i = c_0^i c_1^i \cdots c_{l_i}^i c_0^i$ be a Hamiltonian cycle for K_i, for $1 \leq i \leq r$. We proceed with the following Claims 1 and 2.

Claim 1. *For any $1 \leq i \leq r$, if c_j^i is a vertex of C^i, for some $j \in \{0, 1, \ldots, l_i\}$, such that $\deg_G\left(c_j^i\right) = 2$, then $\deg_G\left(c_{j+1}^i\right) \geq 3$ and $\deg_G\left(c_{j-1}^i\right) \geq 3$, where the calculations in $j+1$ and $j-1$ are taken modulo l_i.*

Proof. Assume that $\deg_G\left(c_j^i\right) = 2$ for some $j \in \{0, 1, \ldots, l_i\}$. Suppose that $\deg_G\left(c_{j+1}^i\right) = 2$. Let $G' = G - c_j^i c_{j+1}^i$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of G, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m-1) - 3n + 3)/2 - (r-1) = (4m-3n+3)/2 - r - 1$. Let S' be a $fd_1(G')$-set. If $\left|S' \cap \left\{c_j^i, c_{j+1}^i\right\}\right| \in \{0, 2\}$, then S' is a 1FD-set for G of cardinality at most $(4m-3n+3)/2-r-1$, and so $fd_1(G) \leq (4m-3n+3)/2-r-1$, a contradiction. Thus $\left|S' \cap \left\{c_j^i, c_{j+1}^i\right\}\right| = 1$. Assume that $c_j^i \in S'$. Then $c_{j+1}^i \notin S'$, and $c_{j+2}^i \in S'$, since S' is a dominating set. Thus $\left\{c_{j+1}^i\right\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-r$ and so $fd_1(G) \leq (4m-3n+3)/2-r$, a contradiction. Next assume that $c_{j+1}^i \in S'$. Then $c_j^i \notin S'$ and $c_{j-1}^i \in S'$. Thus $\left\{c_j^i\right\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-r$. So $fd_1(G) \leq (4m-3n+3)/2-r$, a contradiction. Hence $\deg_G\left(c_{j+1}^i\right) \geq 3$. Similarly, $\deg_G\left(c_{j-1}^i\right) \geq 3$. □

Claim 2. *If c_j^i is a vertex of C^i, for some $j \in \{0,1,\ldots,l_i\}$, such that* $\deg_G\left(c_j^i\right) =$ *2, then non of c_{j+1}^i and c_{j-1}^i is a support vertex of G.*

Proof. Assume that $\deg_G\left(c_j^i\right) = 2$ for some $j \in \{0,1,\ldots,l_i\}$. Suppose that c_{j+1}^i is a support vertex of G. Let $G' = G - c_j^i c_{j-1}^i$. Clearly $r-1 \leq r(G') \leq r$. By the choice of G, $fd_1(G') \leq (4m(G')-3n(G')+3)/2 - r(G') \leq (4(m-1)-3n+3)/2-(r-1) = (4m-3n+3)/2-r-1$. Let S' be a $fd_1(G')$-set. By Observation 2, $c_{j+1}^i \in S'$, since c_{j+1}^i is a strong support vertex of G'. If $c_{j-1}^i \notin S'$, then S' is a 1FD-set for G of cardinality at most $(4m-3n+3)/2-r-1$ and so $fd_1(G) \leq (4m-3n+3)/2-r-1$, a contradiction. Thus $c_{j-1}^i \in S'$ and so $\left\{c_j^i\right\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-r$, and so $fd_1(G) \leq (4m-3n+3)/2-r$, a contradiction. Hence c_{j+1}^i is not a support vertex of G. Similarly, c_{j-1}^i is not a support vertex of G. □

We consider the following cases.

Case 1. $r = 1$. First assume that $V(G) = \left\{c_0^1, c_1^1, \ldots, c_{l_1}^1\right\}$ and so $n = l_1 + 1$. By Claim 1, at least $\lceil n/2 \rceil$ vertices of C^1 are of degree at least 3. Now, we can easily see that $m = \frac{1}{2}\sum_{v \in V(G)} \deg(v) \geq n + \lceil n/2 \rceil/2$. (Since $\delta(G) \geq 2$ and at least $\lceil n/2 \rceil$ vertices of G are of degree at least 3, we have $\sum_{v \in V(G)} \deg(v) \geq 2n + \lceil n/2 \rceil$.) Thus $m \geq n + \lceil n/2 \rceil/2$. If n is even, then $n \leq (4m-3n)/2$ and if n is odd, then $n \leq (4m-3n-1)/2$. We thus obtain that $n \leq (4m-3n+3)/2-1$. Now $V(G)$ is a 1FD-set in G of cardinality n, and thus $fd_1(G) \leq (4m-3n+3)/2-1$, a contradiction. We deduce that $V(G) \neq \left\{c_0^1, c_1^1, \ldots, c_{l_1}^1\right\}$. Since $r = 1$, there is a vertex of degree one in G. Let v_d be a leaf of G such that $d(v_d, C^1)$ is maximum. Let $v_0 v_1 \cdots v_d$ be the shortest path from v_d to a vertex $v_0 \in C^1$. Clearly, $\{v_0, v_1, \ldots, v_d\} \cap V(C^1) = \{v_0\}$.

Assume that $d \geq 2$. Suppose that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of G, $fd_1(G') \leq (4m(G')-3n(G')+3)/2 - r(G') = (4(m-2)-3(n-2)+3)/2-1 = (4m-3n+3)/2-2$. Let S' be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-1$ and so $fd_1(G) \leq (4m-3n+3)/2-1$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most

$(4m-3n+3)/2-1$ and so $fd_1(G) \le (4m-3n+3)/2-1$, a contradiction. Thus assume that $\deg_G(v_{d-1}) \ge 3$. Clearly any vertex of $N_G(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let G' be obtained from G by removing all leaves adjacent to v_{d-1}. Clearly $r(G') = r$. By the choice of G, $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') \le (4(m-2) - 3(n-2) + 3)/2 - 1 = (4m-3n+3)/2 - 2$. Let S' be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then S' is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-2$ and so $fd_1(G) \le (4m-3n+3)/2-2$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Now $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m-3n+3)/2-1$ and so $fd_1(G) \le (4m-3n+3)/2-1$, a contradiction.

We next assume that $d = 1$. Let $D_1 = \{c_j^1 \mid \deg_G(c_j^1) = 2\}$ and $D_2 = \{c_j^1 \mid c_j^1$ is a support vertex of $G\}$ and $D_3 = \{c_j^1 \mid \deg_G(c_j^1) \ge 3$ and c_j^1 is not a support vertex of $G\}$. Clearly $|D_1| + |D_2| + |D_3| = l_1 + 1$. Since $d = 1$, we have $|D_2| \ge 1$. By Claims 1 and 2, $|D_1| \le |D_3|$. Observe that $m = \frac{1}{2}\sum_{v \in V(G)} \deg(v) \ge n + |D_3|/2$. Clearly $n \ge l_1 + 1 + |D_2|$. Thus

$$\begin{aligned}(4m-3n+3)/2-1 &\ge (4(n+|D_3|/2) - 3n + 3)/2 - 1\\ &\ge (l_1 + 1 + |D_2| + 2|D_3| + 3)/2 - 1\\ &\ge (l_1 + 1 + |D_1| + |D_2| + |D_3| + 3)/2 - 1\\ &= l_1 + 3/2 > l_1 + 1.\end{aligned}$$

Evidently, $\{c_0^1, \dots, c_{l_1}^1\}$ is a $fd_1(G)$-set of cardinality l_1+1. Thus $fd_1(G) < (4m-3n+3)/2 - r$, a contradiction.

Case 2. $r \ge 2$. By Observation 1, G has at least two leaf-blocks. Let K_i be a leaf-block of G, where $i \in \{1, 2, \dots, r\}$. By relabeling of the vertices of C^i we may assume that c_0^i is a special cut-vertex of G. Let G' be the graph obtained by removal of all edges $c_0^i c_j^i$, with $c_j^i \in \{c_1^i, \dots, c_{l_i}^i\}$. Clearly G' has two components. Let G_1' be the component of G' containing c_1^i, and G_2' be the component of G' containing c_0^i. Clearly, $\{c_1^i, c_2^i, \dots, c_{l_i}^i\} \subseteq V(G_1')$. We consider the following subcases.

Subcase 2.1. $V(G_1') = \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$. Let $G_1^* = G[V(G_1') \cup \{c_0^i\}]$. Clearly $n(G_1^*) = l_i + 1$. By Claim 1, at least $\lfloor l_i/2 \rfloor$ vertices of $C^i - c_0^i$ are of degree at least 3.

Assume that l_i is even. Thus at least $l_i/2$ vertices of $C^i - c_0^i$ are of degree at least 3. Now, we can easily see that $m(G_1^*) = \frac{1}{2}\sum_{v \in V(G_1^*)} \deg(v) \ge l_i + 1 + l_i/4$. Let $G_2^* = G[V(G_2') \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_1}^i c_1^i\}$. Clearly $n = n(G_2^*) + l_i - 2$, $m = m(G_2^*) + m(G_1^*) - 2$ and $r(G_2^*) = r - 1$. By the choice of G, $fd_1(G_2^*) \le (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$. Let S'' be a $fd_1(G_2^*)$-set. By Observation 2, $c_0^i \in S''$, since c_0^i is a strong support vertex of G_2^* . Then $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$ is a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$\begin{aligned}
&(4m - 3n + 3)/2 - r\\
&\geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r\\
&= (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1\\
&\geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\end{aligned}$$

Thus $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

Assume next that l_i is odd. Observe that at least $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3. Now, we can easily see that $m(G_1^*) = \frac{1}{2}\sum_{v\in V(G_1^*)} \deg(v) \geq l_i + 1 + (l_i - 1)/4$. We show that $m(G_1^*) = l_i + 1 + (l_i - 1)/4$. Suppose that $m(G_1^*) > l_i + 1 + (l_i - 1)/4$. Then $m(G_1^*) \geq l_i + 1 + (l_i - 1)/4 + 1/4$. Let $G_2^* = G\big[G_2' \cup \{c_1^i, c_{l_i}^i\}\big] - \{c_{l_i}^i c_1^i\}$. Clearly $n = n(G_2^*) + l_i - 2$, $m = m(G_2^*) + m(G_1^*) - 2$ and $r(G_2^*) = r - 1$. By the choice of G, $fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$. Let S'' be a $fd_1(G_2^*)$-set. By Observation 2, $c_0^i \in S''$, since c_0^i is a strong support vertex of G_2^*. Then $S'' \cup \{c_1^i, c_2^i, \ldots, c_{l_i}^i\}$ is a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$\begin{aligned}
&(4m - 3n + 3)/2 - r\\
&\geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r\\
&= (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1\\
&\geq |S''| + (4(l_i + 1 + (l_i - 1)/4 + 1/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\end{aligned}$$

Thus $fd_1(G) \leq (4m-3n+3)/2-r$, a contradiction. We thus obtain that $m(G_1^*) = l_i + 1 + (l_i - 1)/4$. Note that $|E(G_1^*) \cap E(C^i)| = l_i + 1$. Hence $|E(G_1^*) - E(C^i)| = (l_i - 1)/4$. Since $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3, we thus obtain that precisely $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree 3, and so $(l_i + 1)/2$ vertices of $C^i - c_0^i$ are of degree two. Now Claim 1 implies that $\deg_G(c_1^i) = \deg_G(c_{l_i}^i) = 2$. Thus we obtain that $\deg_{G_1^*}(c_0^i) = 2$. Let $A_1 = \{c_j \mid \deg_G(c_j^i) = 2 \text{ for } 1 \leq j \leq l_i\}$ and $A_2 = \{c_1^i, c_2^i, \ldots, c_{l_i}^i\} - A_1$. Clearly $|A_1| = (l_i + 1)/2$ and $|A_2| = (l_i - 1)/2$. Note that $|A_2|$ is even, since the number of odd vertices in every graph (here G_1^*) is even. Thus $|A_1|$ is odd, since l_i is odd and $|A_1| + |A_2| = l_i$. Then $|A_1| \geq 3$, since $c_1^i, c_{l_i}^i \in A_1$. Now Claim 1 implies that $A_1 = \{c_1^i, c_3^i, \ldots, c_{(l_i+1)/2}^i, \ldots, c_{l_i}^i\}$ and $A_2 = \{c_2^i, c_4^i, \ldots, c_{l_i-1}^i\}$.

Fact 1. *There are two adjacent vertices $c_s^i, c_t^i \in A_2$ such that $|s - t| = 2$.*

Proof. Note that $l_i \equiv 1 \pmod 4$, since $\frac{l_i-1}{2}$ is even. If $l_i = 5$, then $c_2^i, c_4^i \in A_2$ are the desired vertices, since they are the only vertices of G_1^* of degree three. Thus assume that $l_i \geq 9$. If $\left\{c^i_{\frac{l_i+1}{2}+1}, c^i_{\frac{l_i+1}{2}-3}\right\} \cap N\left(c^i_{\frac{l_i+1}{2}-1}\right) \neq \emptyset$, then the desired pairs are $c^i_{\frac{l_i+1}{2}-1}$ and the vertex of $\left\{c^i_{\frac{l_i+1}{2}+1}, c^i_{\frac{l_i+1}{2}-3}\right\} \cap N\left(c^i_{\frac{l_i+1}{2}-1}\right)$. Thus assume that $\left\{c^i_{\frac{l_i+1}{2}+1}, c^i_{\frac{l_i+1}{2}-3}\right\} \cap N\left(c^i_{\frac{l_i+1}{2}-1}\right) = \emptyset$. Clearly there is a vertex $c^i_t \in A_2$ such that c^i_t is adjacent to $c^i_{\frac{l_i+1}{2}-1}$. Without loss of generality, assume that $t < \frac{l_i+1}{2} - 3$. Since G is an outerplanar graph, $\left|A_2 \cap \left\{c^i_h : t+2 \leq h \leq \frac{l_i+1}{2} - 3\right\}\right|$ is even. Furthermore, since G is an outerplanar graph, any vertex of $A_2 \cap \left\{c^i_h : t+2 \leq h \leq \frac{l_i+1}{2} - 3\right\}$ is adjacent to a vertex of $A_2 \cap \left\{c^i_h : t+2 \leq h \leq \frac{l_i+1}{2} - 3\right\}$. Consequently, there are two pairs $c^i_{h_1}, c^i_{h_2} \in A_2 \cap \left\{c^i_h : t+2 \leq h \leq \frac{l_i+1}{2} - 3\right\}$ such that $c^i_{h_1} \in N\left(c^i_{h_2}\right)$ and $|h_1 - h_2| = 2$. $\square$

Let c^i_t and c^i_{t+2} be two adjacent vertices of A_2 according to Fact 1. Clearly, $\deg\left(c^i_{t+1}\right) = 2$. Let $G^* = G - c^i_t c^i_{t-1} - c^i_t c^i_{t+1}$. Clearly $n(G^*) = n$, $m(G^*) = m - 2$ and $r - 1 \leq r(G^*) \leq r$. By the choice of G, $fd_1(G^*) \leq (4m(G^*) - 3n(G^*) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3$. Let S^* be a $fd_1(G^*)$-set. Since c^i_{t+2} is a strong support vertex of G^*, by Observation 2, we have $c^i_{t+2} \in S^*$. If $c^i_{t-1} \notin S^*$, then S^* is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 3$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 3$, a contradiction. Thus $c^i_{t-1} \in S'$. Then $S' \cup \left\{c^i_t, c^i_{t+1}\right\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction.

Subcase 2.2. $V(G'_1) \neq \left\{c^i_1, c^i_2, \ldots, c^i_{l_i}\right\}$. Since K_i is a leaf-block of G, $G'_1 - C_i$ has some vertex of degree at most one. Let v_d be a leaf of G'_1 such that $d(v_d, C^i - c^i_0)$ is as maximum as possible, and the shortest path from v_d to C^i does not contain c^i_0. Let $v_0 v_1 \cdots v_d$ be the shortest path from v_d to a vertex $v_0 \in C^i$.

Suppose that $d \geq 2$. Assume that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of G, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4(m-2) - 3(n-2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. We deduce that $\deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let G' be obtained from G by removing all leaves adjacent to v_{d-1}. Clearly $r(G') = r$. By the choice of G, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m-2) - 3(n-2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$-set. If $v_{d-1} \in S'$, then S' is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Now $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

We thus assume that $d = 1$. Let $D_1 = \left\{c^i_j \mid \deg_G(c^i_j) = 2\right\}$, $D_2 = \left\{c^i_j \mid c^i_j\right.$

is a support vertex of $G\big\}$ and $D_3 = \big\{c^i_j \mid \deg_G(c^i_j) \geq 3$ and c^i_j is not a support vertex of $G\big\}$. Clearly $|D_1 \mid + |D_2 \mid + |D_3 \mid = l_i$. Observe that $|D_2 \mid \geq 1$, since $d = 1$. Thus by Claims 1 and 2, $|D_1 \mid \leq |D_3 \mid$. Let $G^*_1 = G\big[G'_1 \cup \big\{c^i_0\big\}\big]$. Observe that $m(G^*_1) = \frac{1}{2}\sum_{v\in V(G^*_1)} \deg(v) \geq n(G^*_1) + |D_3 \mid /2$. Then $n(G^*_1) \geq l_i + 1 + |D_2 \mid$. Let $G^*_2 = \big[G'_2 \cup \big\{c^i_1, c^i_{l_i}\big\}\big] - \big\{c^i_{l_i} c^i_1\big\}$. Clearly $n = n(G^*_2) + n(G^*_1) - 3$, $m = m(G^*_2) + m(G^*_1) - 2$ and $r(G^*_2) = r - 1$. By the choice of G, $fd_1(G^*_2) \leq (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2)$. Let S'' be a $fd_1(G^*_2)$-set. By Observation 2, $c^i_0 \in S''$, since c^i_0 is a strong support vertex of G^*_2. Then $S'' \cup \big\{c^i_1, c^i_2, \ldots, c^i_{l_i}\big\}$ is a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$
\begin{aligned}
&(4m - 3n + 3)/2 - r\\
&\geq (4(m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r\\
&= (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3n(G^*_1) + 1)/2 - 1\\
&\geq |S''| + (4(n(G^*_1) + |D_3 \mid /2) - 3n(G^*_1) + 1)/2 - 1\\
&= |S''| + (n(G^*_1) + 2|D_3 \mid + 1)/2 - 1\\
&\geq |S''| + (l_i + 1 + |D_2 \mid + 2|D_3 \mid + 1)/2 - 1\\
&\geq (l_i + |D_2 \mid + |D_3 \mid + |D_1 \mid)/2 \geq |S''| + l_i.
\end{aligned}
$$

Thus $fd_1(G) \leq |S''| + l_i \leq (4m - 3n + 3)/2 - r$, a contradiction.

To the sharpness, consider a cycle C_5. ■

3. Concluding Remarks

As it is noted, Caro *et al.* [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph G of order n. They also proved that $fd(G) \leq n - 2$ for any connected graph G of order $n \geq 3$. It is worth-noting that the bound of Theorem 5 improves the bound $n - 2$ when $4m < 5n + 2r - 7$. It is also known that every maximal outerplanar graph G of order at least 3 is 2-connected [7], and thus $r(G) = 1$. Therefore, the bound of Theorem 5 improves the bound $17n/19$ when $4m < \frac{91n}{19} - 1$. We have the following conjecture.

Conjecture 6. *If G is a graph of order n and size m with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$.*

References

[1] Y. Caro, A. Hansberg and M.A. Henning, *Fair domination in graphs*, Discrete Math. **312** (2012) 2905–2914.
doi:10.1016/j.disc.2012.05.006

[2] B. Chaluvaraju, M. Chellali and K.A. Vidya, *Perfect k-domination in graphs*, Australas. J. Combin. **48** (2010) 175–184.

[3] B. Chaluvaraju and K. Vidya, *Perfect dominating set graph of a graph*, Adv. Appl. Discrete Math. **2** (2008) 49–57.

[4] E.J. Cockayne, B.L. Hartnell, S.T. Hedetniemi and R. Laskar, *Perfect domination in graphs*, J. Comb. Inf. Syst. Sci. **18** (1993) 136–148.

[5] I.J. Dejter, *Perfect domination in regular grid graphs*, Australas. J. Combin. **42** (2008) 99–114.

[6] I.J. Dejter and A.A. Delgado, *Perfect domination in rectangular grid graphs*, J. Combin. Math. Combin. Comput. **70** (2009) 177–196.

[7] I. Fabrici, *Light graphs in families of outerplanar graphs*, Discrete Math. **307** (2007) 866–872.
doi:10.1016/j.disc.2005.11.058

[8] M.R. Fellows and M.N. Hoover, *Perfect domination*, Australas. J. Combin. **3** (1991) 141–150.

[9] M. Hajian and N. Jafari Rad, *Trees and unicyclic graphs with large fair domination number*, Util. Math., to appear.

[10] M. Hajian and N. Jafari Rad, *Fair domination number in cactus graphs*, Discuss. Math. Graph Theory **39** (2019) 489–503.
doi:10.7151/dmgt.2088

[11] A. Hansberg, *Reviewing some results on fair domination in graphs*, Electron. Notes Discrete Math. **43** (2013) 367–373.
doi:10.1016/j.endm.2013.07.054

[12] H. Hatami and P. Hatami, *Perfect dominating sets in the Cartesian products of prime cycles*, Electron. J. Combin. **14** (2007) #N8.

[13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[14] J. Leydold and P.F. Stadler, *Minimal cycle bases of outerplanar graphs*, Electron. J. Combin. **5** (1998) #R16.

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