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# A NOTE ON THE FAIR DOMINATION NUMBER IN OUTERPLANAR GRAPHS

#### Majid Hajian

Department of Mathematics Shahrood University of Technology Shahrood, Iran

#### AND

NADER JAFARI RAD

Department of Mathematics Shahed University, Tehran, Iran

e-mail: n.jafarirad@gmail.com

#### Abstract

For  $k \geq 1$ , a k-fair dominating set (or just kFD-set), in a graph G is a dominating set S such that  $|N(v) \cap S| = k$  for every vertex  $v \in V - S$ . The k-fair domination number of G, denoted by  $fd_k(G)$ , is the minimum cardinality of a kFD-set. A fair dominating set, abbreviated FD-set, is a kFD-set for some integer  $k \geq 1$ . The fair domination number, denoted by fd(G), of G that is not the empty graph, is the minimum cardinality of an FD-set in G. In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

Keywords: fair domination, outerplanar graph, unicyclic graph.

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# 1. INTRODUCTION

For notation and graph theory terminology not given here, we follow [13]. Specifically, let G be a simple graph with vertex set V(G) = V of order |V| = n and let v be a vertex in V. The open neighborhood of v is  $N_G(v) = \{u \in V \mid uv \in E(G)\}$  and the closed neighborhood of v is  $N_G[v] = \{v\} \cup N_G(v)$ . If the graph G is

clear from the context, then we simply write N(v) rather than  $N_G(v)$ . The degree of a vertex v, is  $\deg(v) = |N(v)|$ . A vertex of degree one is called a *leaf* and its neighbor a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set  $S \subseteq V$ , its open neighborhood is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and its closed neighborhood is the set  $N[S] = N(S) \cup S$ . The distance d(u, v) between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. A graph G of order at least three is 2-connected if the deletion of any vertex does not disconnect the graph. A cutvertex in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a block. A graph Gis outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph G is Hamiltonian if there is a spanning cycle in G. For a subset S of vertices of G, we denote by G[S] the subgraph of G induced by S.

A subset  $S \subseteq V$  is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of G. A vertex v is said to be *dominated* by a set S if  $N[v] \cap S \neq \emptyset$ .

Caro *et al.* [1] studied the concept of fair domination in graphs. For  $k \ge 1$ , a k-fair dominating set, abbreviated kFD-set, in G is a dominating set S such that  $|N(v) \cap D| = k$  for every vertex  $v \in V - D$ . The k-fair domination number of G, denoted by  $fd_k(G)$ , is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality  $fd_k(G)$  is called a  $fd_k(G)$ -set. A fair dominating set, abbreviated FD-set, in G is a kFD-set for some integer  $k \geq 1$ . The fair domination number, denoted by fd(G), of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality fd(G) is called a fd(G)-set. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A perfect dominating set in a graph G is a dominating set S such that every vertex in V(G) - S is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows et al. [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro *et al.* [1] proved that fd(G) < 17n/19 for any maximal outerplanar graph G of order n, and among open problems posed by Caro *et al.* [1], one asks to find fd(G) for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block K in an outerplanar graph G a strong-block if K contains

at least three vertices. We call a vertex w in a strong-block K of an outerplanar graph G a special cut-vertex if w belongs to a shortest path from K to a strongblock  $K' \neq K$ . We call a strong-block K in an outerplanar graph G a leaf-block if K contains exactly one special cut-vertex. We denote by r(G) the number of strong-blocks of a graph G. The following is straightforward.

**Observation 1.** Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.

We make use of the following.

**Observation 2** (Caro *et al.* [1]). Every 1FD-set in a graph contains all its strong support vertices.

**Theorem 3** (Leydolda et al. [14]). An outerplanar graph G is Hamiltonian if and only if it is 2-connected.

**Theorem 4** (Hajian *et al.* [9]). If G is a unicyclic graph of order n, then  $fd_1(G) \leq (n+1)/2$ .

# 2. Main Result

**Theorem 5.** If G is an outerplanar graph of order n and size m with  $r \ge 1$  strong-blocks, then  $fd(G) \le (4m - 3n + 3)/2 - r$ . This bound is sharp.

**Proof.** Let G be an outerplanar graph of order n and size m with  $r \ge 1$  strongblocks. We prove that  $fd_1(G) \le (4m - 3n + 3)/2 - r$ . The result follows from Theorem 4 if G is a unicyclic graph. Thus assume that G is not a unicyclic graph. Suppose to the contrary that  $fd_1(G) > (4m - 3n + 3)/2 - r$ . Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is as minimum as possible. Let  $K_1, K_2, \ldots, K_r$  be the r strong-blocks of G. By Theorem 3,  $K_j$  is Hamiltonian, for  $1 \le j \le r$ . Let  $C^i = c_0^i c_1^i \cdots c_{l_i}^i c_0^i$  be a Hamiltonian cycle for  $K_i$ , for  $1 \le i \le r$ . We proceed with the following Claims 1 and 2.

**Claim 1.** For any  $1 \le i \le r$ , if  $c_j^i$  is a vertex of  $C^i$ , for some  $j \in \{0, 1, ..., l_i\}$ , such that  $\deg_G(c_j^i) = 2$ , then  $\deg_G(c_{j+1}^i) \ge 3$  and  $\deg_G(c_{j-1}^i) \ge 3$ , where the calculations in j + 1 and j - 1 are taken modulo  $l_i$ .

**Proof.** Assume that  $\deg_G(c_j^i) = 2$  for some  $j \in \{0, 1, \ldots, l_i\}$ . Suppose that  $\deg_G(c_{j+1}^i) = 2$ . Let  $G' = G - c_j^i c_{j+1}^i$ . Clearly  $r - 1 \le r(G') \le r$ . By the choice of G,  $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') \le (4(m-1) - 3n + 3)/2 - (r-1) = (4m - 3n + 3)/2 - r - 1$ . Let S' be a  $fd_1(G')$ -set. If  $|S' \cap \{c_j^i, c_{j+1}^i\}| \in \{0, 2\}$ ,

then S' is a 1FD-set for G of cardinality at most (4m - 3n + 3)/2 - r - 1, and so  $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$ , a contradiction. Thus  $|S' \cap \{c_j^i, c_{j+1}^i\}| = 1$ . Assume that  $c_j^i \in S'$ . Then  $c_{j+1}^i \notin S'$ , and  $c_{j+2}^i \in S'$ , since S' is a dominating set. Thus  $\{c_{j+1}^i\} \cup S'$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r and so  $fd_1(G) \leq (4m - 3n + 3)/2 - r$ , a contradiction. Next assume that  $c_{j+1}^i \in S'$ . Then  $c_j^i \notin S'$  and  $c_{j-1}^i \in S'$ . Thus  $\{c_j^i\} \cup S'$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r. So  $fd_1(G) \leq (4m - 3n + 3)/2 - r$ , a contradiction. Hence  $\deg_G(c_{j+1}^i) \geq 3$ . Similarly,  $\deg_G(c_{j-1}^i) \geq 3$ .

**Claim 2.** If  $c_j^i$  is a vertex of  $C^i$ , for some  $j \in \{0, 1, \ldots, l_i\}$ , such that  $\deg_G (c_j^i) = 2$ , then non of  $c_{j+1}^i$  and  $c_{j-1}^i$  is a support vertex of G.

**Proof.** Assume that  $\deg_G (c_j^i) = 2$  for some  $j \in \{0, 1, \ldots, l_i\}$ . Suppose that  $c_{j+1}^i$  is a support vertex of G. Let  $G' = G - c_j^i c_{j-1}^i$ . Clearly  $r - 1 \le r(G') \le r$ . By the choice of G,  $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') \le (4(m-1) - 3n+3)/2 - (r-1) = (4m-3n+3)/2 - r - 1$ . Let S' be a  $fd_1(G')$ -set. By Observation 2,  $c_{j+1}^i \in S'$ , since  $c_{j+1}^i$  is a strong support vertex of G'. If  $c_{j-1}^i \notin S'$ , then S' is a 1FD-set for G of cardinality at most (4m-3n+3)/2 - r - 1 and so  $fd_1(G) \le (4m-3n+3)/2 - r - 1$ , a contradiction. Thus  $c_{j-1}^i \in S'$  and so  $\{c_j^i\} \cup S'$  is a 1FD-set in G of cardinality at most (4m-3n+3)/2 - r, and so  $fd_1(G) \le (4m-3n+3)/2 - r$ , a contradiction. Hence  $c_{j+1}^i$  is not a support vertex of G.

We consider the following cases.

Case 1. r = 1. First assume that  $V(G) = \{c_0^1, c_1^1, \ldots, c_{l_1}^1\}$  and so  $n = l_1 + 1$ . By Claim 1, at least  $\lceil n/2 \rceil$  vertices of  $C^1$  are of degree at least 3. Now, we can easily see that  $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \ge n + \lceil n/2 \rceil/2$ . (Since  $\delta(G) \ge 2$  and at least  $\lceil n/2 \rceil$  vertices of G are of degree at least 3, we have  $\sum_{v \in V(G)} \deg(v) \ge 2n + \lceil n/2 \rceil$ .) Thus  $m \ge n + \lceil n/2 \rceil/2$ . If n is even, then  $n \le (4m - 3n)/2$  and if n is odd, then  $n \le (4m - 3n - 1)/2$ . We thus obtain that  $n \le (4m - 3n + 3)/2 - 1$ . Now V(G) is a 1FD-set in G of cardinality n, and thus  $fd_1(G) \le (4m - 3n + 3)/2 - 1$ , a contradiction. We deduce that  $V(G) \ne \{c_0^1, c_1^1, \ldots, c_{l_1}^1\}$ . Since r = 1, there is a vertex of degree one in G. Let  $v_d$  be a leaf of G such that  $d(v_d, C^1)$  is maximum. Let  $v_0v_1\cdots v_d$  be the shortest path from  $v_d$  to a vertex  $v_0 \in C^1$ . Clearly,  $\{v_0, v_1, \ldots, v_d\} \cap V(C^1) = \{v_0\}$ .

Assume that  $d \ge 2$ . Suppose that  $\deg_G(v_{d-1}) = 2$ . Let  $G' = G - \{v_d, v_{d-1}\}$ . Clearly r(G') = r. By the choice of G,  $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') = (4(m-2) - 3(n-2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$ . Let S' be a  $fd_1(G')$ -set. If  $v_{d-2} \notin S'$ , then  $S' \cup \{v_d\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - 1 and so  $fd_1(G) \le (4m - 3n + 3)/2 - 1$ , a contradiction. Thus  $v_{d-2} \in S'$ . Then  $S' \cup \{v_{d-1}\}$  is a 1FD-set in G of cardinality at most

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(4m - 3n + 3)/2 - 1 and so  $fd_1(G) \leq (4m - 3n + 3)/2 - 1$ , a contradiction. Thus assume that  $\deg_G(v_{d-1}) \geq 3$ . Clearly any vertex of  $N_G(v_{d-1}) - \{v_{d-2}\}$  is a leaf. Let G' be obtained from G by removing all leaves adjacent to  $v_{d-1}$ . Clearly r(G') = r. By the choice of G,  $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m-2) - 3(n-2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2$ . Let S' be a  $fd_1(G')$ -set. If  $v_{d-1} \in S'$ , then S' is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - 2 and so  $fd_1(G) \leq (4m - 3n + 3)/2 - 2$ , a contradiction. Thus assume that  $v_{d-1} \notin S'$ . Then  $v_{d-2} \in S'$ . Now  $S' \cup \{v_{d-1}\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - 1 and so  $fd_1(G) \leq (4m - 3n + 3)/2 - 1$ , a contradiction.

We next assume that d = 1. Let  $D_1 = \{c_j^1 \mid \deg_G(c_j^1) = 2\}$  and  $D_2 = \{c_j^1 \mid c_j^1 \text{ is a support vertex of } G\}$  and  $D_3 = \{c_j^1 \mid \deg_G(c_j^1) \ge 3 \text{ and } c_j^1 \text{ is not a support vertex of } G\}$ . Clearly  $|D_1| + |D_2| + |D_3| = l_1 + 1$ . Since d = 1, we have  $|D_2| \ge 1$ . By Claims 1 and 2,  $|D_1| \le |D_3|$ . Observe that  $m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \ge n + |D_3|/2$ . Clearly  $n \ge l_1 + 1 + |D_2|$ . Thus

$$\begin{aligned} (4m-3n+3)/2-1 &\geq (4(n+|D_3|/2)-3n+3)/2-1 \\ &\geq (l_1+1+|D_2|+2|D_3|+3)/2-1 \\ &\geq (l_1+1+|D_1|+|D_2|+|D_3|+3)/2-1 \\ &= l_1+3/2 > l_1+1. \end{aligned}$$

Evidently,  $\{c_0^1, \ldots, c_{l_1}^1\}$  is a  $fd_1(G)$ -set of cardinality  $l_1+1$ . Thus  $fd_1(G) < (4m-3n+3)/2 - r$ , a contradiction.

Case 2.  $r \geq 2$ . By Observation 1, G has at least two leaf-blocks. Let  $K_i$  be a leaf-block of G, where  $i \in \{1, 2, ..., r\}$ . By relabeling of the vertices of  $C^i$  we may assume that  $c_0^i$  is a special cut-vertex of G. Let G' be the graph obtained by removal of all edges  $c_0^i c_j^i$ , with  $c_j^i \in \{c_1^i, \ldots, c_{l_i}^i\}$ . Clearly G' has two components. Let  $G'_1$  be the component of G' containing  $c_1^i$ , and  $G'_2$  be the component of G' containing  $c_0^i$ . Clearly,  $\{c_1^i, c_2^i, \ldots, c_{l_i}^i\} \subseteq V(G'_1)$ . We consider the following subcases.

Subcase 2.1.  $V(G'_1) = \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$ . Let  $G_1^* = G[V(G'_1) \cup \{c_0^i\}]$ . Clearly  $n(G_1^*) = l_i + 1$ . By Claim 1, at least  $\lfloor l_i/2 \rfloor$  vertices of  $C^i - c_0^i$  are of degree at least 3.

Assume that  $l_i$  is even. Thus at least  $l_i/2$  vertices of  $C^i - c_0^i$  are of degree at least 3. Now, we can easily see that  $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \ge l_i + 1 + l_i/4$ . Let  $G_2^* = G[V(G_2') \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_1}^i c_1^i\}$ . Clearly  $n = n(G_2^*) + l_i - 2$ ,  $m = m(G_2^*) + m(G_1^*) - 2$  and  $r(G_2^*) = r - 1$ . By the choice of G,  $fd_1(G_2^*) \le (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$ . Let S'' be a  $fd_1(G_2^*)$ -set. By Observation 2,  $c_0^i \in S''$ , since  $c_0^i$  is a strong support vertex of  $G_2^*$ . Then  $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$  is

a 1FD-set for G of cardinality  $|S''| + l_i$ . On the other hand

$$\begin{aligned} (4m - 3n + 3)/2 - r \\ &\geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\ &= (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1 \\ &\geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i. \end{aligned}$$

Thus  $fd_1(G) \leq (4m - 3n + 3)/2 - r$ , a contradiction.

Assume next that  $l_i$  is odd. Observe that at least  $(l_i - 1)/2$  vertices of  $C^i - c_0^i$  are of degree at least 3. Now, we can easily see that  $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \ge l_i + 1 + (l_i - 1)/4$ . We show that  $m(G_1^*) = l_i + 1 + (l_i - 1)/4$ . Suppose that  $m(G_1^*) > l_i + 1 + (l_i - 1)/4$ . Then  $m(G_1^*) \ge l_i + 1 + (l_i - 1)/4 + 1/4$ . Let  $G_2^* = G[G'_2 \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_i}^i c_1^i\}$ . Clearly  $n = n(G_2^*) + l_i - 2$ ,  $m = m(G_2^*) + m(G_1^*) - 2$  and  $r(G_2^*) = r - 1$ . By the choice of G,  $fd_1(G_2^*) \le (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$ . Let S'' be a  $fd_1(G_2^*)$ -set. By Observation 2,  $c_0^i \in S''$ , since  $c_0^i$  is a strong support vertex of  $G_2^*$ . Then  $S'' \cup \{c_1^i, c_2^i, \ldots, c_{l_i}^i\}$  is a 1FD-set for G of cardinality  $|S''| + l_i$ . On the other hand

$$\begin{aligned} (4m - 3n + 3)/2 - r \\ &\geq (4(m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\ &= (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3(l_i + 1) + 1)/2 - 1 \\ &\geq |S''| + (4(l_i + 1 + (l_i - 1)/4 + 1/4) - 3l_i - 2)/2 - 1 = |S''| + l_i. \end{aligned}$$

Thus  $fd_1(G) \leq (4m-3n+3)/2-r$ , a contradiction. We thus obtain that  $m(G_1^*) = l_i + 1 + (l_i - 1)/4$ . Note that  $|E(G_1^*) \cap E(C^i)| = l_i + 1$ . Hence  $|E(G_1^*) - E(C^i)| = (l_i - 1)/4$ . Since  $(l_i - 1)/2$  vertices of  $C^i - c_0^i$  are of degree at least 3, we thus obtain that precisely  $(l_i - 1)/2$  vertices of  $C^i - c_0^i$  are of degree 3, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 3, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 3, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 4, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 4, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 5, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 5, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and so  $(l_i + 1)/2$  vertices of  $C^i - c_0^i$  are of degree 6, and  $A_2 = \{c_1^i, c_2^i, \ldots, c_{l_i}^i\} - A_1$ . Clearly  $|A_1| = (l_i + 1)/2$  and  $|A_2| = (l_i - 1)/2$ . Note that  $|A_2|$  is even, since the number of odd vertices in every graph (here  $G_1^*)$  is even. Thus  $|A_1|$  is odd, since  $l_i$  is odd and  $|A_1| + |A_2| = l_i$ . Then  $|A_1| \ge 3$ , since  $c_1^i, c_{l_i}^i \in A_1$ . Now Claim 1 implies that  $A_1 = \{c_1^i, c_3^i, \ldots, c_{(l_i+1)/2}^i, \ldots, c_{l_i}^i\}$  and  $A_2 = \{c_2^i, c_4^i, \ldots, c_{l_i-1}^i\}$ .

**Fact 1.** There are two adjacent vertices  $c_s^i, c_t^i \in A_2$  such that |s - t| = 2.

**Proof.** Note that  $l_i \equiv 1 \pmod{4}$ , since  $\frac{l_i-1}{2}$  is even. If  $l_i = 5$ , then  $c_2^i, c_4^i \in A_2$  are the desired vertices, since they are the only vertices of  $G_1^*$  of degree three. Thus assume that  $l_i \geq 9$ . If  $\left\{c_{l_i+1}^i+1, c_{l_i+1}^i-3\right\} \cap N\left(c_{l_i+1}^i-1\right) \neq \emptyset$ , then the desired pairs

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are  $c_{\frac{l_{i+1}}{2}-1}^{i}$  and the vertex of  $\left\{c_{\frac{l_{i+1}}{2}+1}^{i}, c_{\frac{l_{i+1}}{2}-3}^{i}\right\} \cap N\left(c_{\frac{l_{i+1}}{2}-1}^{i}\right)$ . Thus assume that  $\left\{c_{\frac{l_{i+1}}{2}+1}^{i}, c_{\frac{l_{i+1}}{2}-3}^{i}\right\} \cap N\left(c_{\frac{l_{i+1}}{2}-1}^{i}\right) = \emptyset$ . Clearly there is a vertex  $c_{t}^{i} \in A_{2}$  such that  $c_{t}^{i}$  is adjacent to  $c_{\frac{l_{i+1}}{2}-1}^{i}$ . Without loss of generality, assume that  $t < \frac{l_{i+1}}{2} - 3$ . Since G is an outerplanar graph,  $\left|A_{2} \cap \left\{c_{h}^{i}: t+2 \le h \le \frac{l_{i+1}}{2} -3\right\}\right|$  is even. Furthermore, since G is an outerplanar graph, any vertex of  $A_{2} \cap \left\{c_{h}^{i}: t+2 \le h \le \frac{l_{i+1}}{2} -3\right\}$  is adjacent to a vertex of  $A_{2} \cap \left\{c_{h}^{i}: t+2 \le h \le \frac{l_{i+1}}{2} -3\right\}$ . Consequently, there are two pairs  $c_{h_{1}}^{i}, c_{h_{2}}^{i} \in A_{2} \cap \left\{c_{h}^{i}: t+2 \le h \le \frac{l_{i+1}}{2} -3\right\}$  such that  $c_{h_{1}}^{i} \in N(c_{h_{2}}^{i})$  and  $|h_{1}-h_{2}|=2$ .

Let  $c_t^i$  and  $c_{t+2}^i$  be two adjacent vertices of  $A_2$  according to Fact 1. Clearly, deg  $(c_{t+1}^i) = 2$ . Let  $G^* = G - c_t^i c_{t-1}^i - c_t^i c_{t+1}^i$ . Clearly  $n(G^*) = n$ ,  $m(G^*) = m - 2$ and  $r - 1 \leq r(G^*) \leq r$ . By the choice of G,  $fd_1(G^*) \leq (4m(G^*) - 3n(G^*) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3$ . Let  $S^*$  be a  $fd_1(G^*)$ -set. Since  $c_{t+2}^i$  is a strong support vertex of  $G^*$ , by Observation 2, we have  $c_{t+2}^i \in S^*$ . If  $c_{t-1}^i \notin S^*$ , then  $S^*$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r - 3 and so  $fd_1(G) \leq (4m - 3n + 3)/2 - r - 3$ , a contradiction. Thus  $c_{t-1}^i \in S'$ . Then  $S' \cup \{c_t^i, c_{t+1}^i\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r - 1and so  $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$ , a contradiction.

Subcase 2.2.  $V(G'_1) \neq \{c_1^i, c_2^i, \ldots, c_{l_i}^i\}$ . Since  $K_i$  is a leaf-block of  $G, G'_1 - C_i$  has some vertex of degree at most one. Let  $v_d$  be a leaf of  $G'_1$  such that  $d(v_d, C^i - c_0^i)$  is as maximum as possible, and the shortest path from  $v_d$  to  $C^i$  does not contain  $c_0^i$ . Let  $v_0v_1\cdots v_d$  be the shortest path from  $v_d$  to a vertex  $v_0 \in C^i$ .

Suppose that  $d \ge 2$ . Assume that  $\deg_G(v_{d-1}) = 2$ . Let  $G' = G - \{v_d, v_{d-1}\}$ . Clearly r(G') = r. By the choice of G,  $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') = (4(m-2) - 3(n-2) + 3)/2 - r = (4m - 3n + 3)/2 - r - 1$ . Let S' be a  $fd_1(G')$ -set. If  $v_{d-2} \notin S'$ , then  $S' \cup \{v_d\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r and so  $fd_1(G) \le (4m - 3n + 3)/2 - r$ , a contradiction. Thus  $v_{d-2} \in S'$ . Then  $S' \cup \{v_{d-1}\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r and so  $fd_1(G) \le (4m - 3n + 3)/2 - r$ , a contradiction. We deduce that  $\deg_G(v_{d-1}) \ge 3$ . Clearly any vertex of  $N_G(v_{d-1}) - \{v_{d-2}\}$  is a leaf. Let G' be obtained from G by removing all leaves adjacent to  $v_{d-1}$ . Clearly r(G') = r. By the choice of G,  $fd_1(G') \le (4m(G') - 3n(G') + 3)/2 - r(G') \le (4(m-2)-3(n-2)+3)/2 - r = (4m-3n+3)/2 - r - 1$ . Let S' be a  $fd_1(G')$ -set. If  $v_{d-1} \in S'$ , then S' is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r = (4m - 3n + 3)/2 - r - 1. Let S' be a  $fd_1(G')$ -set. If  $v_{d-1} \in S'$ , then S' is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r - 1 and so  $fd_1(G) \le (4m - 3n + 3)/2 - r - 1$ , a contradiction. Thus assume that  $v_{d-1} \notin S'$ . Then  $v_{d-2} \in S'$ . Now  $S' \cup \{v_{d-1}\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r - 1 and so  $fd_1(G) \le (4m - 3n + 3)/2 - r - 1$ , a contradiction. Thus assume that  $v_{d-1} \notin S'$ . Then  $v_{d-2} \in S'$ . Now  $S' \cup \{v_{d-1}\}$  is a 1FD-set in G of cardinality at most (4m - 3n + 3)/2 - r - 1 and so  $fd_1(G) \le (4m - 3n + 3)/2 - r - 3$ .

We thus assume that d = 1. Let  $D_1 = \{c_i^i \mid \deg_G(c_i^i) = 2\}, D_2 = \{c_i^i \mid c_i^i \mid c_i^j \mid c_i^$ 

is a support vertex of G and  $D_3 = \{c_j^i \mid \deg_G(c_j^i) \geq 3 \text{ and } c_j^i \text{ is not a support vertex of } G\}$ . Clearly  $|D_1| + |D_2| + |D_3| = l_i$ . Observe that  $|D_2| \geq 1$ , since d = 1. Thus by Claims 1 and 2,  $|D_1| \leq |D_3|$ . Let  $G_1^* = G[G_1' \cup \{c_0^i\}]$ . Observe that  $m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \geq n(G_1^*) + |D_3| / 2$ . Then  $n(G_1^*) \geq l_i + 1 + |D_2|$ . Let  $G_2^* = [G_2' \cup \{c_1^i, c_{l_i}^i\}] - \{c_{l_i}^i c_1^i\}$ . Clearly  $n = n(G_2^*) + n(G_1^*) - 3$ ,  $m = m(G_2^*) + m(G_1^*) - 2$  and  $r(G_2^*) = r - 1$ . By the choice of G,  $fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)$ . Let S'' be a  $fd_1(G_2^*)$ -set. By Observation 2,  $c_0^i \in S''$ , since  $c_0^i$  is a strong support vertex of  $G_2^*$ . Then  $S'' \cup \{c_1^i, c_2^i, \dots, c_{l_i}^i\}$  is a 1FD-set for G of cardinality  $|S''| + l_i$ . On the other hand

$$\begin{split} (4m-3n+3)/2 - r \\ &\geq (4(m(G_2^*)+m(G_1^*)-2)-3(n(G_2^*)+n(G_1^*)-3)+3)/2 - r \\ &= (4m(G_2^*)-3n(G_2^*)+3)/2 - r(G_2^*) + (4m(G_1^*)-3n(G_1^*)+1)/2 - 1 \\ &\geq |S''| + (4(n(G_1^*)+|D_3|/2)-3n(G_1^*)+1)/2 - 1 \\ &= |S''| + (n(G_1^*)+2|D_3|+1)/2 - 1 \\ &\geq |S''| + (l_i+1+|D_2|+2|D_3|+1)/2 - 1 \\ &\geq (l_i+|D_2|+|D_3|+|D_1|)/2 \geq |S''| + l_i. \end{split}$$

Thus  $fd_1(G) \leq |S''| + l_i \leq (4m - 3n + 3)/2 - r$ , a contradiction. To the sharpness consider a cycle  $C_2$ 

To the sharpness, consider a cycle  $C_5$ .

# 3. Concluding Remarks

As it is noted, Caro *et al.* [1] proved that fd(G) < 17n/19 for any maximal outerplanar graph G of order n. They also proved that  $fd(G) \le n-2$  for any connected graph G of order  $n \ge 3$ . It is worth-noting that the bound of Theorem 5 improves the bound n-2 when 4m < 5n + 2r - 7. It is also known that every maximal outerplanar graph G of order at least 3 is 2-connected [7], and thus r(G) = 1. Therefore, the bound of Theorem 5 improves the bound 17n/19 when  $4m < \frac{91n}{19} - 1$ . We have the following conjecture.

**Conjecture 6.** If G is a graph of order n and size m with  $r \ge 1$  strong-blocks, then  $fd(G) \le (4m - 3n + 3)/2 - r$ .

#### References

 Y. Caro, A. Hansberg and M.A. Henning, *Fair domination in graphs*, Discrete Math. **312** (2012) 2905–2914. doi:10.1016/j.disc.2012.05.006

- [2] B. Chaluvaraju, M. Chellali and K.A. Vidya, *Perfect k-domination in graphs*, Australas. J. Combin. 48 (2010) 175–184.
- [3] B. Chaluvaraju and K. Vidya, Perfect dominating set graph of a graph, Adv. Appl. Discrete Math. 2 (2008) 49–57.
- [4] E.J. Cockayne, B.L. Hartnell, S.T. Hedetniemi and R. Laskar, *Perfect domination in graphs*, J. Comb. Inf. Syst. Sci. 18 (1993) 136–148.
- [5] I.J. Dejter, Perfect domination in regular grid graphs, Australas. J. Combin. 42 (2008) 99–114.
- [6] I.J. Dejter and A.A. Delgado, Perfect domination in rectangular grid graphs, J. Combin. Math. Combin. Comput. 70 (2009) 177–196.
- [7] I. Fabrici, Light graphs in families of outerplanar graphs, Discrete Math. 307 (2007) 866-872. doi:10.1016/j.disc.2005.11.058
- [8] M.R. Fellows and M.N. Hoover, *Perfect domination*, Australas. J. Combin. 3 (1991) 141–150.
- [9] M. Hajian and N. Jafari Rad, *Trees and unicyclic graphs with large fair domination number*, Util. Math., to appear.
- [10] M. Hajian and N. Jafari Rad, Fair domination number in cactus graphs, Discuss. Math. Graph Theory 39 (2019) 489–503. doi:10.7151/dmgt.2088
- [11] A. Hansberg, Reviewing some results on fair domination in graphs, Electron. Notes Discrete Math. 43 (2013) 367–373. doi:10.1016/j.endm.2013.07.054
- [12] H. Hatami and P. Hatami, Perfect dominating sets in the Cartesian products of prime cycles, Electron. J. Combin. 14 (2007) #N8.
- [13] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).
- [14] J. Leydold and P.F. Stadler, Minimal cycle bases of outerplanar graphs, Electron. J. Combin. 5 (1998) #R16.

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