# LIGHT MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND NO 6-VERTICES ${ }^{1}$ 

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#### Abstract

In 1940, Lebesgue gave an approximate description of the neighborhoods of 5 -vertices in the class $\mathbf{P}_{\mathbf{5}}$ of 3-polytopes with minimum degree 5 .

Given a 3-polytope $P$, by $w(P)$ denote the minimum of the degree-sum (weight) of the neighborhoods of 5 -vertices (minor 5 -stars) in $P$.

In 1996, Jendrol' and Madaras showed that if a polytope $P$ in $\mathbf{P}_{\mathbf{5}}$ is allowed to have a 5 -vertex adjacent to four 5 -vertices, then $w(P)$ can be arbitrarily large.

For each $P$ in $\mathbf{P}_{\mathbf{5}}$ without vertices of degree 6 and 5 -vertices adjacent to four 5 -vertices, it follows from Lebesgue's Theorem that $w(P) \leq 68$. Recently, this bound was lowered to $w(P) \leq 55$ by Borodin, Ivanova, and Jensen and then to $w(P) \leq 51$ by Borodin and Ivanova.

In this note, we prove that every such polytope $P$ satisfies $w(P) \leq 44$, which bound is sharp.

Keywords: planar map, planar graph, 3-polytope, structural properties, 5 -star, weight, height.


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## 1. Introduction

The degree of a vertex or face $x$ in a convex finite 3 -dimensional polytope (called a 3 -polytope) is denoted by $d(x)$. A $k$-vertex is a vertex $v$ with $d(v)=k$. A $k^{+}$-vertex ( $k^{-}$-vertex) is one of degree at least $k$ (at most $k$ ). Similar notation is used for the faces. A 3-polytope with minimum degree 5 is denoted by $P_{5}$, and the set of such 3-polytopes is $\mathbf{P}_{\mathbf{5}}$.

The weight of a subgraph $S$ of $P_{5}$ is the degree sum of the vertices of $S$ in $P_{5}$, and the height of $S$ is the maximum degree of the vertices of $S$ in $P_{5}$. A $k$-star, a star with $k$ rays, is minor if its center $v$ has degree at most 5 . In particular, the neighborhoods of 5 -vertices are minor 5 -stars and vice versa. All stars considered in this note are minor.

By $w\left(S_{k}\right)$ and $h\left(S_{k}\right)$ we denote the minimum weight and height, respectively, of minor $k$-stars in a given 3 -polytope $P_{5}$.

In 1904, Wernicke [13] proved that every $P_{5}$ has a 5 -vertex adjacent to a $6^{-}$-vertex. This result was strengthened by Franklin [9] in 1922 to the existence of a 5 -vertex with two $6^{-}$-neighbors. In 1940, in attempts to solve the Four Color Problem, Lebesgue [12, p. 36] gave an approximate description of the neighborhoods of 5 -vertices in $P_{5}$ s. In particular, this description implies the results in $[9,13]$ and shows that there is a 5 -vertex with three $7^{-}$-neighbors.

The bounds $w\left(S_{1}\right) \leq 11$ (Wernicke [13]) and $w\left(S_{2}\right) \leq 17$ (Franklin [9]) are tight. It was proved by Lebesgue [12] that $w\left(S_{3}\right) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [10] to the sharp bound $w\left(S_{3}\right) \leq 23$. Furthermore, Jendrol' and Madaras [10] gave a precise description of minor 3-stars in $P_{5}$ s.

Lebesgue [12] proved $w\left(S_{4}\right) \leq 31$, which was strengthened by Borodin and Woodall [8] to the tight bound $w\left(S_{4}\right) \leq 30$. Note that $w\left(S_{3}\right) \leq 23$ easily implies $w\left(S_{2}\right) \leq 17$ and immediately follows from $w\left(S_{4}\right) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). Recently, Borodin and Ivanova [1] obtained a precise description of 4-stars in $P_{5}$ s.

The more general problem of precisely describing 5 -stars at 5 -vertices in $P_{5} \mathrm{~s}$ inspired by Lebesgue's Theorem is still widely open.

Jendrol' and Madaras [10] showed that if a polytope $P_{5}$ has a 5 -vertex adjacent to four 5 -vertices, called a minor $(5,5,5,5, \infty)$-star, then $h\left(S_{5}\right)$ and hence $w\left(S_{5}\right)$ can be arbitrarily large. Therefore, in what follows we consider $P_{5}$ s without minor ( $5,5,5,5, \infty$ )-stars.

Recently, precise upper bounds for the height and weight of minor 5 -stars have been obtained for some restricted subclasses in $\mathbf{P}_{\mathbf{5}}$. A lot of earlier results on the structure of stars in 3 -polytopes can be found in [11].

For every $P_{5}$ having no vertices of degree from 6 to 9 , Lebegue's bounds $h\left(S_{5}\right) \leq 14$ and $w\left(S_{5}\right) \leq 44$ were improved by Borodin and Ivanova [3] to the sharp bounds $h\left(S_{5}\right) \leq 12$ and $w\left(S_{5}\right) \leq 42$.

For each $P_{5}$ with no 6- to 8-vertices, it follows from Lebesgue's Theorem that $h\left(S_{5}\right) \leq 17$ and $w\left(S_{5}\right) \leq 46$, which bounds were improved in Borodin, Ivanova and Nikiforov [7] to the best possible bounds $h\left(S_{5}\right) \leq 12$ and $w\left(S_{5}\right) \leq 42$.

Under the absence of 6 - and 7 -vertices, Lebegue's bound $h\left(S_{5}\right) \leq 23$ was improved by Borodin et al. [5] to the sharp bound $h\left(S_{5}\right) \leq 14$.

For each $P_{5}$ with no 6-vertices, it follows from Lebesgue's Theorem that $h\left(S_{5}\right) \leq 41$. This bound was lowered to $h\left(S_{5}\right) \leq 28$ by Borodin, Ivanova, and Jensen [4], then to $h\left(S_{5}\right) \leq 23$ in Borodin-Ivanova [2], and finally to the tight bound $h\left(S_{5}\right) \leq 17$ by Borodin, Ivanova, and Nikiforov [6].

As for the minimum weight of minor 5 -stars in $P_{5}$ s under the absence of 6 -vertices, Lebesgue's bound $w\left(S_{5}\right) \leq 68$ was lowered to $w\left(S_{5}\right) \leq 55$ by Borodin, Ivanova, and Jensen [4] and then to $w\left(S_{5}\right) \leq 51$ in Borodin-Ivanova [2]. The purpose of this paper is to prove the following fact.

Theorem 1. Every 3-polytope with minimum degree 5 and neither 6-vertices nor minor $(5,5,5,5, \infty)$-stars has a minor 5 -star with weight at most 44 , which bound is best possible.

We note that a light minor 5 -star ensured by Theorem 1 has height at most $44-4 \times 5-7=17$. The tightness of the bounds 44 and 17 is confirmed by a construction in [6].

## 2. Proof of Theorem 1

## Discharging.

Suppose that a 3-polytope $P_{5}^{\prime}$ is a counterexample to the main statement of Theorem 1. Thus each minor 5 -star in $P_{5}^{\prime}$ has weight at least 45 and at most three 5 -vertices.

Let $P_{5}$ be a counterexample with the maximum number of edges on the same set of vertices as $P_{5}^{\prime}$.
Remark 2. $P_{5}$ has no $4^{+}$-face with two nonconsecutive $7^{+}$-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with greater number of edges.

Let $V, E$, and $F$ be the sets of vertices, edges, and faces of $P_{5}$. Euler's formula $|V|-|E|+|F|=2$ implies

$$
\begin{equation*}
\sum_{v \in V}(d(v)-6)+\sum_{f \in F}(2 d(f)-6)=-12 . \tag{1}
\end{equation*}
$$

We assign an initial charge $\mu(v)=d(v)-6$ to each $v \in V$ and $\mu(f)=$ $2 d(f)-6$ to each $f \in F$, so that only 5 -vertices have a negative initial charge.

Using the properties of $P_{5}$ as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12 .

The final charge $\mu^{\prime}(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1-R11 below (see Figure 1).

For a vertex $v$, let $v_{1}, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a fixed cyclic order. If $f$ is a face, then $v_{1}, \ldots, v_{d(f)}$ are the vertices incident with $f$ in the same cyclic order.

If $d$ is an integer with $8 \leq d \leq 15$, then we put $\xi_{d}=\frac{d-6}{d}$.
A vertex is simplicial if it is completely surrounded by 3 -faces. A simplicial 5 -vertex $v$ is helpful if $d\left(v_{1}\right) \geq 12, d\left(v_{2}\right)=d\left(v_{4}\right)=5, d\left(v_{3}\right)=7$, and $d\left(v_{5}\right) \geq 12$ (see Figure 1, R10). A simplicial 5-vertex $v$ is strong if $d\left(v_{1}\right)=d\left(v_{2}\right)=5$, $7 \leq d\left(v_{3}\right) \leq 11$, and $7 \leq d\left(v_{5}\right) \leq 11$ (so $\left.d\left(v_{4}\right) \geq 45-2 \times 11-3 \times 5 \geq 8\right)$ (see Figure 1, R11).
R1. Each $4^{+}$-face gives $\frac{1}{2}$ to each incident 5 -vertex.
$\mathbf{R 2}$. If a 5 -vertex $v$ is incident with precisely one $4^{+}$-face, then $v$ receives $\frac{1}{2}$ from each adjacent $16^{+}$-vertex.
R3. A simplicial 5-vertex $v$ with at least two $12^{+}$-neighbors receives $\frac{1}{2}$ from each adjacent $16^{+}$-vertex.
R4. A simplicial 5 -vertex $v$ with $d\left(v_{4}\right) \neq 5, d\left(v_{5}\right) \geq 16$, and no other $12^{+}$_ neighbors receives the following charge from $v_{5}$ :
(a) if $d\left(v_{1}\right) \neq 5$, then 1 , and
(b) if $d\left(v_{1}\right)=5$, then $\frac{3}{4}$ provided that $d\left(v_{5}\right) \leq 17$ or $\frac{5}{6}$ otherwise.

R5. A simplicial 5-vertex $v$ with $d\left(v_{5}\right) \geq 18, d\left(v_{1}\right)=d\left(v_{4}\right)=5$, and $7 \leq d\left(v_{2}\right) \leq$ $d\left(v_{3}\right) \leq 11$ receives $\frac{2}{3}$ from $v_{5}$.
R6. A simplicial 5 -vertex $v$ with $16 \leq d\left(v_{5}\right) \leq 17, d\left(v_{1}\right)=d\left(v_{4}\right)=5$, and $7 \leq d\left(v_{2}\right) \leq d\left(v_{3}\right) \leq 11$ receives from $v_{5}$ :
(n) $\frac{5}{8}$ if neither $v_{2}$ nor $v_{3}$ is a 7 -vertex having six simplicial 5 -neighbors ("normally"), and
(e) $\frac{2}{3}$ otherwise ("as an exception").

R7. A simplicial 5-vertex $v$ with $d\left(v_{5}\right) \geq 16, d\left(v_{1}\right)=d\left(v_{2}\right)=d\left(v_{4}\right)=5$, and $7 \leq d\left(v_{3}\right) \leq 11$ receives the following charge from $v_{5}$.
(n) If $v_{1}$ is not simplicial or $v_{2}$ is not strong (that is "normal"), then $\frac{3}{4}$ if $d\left(v_{5}\right) \leq$ 17 or $\frac{5}{6}$ otherwise.
(e) If $v_{1}$ is simplicial and $v_{2}$ is strong (which is "an exception"), then $\frac{5}{8}$ if $d\left(v_{5}\right) \leq$ 17 or $\frac{2}{3}$ otherwise.

R8. A $d$-vertex $v$ with $8 \leq d(v) \leq 15$ gives its 5 -neighbor $v_{2}$ :
(a) $\xi_{d}$ if $d\left(v_{1}\right)=d\left(v_{3}\right)=5$,
(b) $\frac{3 \xi_{d}}{2}$ if $d\left(v_{1}\right)=5$ and $d\left(v_{3}\right) \neq 5$, and
(c) $2 \xi_{d}$ if $d\left(v_{1}\right) \neq 5$ and $d\left(v_{3}\right) \neq 5$.

R9. A 7 -vertex $v$ gives each adjacent simplicial 5 -vertex:
(n) $\frac{1}{5}$ "as a norm", that is if $v$ has at most five simplicial 5 -neighbors, or
(e) $\frac{1}{6}$ "as an exception".

R10. A 7 -vertex $v$ receives $\frac{1}{6}$ from each helpful 5 -neighbor.
R11. A strong 5-vertex gives $\frac{1}{6}$ to each 5-neighbor.


Figure 1. Rules of discharging.

Checking $\mu^{\prime}(x) \geq 0$ whenever $x \in V \cup F$.
If $f$ is a $4^{+}$-face, then $\mu^{\prime}(f) \geq 2 d(f)-6-d(f) \times \frac{1}{2}=\frac{3(d(f)-4)}{2} \geq 0$ by R1.

Now suppose $v \in V$.
Case 1. $d(v) \geq 18$. We know that $v$ gives one of the charges in $\left\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$ to each adjacent 5 -vertex incident with at least four 3 -faces by R2-R7. Since $d(v)-6 \geq \frac{2 d(v)}{3}$, it suffices to average these donations so that each $5^{+}$-neighbor will receive at most $\frac{2}{3}$ from $v$.

To this end, we first switch $\frac{1}{6}$ from 1 given to a 5 -neighbor $v_{k}$ by R 4 (a) to each of the $7^{+}$-neighbors $v_{k-1}$ and $v_{k+1}$ (hereafter, addition modulo $d(v)$ ). As a result, the averaged donation from $v$ to $v_{k}$ becomes $1-2 \times \frac{1}{6}=\frac{2}{3}$.

Next, if $\frac{5}{6}$ is given to a 5 -neighbor $v_{k}$ by R4(b), then we switch $\frac{1}{6}$ to its common $7^{+}$-neighbor with $v$.

Finally, the donation of $\frac{5}{6}$ by $\mathrm{R} 7(\mathrm{n})$ happens to a simplicial 5 -neighbor $v_{k}$ of $v$ having cyclic neighbors $v_{k-1}, x_{k}, y_{k}, v_{k+1}$ with $7 \leq d\left(x_{k}\right) \leq 11$ and 5 -neighbors $v_{k-1}, y_{k}, v_{k+1}$, where either $v_{k+1}$ is not simplicial or $y_{k}$ is not strong.

If $v_{k+1}$ is not simplicial, then we switch $\frac{1}{6}$ from $v_{k}$ to $v_{k+1}$ and note that the latter receives at most $\frac{1}{2}$ from $v$ by R2.

From now on suppose that $v_{k+1}$ is simplicial, and let $z_{k}$ be the vertex conjugated with $v_{k}$ with respect to the edge $y_{k} v_{k+1}$. Since $v_{k}$ receives $\frac{5}{6}$ by $\mathrm{R} 7(\mathrm{n})$ by our assumption, it follows that $d\left(z_{k}\right) \notin\{7, \ldots, 11\}$, for otherwise $y_{k}$ is strong since it has the fifth neighbor of degree at least $w\left(S_{5}\right)-3 \times 5-2 \times 11=8$ and is simplicial in view of Remark 2 .

If $d\left(z_{k}\right) \geq 12$, then we switch $\frac{1}{6}$ from $v_{k}$ to $v_{k+1}$, where $v_{k+1}$ this time receives $\frac{1}{2}$ by R3. Note that $v_{k+2}$ receives $\frac{1}{2}$ by R2 or R3, which implies that $\frac{1}{6}$ is switched to $v_{k+1}$ only once.

It remains to assume that $d\left(z_{k}\right)=5$. This implies that $d\left(v_{k+2}\right) \geq 7$ since $v_{k+1}$ cannot have four 5 -neighbors. Here, we switch $\frac{1}{6}$ from $v_{k}$ to $v_{k+2}$. (Of course, $v_{k+1}$ also switches $\frac{1}{6}$ from its $\frac{5}{6}$ obtained by R4(b) to $v_{k+2}$, as said above.)

It is not hard to see that no 5 -vertex $v_{k+1}$ can receive $\frac{1}{6}$ in the course of our averaging both from $v_{k}$ and $v_{k+2}$ since then $v_{k+1}$ would have four 5 -neighbors, which is impossible.

As a result, the averaged donation of $v$ to each 5-neighbor becomes at most $1-2 \times \frac{1}{6}=\frac{5}{6}-\frac{1}{6}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$ and that to each $7^{+}$-neighbor is at most $0+4 \times \frac{1}{6}=\frac{2}{3}$, as desired.

Case 2. $16 \leq d(v) \leq 17$. We now show that the neighbors of $v$ receive from $v$ by R2-R7 at most $\frac{5}{8}$ on the average, which implies that $\mu^{\prime}(v) \geq d(v)-6-\frac{5 d(v)}{8}=$ $\frac{3(d(v)-16)}{8} \geq 0$. We proceed similarly to Case 1 with a 5 -vertex $v_{k}$ getting more than $\frac{5}{8}$ from $v$ by R4, R6(e) or $\mathrm{R} 7(\mathrm{n})$.

If $v_{k}$ is as in $\mathrm{R} 4(\mathrm{a})$, then we shift $\frac{1}{4}$ from 1 obtained by $v_{k}$ to each of the $7^{+}$-vertices $v_{k-1}$ and $v_{k+1}$. In R4(b), we shift $\frac{1}{8}$ from $\frac{3}{4}$ to a unique $7^{+}$-vertex in $\left\{v_{k-1}, v_{k+1}\right\}$.

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Now consider R6(e), which has no analogues in Case 1. By symmetry, we can assume that $v_{k+1}$ lies in a common 3 -face with $v_{k}$ and a 7 -vertex having six simplicial 5 -neighbors. Now $d\left(v_{k+2}\right) \geq 7$ as $v_{k+1}$ cannot have three 5 -neighbors in addition to a 7 -neighbor and a $17^{-}$-neighbor since $w\left(S_{5}\right) \geq 45$ by assumption. Recall that $v_{k+1}$ receives at most $\frac{3}{4}$ by $\mathrm{R} 4(\mathrm{~b}), \mathrm{R} 3$, or $\mathrm{R} 7(\mathrm{n})$ and that $\frac{1}{8}$ was already switched from $v_{k+1}$ to $v_{k+2}$ in the previous paragraph. Here, we also switch $\frac{1}{8}$ from $\frac{2}{3}$ received by $v_{k}$ to $v_{k+2}$.

In the situation of $\mathrm{R} 7(\mathrm{n})$, let $v_{k+1}$ lie in a 3 -face incident with three 5 -vertices. Arguing as in Case 1, we see that either $v_{k+1}$ receives $\frac{1}{2}$ from $v$, in which case we switch $\frac{1}{8}$ from $v_{k}$ to $v_{k+1}$, or we have $d\left(v_{k+2}\right) \geq 7$, in which case we switch $\frac{1}{8}$ from $v_{k}$ to $v_{k+2}$.

As a result of this averaging, each 5 -neighbor of $v$ receives at most $1-2 \times \frac{1}{4}<$ $\frac{3}{4}-\frac{1}{8}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$, while each $7^{+}$-neighbor receives at most $4 \times \frac{1}{8}=\frac{1}{4}+2 \times \frac{1}{8}=$ $2 \times \frac{1}{4}<\frac{5}{8}$ from $v$, as desired.

Case 3 . $8 \leq d(v) \leq 15$. To satisfy R8, we first send $\xi_{d(v)}$ to each neighbor $v_{k}$, and then each $7^{+}$-neighbor $v_{k}$ transfers $\frac{\xi_{d(v)}}{2}$ to each 5 -vertex in $\left\{v_{k-1}, v_{k+1}\right\}$. This shows that $\mu^{\prime}(v) \geq d(v)-6-d(v) \times \xi_{d(v)}=0$.

Case 4. $d(v)=7$. If $v$ has at most five simplicial 5 -neighbors, then $\mu^{\prime}(v) \geq$ $7-6-5 \times \frac{1}{5}=0$ by $\operatorname{R} 9(\mathrm{n})$. If $v$ has precisely six simplicial 5 -neighbors, then $\mu^{\prime}(v) \geq 1-6 \times \frac{1}{6}=0$ by R9(e).

Finally, suppose $v$ is completely surrounded by simplicial 5 -vertices. This implies that there is a 7 -cycle $C_{7}=w_{1} \cdots w_{7}$ avoiding $v$, where each $v_{k}$ lies in a 3 -face $w_{k} v_{k} w_{k+1}$ (addition modulo 7 ). Note that $d\left(w_{k}\right)+d\left(w_{k+1}\right) \geq 45-3 \times 5-7=$ 23 whenever $1 \leq k \leq 7$. By the oddness of $7, v$ has a helpful neighbor, which gives $\frac{1}{6}$ to $v$ by R10. As a result, we have $\mu^{\prime}(v) \geq 1+\frac{1}{6}-7 \times \frac{1}{6}=0$ in view of $\mathrm{R} 9(\mathrm{e})$, as required.

Case 5. $d(v)=5$. If $v$ is incident with at least two $4^{+}$-faces, then $\mu^{\prime}(v) \geq$ $5-6+2 \times \frac{1}{2}=0$ by R1.

If $v$ is incident with precisely one $4^{+}$-face, then we are done when $v$ has a $12^{+}$-neighbor since $v$ receives $\frac{1}{2}$ by R1 and at least $\frac{1}{2}$ by R2 or R8.

So suppose otherwise. Note that $v$ then has two $8^{+}$-neighbors, for otherwise $v$ would have an $11^{-}$-neighbor and four $7^{-}$-neighbors, which implies $w\left(S_{5}\right) \leq$ $5+4 \times 7+11<45$, a contradiction. Thus $\mu^{\prime}(v) \geq-1+\frac{1}{2}+2 \times \frac{1}{4}=0$ by R1 and R8.

From now on we can assume that $v$ is simplicial.
Subcase 5.1. $v$ is helpful, with $d\left(v_{1}\right) \geq 12, d\left(v_{2}\right)=d\left(v_{4}\right)=5, d\left(v_{3}\right)=7$, and $d\left(v_{5}\right) \geq 12$. Now $v$ receives $\frac{1}{2}$ from each of $v_{1}, v_{5}$ by R3 and/or R8. Also $v$ receives at least $\frac{1}{6}$ from $v_{3}$ by R9 and returns $\frac{1}{6}$ to $v_{3}$ by R10. This implies $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}+\frac{1}{6}-\frac{1}{6}=0$, as desired.

Subcase 5.2. $v$ is strong, with $d\left(v_{1}\right)=d\left(v_{2}\right)=5,7 \leq d\left(v_{3}\right) \leq d\left(v_{5}\right) \leq 11$, and $d\left(v_{4}\right) \geq 45-3 \times 5-2 \times 11=8$. Now $v$ must collect the total of at least $\frac{4}{3}$ from $v_{3}, v_{4}, v_{5}$ in order to be able to give $2 \times \frac{1}{6}$ to $v_{1}, v_{2}$ according to R11 (and leave 1 for itself).

We are easily done if $d\left(v_{4}\right) \geq 12$, for then $v_{4}$ gives $v$ at least 1 by R4(a) or R8(c) while each of $v_{3}, v_{5}$ gives at least $\frac{1}{6}$ by R8 and R9.

So suppose $d\left(v_{4}\right) \leq 11$. Since $d\left(v_{3}\right)+d\left(v_{4}\right)+d\left(v_{5}\right) \geq w\left(S_{5}\right)-3 \times 5=30$, this implies that $v$ has no neighbors of degree less than $30-2 \times 11=8$. If $d\left(v_{4}\right)=8$, then $d\left(v_{3}\right)=d\left(v_{5}\right)=11$, which implies that $v$ receives $\frac{1}{2}$ from $v_{4}$ by R8(c) and $2 \times \frac{15}{22}$ from $v_{3}, v_{5}$ by R8(b), as desired. If $d\left(v_{4}\right) \geq 9$, then $v$ receives at least $\frac{2}{3}$ from $v_{4}$ by R8(c) and at least $2 \times \frac{3}{8}$ from $v_{3}, v_{5}$ by $\mathrm{R} 8(\mathrm{~b})$, and we are done.

Subcase 5.3. $v$ does not give charge away by R10 and R11. So we must check that $v$ collects the total of at least 1 from its neighbors by R3-R9. If $v$ has at least two $12^{+}$-neighbors, then $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{2}=0$ by R3 or R8(a),(b). So in what follows we assume that $v$ has at most one $12^{+}$-neighbor, which means that R3 is not applied to $v$.

Subcase 5.3.1. $v$ has at most one 5 -neighbor. Here, $v$ receives at least $\frac{3}{8}$ from an 8 -neighbor and at least $\frac{1}{2}$ from a $9^{+}$-neighbor by R4-R8. This implies, in view of R9, that $\mu^{\prime}(v) \geq-1+3 \times \frac{1}{6}+\frac{1}{2}=0$ in the presence of a $9^{+}$-neighbor or $\mu^{\prime}(v) \geq$ $-1+2 \times \frac{1}{6}+2 \times \frac{3}{8}>0$ when $v$ has at least two 8 -neighbors. However, one of this situations is inevitable, since otherwise we would have $w\left(S_{5}\right) \leq 5+4 \times 7+8<45$, which is impossible.

Subcase 5.3.2. $v$ has precisely two 5 -neighbors. Note that the total degree of the three $7^{+}$-neighbors of $v$ is at least $45-3 \times 5=30$.

Suppose $v$ has no 7 -neighbor. Each $8^{+}$-neighbor $v_{2}$ gives $v$ by R4-R8 at least $\frac{1}{4}$ if $d\left(v_{1}\right)=d\left(v_{3}\right)=5$ and at least $\frac{3}{8}$ if $d\left(v_{1}\right) \neq 5$, so $\mu^{\prime}(v) \geq-1+\frac{1}{4}+2 \times \frac{3}{8}=0$, and we are done.

Next suppose $v$ has at least one 7 -neighbor. Now the other two $7^{+}$-neighbors have the total degree at least $30-7=23$, so there is a $12^{+}$-neighbor, say $v_{2}$, among them.

If $v_{2}$ gives $v$ at least $\frac{3}{4}$ to $v$ by R4 or R8, then $\mu^{\prime}(v)>0$, since the other two $7^{+}$-neighbors give at least $2 \times \frac{1}{6}$ by R4-R 9 .

So suppose $d\left(v_{1}\right)=d\left(v_{3}\right)=5$ and $7=d\left(v_{4}\right) \leq d\left(v_{5}\right)$. Now if $d\left(v_{2}\right) \geq 18$, then we have $\mu^{\prime}(v) \geq-1+\frac{2}{3}+2 \times \frac{1}{6}=0$ by R $4-$ R 9 .

For $16 \leq d\left(v_{2}\right) \leq 17$ we are similarly done if $v_{2}$ gives $\frac{2}{3}$ by R6(e), so suppose $\mathrm{R} 6(\mathrm{n})$ is applied to $v_{2}$ rather than $\mathrm{R} 6(\mathrm{e})$. If $d\left(v_{5}\right) \geq 8$, then $\mu^{\prime}(v) \geq-1+\frac{5}{8}+$ $\frac{1}{6}+\frac{3}{8}>0$. It remains to assume that $d\left(v_{4}\right)=d\left(v_{5}\right)=7$ and neither of $v_{4}, v_{5}$ has six simplicial 5 -neighbors (as if we apply R9(e) to $v_{4}$ or $v_{5}$ it would mean we should apply R6(e) to $v$, and then $\mu^{\prime}(v) \geq-1+\frac{2}{3}+2 \times \frac{1}{6}=0$ ). This means that $\mu^{\prime}(v) \geq-1+\frac{5}{8}+2 \times \frac{1}{5}=\frac{1}{40}$ by $\operatorname{R} 6(\mathrm{n})$ and $\operatorname{R} 9(\mathrm{n})$.

Finally, suppose $12 \leq d\left(v_{2}\right) \leq 15$. Now $d\left(v_{5}\right) \geq w\left(S_{5}\right)-3 \times 5-7-15=8$, and it suffices to observe that $v$ receives at least $\frac{1}{2}, \frac{1}{6}, \frac{3}{8}$ from $v_{2}, v_{4}, v_{5}$, respectively, which makes $\mu^{\prime}(v)>0$, as desired.

Subcase 5.3.3. $v$ has precisely three 5 -neighbors. Note that the total degree of the two $7^{+}$-neighbors of $v$ is at least $45-4 \times 5=25$.

First suppose $7 \leq d\left(v_{1}\right) \leq d\left(v_{2}\right)$. By the above assumption that R3 is not applied, we have $d\left(v_{1}\right) \leq 11$, which implies that $v$ has a $14^{+}$-neighbor. Note that $v_{2}$ gives $v$ at least $\frac{3}{4}$ by $\mathrm{R} 4(\mathrm{~b})$ or R8, while $v_{1}$ gives $v$ at least $\frac{3}{8}$ by R 8 if $d\left(v_{1}\right) \geq 8$, and then we have $\mu^{\prime}(v) \geq 0$. But if $d\left(v_{1}\right)=7$, then $d\left(v_{2}\right) \geq 25-7=18$, and $\mu^{\prime}(v) \geq-1+\frac{5}{6}+\frac{1}{6}=0$ by R4(b) combined with R 9 .

Thus from now on we can assume that $7 \leq d\left(v_{1}\right) \leq 11$ and $d\left(v_{3}\right) \geq 14$. If $d\left(v_{3}\right) \leq 15$, then $v$ receives from $v_{1}$ and $v_{3}$ at least $1=\frac{2}{5}+\frac{3}{5}=\xi_{10}+\xi_{15}<\xi_{11}+\xi_{14}$ by R8(a), as desired.

Next suppose $16 \leq d\left(v_{3}\right) \leq 17$, which implies that $d\left(v_{1}\right) \geq 8$. Since $v_{1}$ gives $v$ at least $\frac{1}{4}$ by $\mathrm{R} 8(\mathrm{a})$ while $v_{3}$ gives either $\frac{3}{4}$ or $\frac{5}{8}$ by R7, we are done unless $v_{3}$ gives $\frac{5}{8}$ by R7(e). The latter happens when $v_{5}$ is strong, in which case $v$ receives $\frac{1}{6}$ from $v_{5}$ by R11, which yields $\mu^{\prime}(v) \geq-1+\frac{1}{4}+\frac{1}{6}+\frac{5}{8}>0$.

Finally, suppose $d\left(v_{3}\right) \geq 18$. Now $v_{1}$ gives $v$ at least $\frac{1}{6}$ by R 9 while $v_{3}$ gives either $\frac{5}{6}$ or $\frac{2}{3}$ by R7. Since the donation of $\frac{2}{3}$ by R7(e) to $v$ is accompanied by receiving $\frac{1}{6}$ by R11 from a strong vertex $v_{5}$, we have $\mu^{\prime}(v) \geq-1+2 \times \frac{1}{6}+\frac{2}{3}=$ $-1+\frac{1}{6}+\frac{5}{6}=0$.

Thus we have proved $\mu^{\prime}(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

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