

A NOTE ON THE FAIR DOMINATION NUMBER IN OUTERPLANAR GRAPHS

MAJID HAJIAN

Department of Mathematics
Shahrood University of Technology
Shahrood, Iran

AND

NADER JAFARI RAD

Department of Mathematics
Shahed University, Tehran, Iran

e-mail: n.jafarirad@gmail.com

Abstract

For $k \geq 1$, a k -fair dominating set (or just k FD-set), in a graph G is a dominating set S such that $|N(v) \cap S| = k$ for every vertex $v \in V - S$. The k -fair domination number of G , denoted by $fd_k(G)$, is the minimum cardinality of a k FD-set. A fair dominating set, abbreviated FD-set, is a k FD-set for some integer $k \geq 1$. The fair domination number, denoted by $fd(G)$, of G that is not the empty graph, is the minimum cardinality of an FD-set in G . In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

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1. INTRODUCTION

For notation and graph theory terminology not given here, we follow [13]. Specifically, let G be a simple graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V . The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is

clear from the context, then we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v , is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v . A graph G of order at least three is *2-connected* if the deletion of any vertex does not disconnect the graph. A *cut-vertex* in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a *block*. A graph G is *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph G is *Hamiltonian* if there is a spanning cycle in G . For a subset S of vertices of G , we denote by $G[S]$ the subgraph of G induced by S .

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A vertex v is said to be *dominated* by a set S if $N[v] \cap S \neq \emptyset$.

Caro *et al.* [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated *kFD-set*, in G is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V - D$. The *k-fair domination number* of G , denoted by $fd_k(G)$, is the minimum cardinality of a *kFD-set*. A *kFD-set* of G of cardinality $fd_k(G)$ is called a *fd_k(G)-set*. A *fair dominating set*, abbreviated *FD-set*, in G is a *kFD-set* for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an *FD-set* in G . An *FD-set* of G of cardinality $fd(G)$ is called a *fd(G)-set*. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G) - S$ is adjacent to exactly one vertex in S . Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro *et al.* [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph G of order n , and among open problems posed by Caro *et al.* [1], one asks to find $fd(G)$ for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block K in an outerplanar graph G a *strong-block* if K contains

at least three vertices. We call a vertex w in a strong-block K of an outerplanar graph G a *special cut-vertex* if w belongs to a shortest path from K to a strong-block $K' \neq K$. We call a strong-block K in an outerplanar graph G a *leaf-block* if K contains exactly one special cut-vertex. We denote by $r(G)$ the number of strong-blocks of a graph G . The following is straightforward.

Observation 1. *Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.*

We make use of the following.

Observation 2 (Caro et al. [1]). *Every 1FD-set in a graph contains all its strong support vertices.*

Theorem 3 (Leydolda et al. [14]). *An outerplanar graph G is Hamiltonian if and only if it is 2-connected.*

Theorem 4 (Hajian et al. [9]). *If G is a unicyclic graph of order n , then $fd_1(G) \leq (n + 1)/2$.*

2. MAIN RESULT

Theorem 5. *If G is an outerplanar graph of order n and size m with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$. This bound is sharp.*

Proof. Let G be an outerplanar graph of order n and size m with $r \geq 1$ strong-blocks. We prove that $fd_1(G) \leq (4m - 3n + 3)/2 - r$. The result follows from Theorem 4 if G is a unicyclic graph. Thus assume that G is not a unicyclic graph. Suppose to the contrary that $fd_1(G) > (4m - 3n + 3)/2 - r$. Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is as minimum as possible. Let $K_1, K_2, \dots, K_r$ be the r strong-blocks of G . By Theorem 3, K_j is Hamiltonian, for $1 \leq j \leq r$. Let $C^i = c_0^i c_1^i \dots c_{l_i}^i c_0^i$ be a Hamiltonian cycle for K_i , for $1 \leq i \leq r$. We proceed with the following Claims 1 and 2.

Claim 1. *For any $1 \leq i \leq r$, if c_j^i is a vertex of C^i , for some $j \in \{0, 1, \dots, l_i\}$, such that $\deg_G(c_j^i) = 2$, then $\deg_G(c_{j+1}^i) \geq 3$ and $\deg_G(c_{j-1}^i) \geq 3$, where the calculations in $j + 1$ and $j - 1$ are taken modulo l_i .*

Proof. Assume that $\deg_G(c_j^i) = 2$ for some $j \in \{0, 1, \dots, l_i\}$. Suppose that $\deg_G(c_{j+1}^i) = 2$. Let $G' = G - c_j^i c_{j+1}^i$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m - 1) - 3n + 3)/2 - (r - 1) = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$ -set. If $|S' \cap \{c_j^i, c_{j+1}^i\}| \in \{0, 2\}$,

then S' is a 1FD-set for G of cardinality at most $(4m - 3n + 3)/2 - r - 1$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $|S' \cap \{c_j^i, c_{j+1}^i\}| = 1$. Assume that $c_j^i \in S'$. Then $c_{j+1}^i \notin S'$, and $c_{j+2}^i \in S'$, since S' is a dominating set. Thus $\{c_{j+1}^i\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Next assume that $c_{j+1}^i \in S'$. Then $c_j^i \notin S'$ and $c_{j-1}^i \in S'$. Thus $\{c_j^i\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$. So $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $\deg_G(c_{j+1}^i) \geq 3$. Similarly, $\deg_G(c_{j-1}^i) \geq 3$. $\square$

Claim 2. *If c_j^i is a vertex of C^i , for some $j \in \{0, 1, \dots, l_i\}$, such that $\deg_G(c_j^i) = 2$, then non of c_{j+1}^i and c_{j-1}^i is a support vertex of G .*

Proof. Assume that $\deg_G(c_j^i) = 2$ for some $j \in \{0, 1, \dots, l_i\}$. Suppose that c_{j+1}^i is a support vertex of G . Let $G' = G - c_j^i c_{j-1}^i$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m - 1) - 3n + 3)/2 - (r - 1) = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$ -set. By Observation 2, $c_{j+1}^i \in S'$, since c_{j+1}^i is a strong support vertex of G' . If $c_{j-1}^i \notin S'$, then S' is a 1FD-set for G of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $c_{j-1}^i \in S'$ and so $\{c_j^i\} \cup S'$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence c_{j+1}^i is not a support vertex of G . Similarly, c_{j-1}^i is not a support vertex of G . $\square$

We consider the following cases.

Case 1. $r = 1$. First assume that $V(G) = \{c_0^1, c_1^1, \dots, c_{l_1}^1\}$ and so $n = l_1 + 1$. By Claim 1, at least $\lceil n/2 \rceil$ vertices of C^1 are of degree at least 3. Now, we can easily see that $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + \lceil n/2 \rceil/2$. (Since $\delta(G) \geq 2$ and at least $\lceil n/2 \rceil$ vertices of G are of degree at least 3, we have $\sum_{v \in V(G)} \deg(v) \geq 2n + \lceil n/2 \rceil$.) Thus $m \geq n + \lceil n/2 \rceil/2$. If n is even, then $n \leq (4m - 3n)/2$ and if n is odd, then $n \leq (4m - 3n - 1)/2$. We thus obtain that $n \leq (4m - 3n + 3)/2 - 1$. Now $V(G)$ is a 1FD-set in G of cardinality n , and thus $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. We deduce that $V(G) \neq \{c_0^1, c_1^1, \dots, c_{l_1}^1\}$. Since $r = 1$, there is a vertex of degree one in G . Let v_d be a leaf of G such that $d(v_d, C^1)$ is maximum. Let $v_0 v_1 \dots v_d$ be the shortest path from v_d to a vertex $v_0 \in C^1$. Clearly, $\{v_0, v_1, \dots, v_d\} \cap V(C^1) = \{v_0\}$.

Assume that $d \geq 2$. Suppose that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4(m - 2) - 3(n - 2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$. Let S' be a $fd_1(G')$ -set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most

$(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. Thus assume that $\deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let G' be obtained from G by removing all leaves adjacent to v_{d-1} . Clearly $r(G') = r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m-2) - 3(n-2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$. Let S' be a $fd_1(G')$ -set. If $v_{d-1} \in S'$, then S' is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - 2$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 2$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Now $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction.

We next assume that $d = 1$. Let $D_1 = \{c_j^1 \mid \deg_G(c_j^1) = 2\}$ and $D_2 = \{c_j^1 \mid c_j^1 \text{ is a support vertex of } G\}$ and $D_3 = \{c_j^1 \mid \deg_G(c_j^1) \geq 3 \text{ and } c_j^1 \text{ is not a support vertex of } G\}$. Clearly $|D_1| + |D_2| + |D_3| = l_1 + 1$. Since $d = 1$, we have $|D_2| \geq 1$. By Claims 1 and 2, $|D_1| \leq |D_3|$. Observe that $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + |D_3|/2$. Clearly $n \geq l_1 + 1 + |D_2|$. Thus

$$\begin{aligned} (4m - 3n + 3)/2 - 1 &\geq (4(n + |D_3|/2) - 3n + 3)/2 - 1 \\ &\geq (l_1 + 1 + |D_2| + 2|D_3| + 3)/2 - 1 \\ &\geq (l_1 + 1 + |D_1| + |D_2| + |D_3| + 3)/2 - 1 \\ &= l_1 + 3/2 > l_1 + 1. \end{aligned}$$

Evidently, $\{c_0^1, \dots, c_{l_1}^1\}$ is a $fd_1(G)$ -set of cardinality $l_1 + 1$. Thus $fd_1(G) < (4m - 3n + 3)/2 - r$, a contradiction.

Case 2. $r \geq 2$. By Observation 1, G has at least two leaf-blocks. Let K_i be a leaf-block of G , where $i \in \{1, 2, \dots, r\}$. By relabeling of the vertices of C^i we may assume that c_0^i is a special cut-vertex of G . Let G' be the graph obtained by removal of all edges $c_0^i c_j^i$, with $c_j^i \in \{c_1^i, \dots, c_{l_i}^i\}$. Clearly G' has two components. Let G'_1 be the component of G' containing c_1^i , and G'_2 be the component of G' containing c_0^i . Clearly, $\{c_1^i, c_2^i, \dots, c_{l_i}^i\} \subseteq V(G'_1)$. We consider the following subcases.

Subcase 2.1. $V(G'_1) = \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$. Let $G_1^* = G[V(G'_1) \cup \{c_0^i\}]$. Clearly $n(G_1^*) = l_i + 1$. By Claim 1, at least $\lfloor l_i/2 \rfloor$ vertices of $C^i - c_0^i$ are of degree at least 3.

Assume that l_i is even. Thus at least $l_i/2$ vertices of $C^i - c_0^i$ are of degree at least 3. Now, we can easily see that $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \geq l_i + 1 + l_i/4$. Let $G_2^* = G[V(G'_2) \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_i}^i c_1^i\}$. Clearly $n = n(G_2^*) + l_i - 2$, $m = m(G_2^*) + m(G_1^*) - 2$ and $r(G_2^*) = r - 1$. By the choice of G , $fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$. Let S'' be a $fd_1(G_2^*)$ -set. By Observation 2, $c_0^i \in S''$, since c_0^i is a strong support vertex of G_2^* . Then $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$ is

a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$\begin{aligned} & (4m - 3n + 3)/2 - r \\ & \geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\ & = (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1 \\ & \geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i. \end{aligned}$$

Thus $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

Assume next that l_i is odd. Observe that at least $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3. Now, we can easily see that $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \geq l_i + 1 + (l_i - 1)/4$. We show that $m(G_1^*) = l_i + 1 + (l_i - 1)/4$. Suppose that $m(G_1^*) > l_i + 1 + (l_i - 1)/4$. Then $m(G_1^*) \geq l_i + 1 + (l_i - 1)/4 + 1/4$. Let $G_2^* = G[G_2' \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_i}^i, c_1^i\}$. Clearly $n = n(G_2^*) + l_i - 2$, $m = m(G_2^*) + m(G_1^*) - 2$ and $r(G_2^*) = r - 1$. By the choice of G , $fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$. Let S'' be a $fd_1(G_2^*)$ -set. By Observation 2, $c_0^i \in S''$, since c_0^i is a strong support vertex of G_2^* . Then $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$ is a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$\begin{aligned} & (4m - 3n + 3)/2 - r \\ & \geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\ & = (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1 \\ & \geq |S''| + (4(l_i + 1 + (l_i - 1)/4 + 1/4) - 3l_i - 2)/2 - 1 = |S''| + l_i. \end{aligned}$$

Thus $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. We thus obtain that $m(G_1^*) = l_i + 1 + (l_i - 1)/4$. Note that $|E(G_1^*) \cap E(C^i)| = l_i + 1$. Hence $|E(G_1^*) - E(C^i)| = (l_i - 1)/4$. Since $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree at least 3, we thus obtain that precisely $(l_i - 1)/2$ vertices of $C^i - c_0^i$ are of degree 3, and so $(l_i + 1)/2$ vertices of $C^i - c_0^i$ are of degree two. Now Claim 1 implies that $\deg_G(c_1^i) = \deg_G(c_{l_i}^i) = 2$. Thus we obtain that $\deg_{G_1^*}(c_0^i) = 2$. Let $A_1 = \{c_j \mid \deg_G(c_j^i) = 2 \text{ for } 1 \leq j \leq l_i\}$ and $A_2 = \{c_1^i, c_2^i, \dots, c_{l_i}^i\} - A_1$. Clearly $|A_1| = (l_i + 1)/2$ and $|A_2| = (l_i - 1)/2$. Note that $|A_2|$ is even, since the number of odd vertices in every graph (here G_1^*) is even. Thus $|A_1|$ is odd, since l_i is odd and $|A_1| + |A_2| = l_i$. Then $|A_1| \geq 3$, since $c_1^i, c_{l_i}^i \in A_1$. Now Claim 1 implies that $A_1 = \{c_1^i, c_3^i, \dots, c_{(l_i+1)/2}^i, \dots, c_{l_i}^i\}$ and $A_2 = \{c_2^i, c_4^i, \dots, c_{l_i-1}^i\}$.

Fact 1. *There are two adjacent vertices $c_s^i, c_t^i \in A_2$ such that $|s - t| = 2$.*

Proof. Note that $l_i \equiv 1 \pmod{4}$, since $\frac{l_i-1}{2}$ is even. If $l_i = 5$, then $c_2^i, c_4^i \in A_2$ are the desired vertices, since they are the only vertices of G_1^* of degree three. Thus assume that $l_i \geq 9$. If $\{c_{\frac{l_i+1}{2}+1}^i, c_{\frac{l_i+1}{2}-3}^i\} \cap N(c_{\frac{l_i+1}{2}-1}^i) \neq \emptyset$, then the desired pairs

are $c_{\frac{l_i+1}{2}-1}^i$ and the vertex of $\{c_{\frac{l_i+1}{2}+1}^i, c_{\frac{l_i+1}{2}-3}^i\} \cap N(c_{\frac{l_i+1}{2}-1}^i)$. Thus assume that $\{c_{\frac{l_i+1}{2}+1}^i, c_{\frac{l_i+1}{2}-3}^i\} \cap N(c_{\frac{l_i+1}{2}-1}^i) = \emptyset$. Clearly there is a vertex $c_t^i \in A_2$ such that c_t^i is adjacent to $c_{\frac{l_i+1}{2}-1}^i$. Without loss of generality, assume that $t < \frac{l_i+1}{2} - 3$. Since G is an outerplanar graph, $|A_2 \cap \{c_h^i : t+2 \leq h \leq \frac{l_i+1}{2} - 3\}|$ is even. Furthermore, since G is an outerplanar graph, any vertex of $A_2 \cap \{c_h^i : t+2 \leq h \leq \frac{l_i+1}{2} - 3\}$ is adjacent to a vertex of $A_2 \cap \{c_h^i : t+2 \leq h \leq \frac{l_i+1}{2} - 3\}$. Consequently, there are two pairs $c_{h_1}^i, c_{h_2}^i \in A_2 \cap \{c_h^i : t+2 \leq h \leq \frac{l_i+1}{2} - 3\}$ such that $c_{h_1}^i \in N(c_{h_2}^i)$ and $|h_1 - h_2| = 2$. $\square$

Let c_t^i and c_{t+2}^i be two adjacent vertices of A_2 according to Fact 1. Clearly, $\deg(c_{t+1}^i) = 2$. Let $G^* = G - c_t^i c_{t-1}^i - c_t^i c_{t+1}^i$. Clearly $n(G^*) = n$, $m(G^*) = m - 2$ and $r - 1 \leq r(G^*) \leq r$. By the choice of G , $fd_1(G^*) \leq (4m(G^*) - 3n(G^*) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3$. Let S^* be a $fd_1(G^*)$ -set. Since c_{t+2}^i is a strong support vertex of G^* , by Observation 2, we have $c_{t+2}^i \in S^*$. If $c_{t-1}^i \notin S^*$, then S^* is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 3$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 3$, a contradiction. Thus $c_{t-1}^i \in S'$. Then $S' \cup \{c_t^i, c_{t+1}^i\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction.

Subcase 2.2. $V(G'_1) \neq \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$. Since K_i is a leaf-block of G , $G'_1 - C_i$ has some vertex of degree at most one. Let v_d be a leaf of G'_1 such that $d(v_d, C^i - c_0^i)$ is as maximum as possible, and the shortest path from v_d to C^i does not contain c_0^i . Let $v_0 v_1 \dots v_d$ be the shortest path from v_d to a vertex $v_0 \in C^i$.

Suppose that $d \geq 2$. Assume that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4(m-2) - 3(n-2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$ -set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. We deduce that $\deg_G(v_{d-1}) \geq 3$. Clearly any vertex of $N_G(v_{d-1}) - \{v_{d-2}\}$ is a leaf. Let G' be obtained from G by removing all leaves adjacent to v_{d-1} . Clearly $r(G') = r$. By the choice of G , $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m-2) - 3(n-2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$. Let S' be a $fd_1(G')$ -set. If $v_{d-1} \in S'$, then S' is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus assume that $v_{d-1} \notin S'$. Then $v_{d-2} \in S'$. Now $S' \cup \{v_{d-1}\}$ is a 1FD-set in G of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.

We thus assume that $d = 1$. Let $D_1 = \{c_j^i \mid \deg_G(c_j^i) = 2\}$, $D_2 = \{c_j^i \mid c_j^i$

is a support vertex of G and $D_3 = \{c_j^i \mid \deg_G(c_j^i) \geq 3 \text{ and } c_j^i \text{ is not a support vertex of } G\}$. Clearly $|D_1| + |D_2| + |D_3| = l_i$. Observe that $|D_2| \geq 1$, since $d = 1$. Thus by Claims 1 and 2, $|D_1| \leq |D_3|$. Let $G_1^* = G[G'_1 \cup \{c_0^i\}]$. Observe that $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \geq n(G_1^*) + |D_3|/2$. Then $n(G_1^*) \geq l_i + 1 + |D_2|$. Let $G_2^* = [G'_2 \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_i}^i c_1^i\}$. Clearly $n = n(G_2^*) + n(G_1^*) - 3$, $m = m(G_2^*) + m(G_1^*) - 2$ and $r(G_2^*) = r - 1$. By the choice of G , $fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$. Let S'' be a $fd_1(G_2^*)$ -set. By Observation 2, $c_0^i \in S''$, since c_0^i is a strong support vertex of G_2^* . Then $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$ is a 1FD-set for G of cardinality $|S''| + l_i$. On the other hand

$$\begin{aligned}
 & (4m - 3n + 3)/2 - r \\
 & \geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\
 & = (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3n(G_1^*) + 1)/2 - 1 \\
 & \geq |S''| + (4(n(G_1^*) + |D_3|/2) - 3n(G_1^*) + 1)/2 - 1 \\
 & = |S''| + (n(G_1^*) + 2|D_3| + 1)/2 - 1 \\
 & \geq |S''| + (l_i + 1 + |D_2| + 2|D_3| + 1)/2 - 1 \\
 & \geq (l_i + |D_2| + |D_3| + |D_1|)/2 \geq |S''| + l_i.
 \end{aligned}$$

Thus $fd_1(G) \leq |S''| + l_i \leq (4m - 3n + 3)/2 - r$, a contradiction.

To the sharpness, consider a cycle C_5 . ■

3. CONCLUDING REMARKS

As it is noted, Caro *et al.* [1] proved that $fd(G) < 17n/19$ for any maximal outerplanar graph G of order n . They also proved that $fd(G) \leq n - 2$ for any connected graph G of order $n \geq 3$. It is worth-noting that the bound of Theorem 5 improves the bound $n - 2$ when $4m < 5n + 2r - 7$. It is also known that every maximal outerplanar graph G of order at least 3 is 2-connected [7], and thus $r(G) = 1$. Therefore, the bound of Theorem 5 improves the bound $17n/19$ when $4m < \frac{91n}{19} - 1$. We have the following conjecture.

Conjecture 6. *If G is a graph of order n and size m with $r \geq 1$ strong-blocks, then $fd(G) \leq (4m - 3n + 3)/2 - r$.*

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