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A NOTE ON LOWER BOUNDS FOR INDUCED RAMSEY NUMBERS

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Abstract

We say that a graph F *strongly arrows* a pair of graphs (G, H) and write $F \xrightarrow{\text{ind}} (G, H)$ if any 2-coloring of its edges with red and blue leads to either a red G or a blue H appearing as induced subgraphs of F. *The induced Ramsey number*, $IR(G, H)$ is defined as $\min\{|V(F)| : F \xrightarrow{\text{ind}} (G, H)\}$. We will consider two aspects of induced Ramsey numbers. Firstly we will show that the lower bound of the induced Ramsey number for a connected graph G with independence number α and a graph H with clique number ω is roughly $\frac{\omega^2\alpha}{2}$. This bound is sharp. Moreover we will also consider the case when G is not connected providing also a sharp lower bound which is linear in both parameters.

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1. Introduction

We say that a graph F *strongly arrows* a pair of graphs (G, H) and write $F \xrightarrow{\text{ind}} (G, H)$ if any 2-coloring of its edges with red and blue leads to either a red G or a blue H appearing as induced subgraphs of F. We call the graph F a strongly arrowing graph. *The induced Ramsey number*, $IR(G, H)$ is defined as $\min\{|V(F)| : F \xrightarrow{\text{ind}} (G, H)\}$. It is a generalization of standard Ramsey numbers $R(G, H)$, where we color the edges of a complete graph and do not require the monochromatic copies to be induced. It is a corollary of the famous theorem of Ramsey that those numbers are always finite.

The existence of the induced Ramsey number is not obvious and it was a subject of intensive studies. Finally that was proved independently by Deuber [5], Erdős, Hajnal and Pósa [8] and Rödl [16, 17]. Since in case of complete graphs an induced subgraph is the same as a subgraph it is obvious that $IR(K_m, K_n) = R(K_m, K_n)$. When at least one of the graphs in the pair is not complete these functions differ.

Little is known about the behaviour of the induced Ramsey numbers. The results are mostly of asymptotic type and concern upper bounds. It is surely motivated by the fact that these ones following from the above mentioned proofs are enormous and Erdős conjectured [7] that there is a positive constant c such that every graph G with n vertices satisfies $IR(G, G) \leq 2^{cn}$. The most recent result in that direction is that of Conlon, Fox and Sudakov [2] who showed that $IR(G, G) \leq 2^{cn \log n}$ improving the earliear result $IR(G, G) \leq 2^{cn(\log n)^2}$ of Kohayakawa, Prömel and Rödl [13].

Moreover these results are generally upper bounds obtained either by probabilistic ([1, 12, 13, 15]) or by constructive methods [14]. A comparision of results of both types can be found in the paper of Shaefer and Shah [18]. The authors show there arrowing graphs for a number of pairs of graphs including trees, complete graphs, bipartite graphs and cycles.

As for the lower bound, it is obvious by the definition

$$IR(G, H) \geq R(G, H) \tag{1}$$

and, as far as we know, it is the only general lower bound known so far.

The main result of this short note is Theorem 1 which establishes a lower bound for the induced Ramsey number in terms of independence and clique numbers. Although the inductive proof of this theorem is not complicated the result is the first step from this standpoint. The theorem is somehow similar in spirit to the result of Chvátal and Harary [3] for Ramsey numbers who observed that for connected G

$$R(G, H) \geq (|V(G)| - 1)(\chi(H) - 1) + 1. \tag{2}$$

To prove (2), consider a 2-edge-coloring of the complete graph on $(|V(G)| - 1)(\chi(H) - 1)$ vertices consisting of $(\chi(H) - 1)$ disjoint red cliques of size $|V(G)| -1$. This coloring has no red G because all red connected components have size $|V(G)| - 1$, and there is no blue H since the partition of this H induced by red cliques would give a coloring of H by $\chi(H) - 1$ colors.

For some graphs the bound in (2) is quite far from the truth. For example Erdős [6] showed that $R(K_n, K_n) \geq \Omega(2^{n/2})$ which is much larger than the quadratic bound we get from (2).

The bound in Theorem 1 is sharp in that sense that for a pair: a star versus a complete graph this lower bound is actually the exact value of the induced Ramsey number. This theorem has its nontrivial application in all cases that the independence (clique) number differs not much from the number of vertices of the graph.

Finally we mention that the only known exact values (not concerning the pairs of small graphs) are for a pair of stars by Harary, Nešetřil and Rödl [11], a path P_3 versus unions of complete or complete multipartite graphs by Kostochka and Sheikh [14], matchings versus complete graphs by Gorgol and Łuczak [10] and for stars versus complete graphs by Gorgol [9]. The two latter will serve as examples of sharpness of our theorems.

2. Notation

In this paper we do not introduce any special notation. A graph G is a subgraph of a graph H (denoted by $G \subset H$) if $V(G) \subset V(H)$ and $E(G) \subset E(H)$. A graph G is an induced subgraph of a graph H (denoted by $G \prec H$) if $V(G) \subset V(H)$ and $E(G) = \{uv \in E(H) : u, v \in V(G)\}$. By $F[S]$ we mean the graph induced by a vertex-set S. Let t be a positive integer and F be a graph. By a symbol tF we mean a graph consisting of t disjoint copies of the graph F. For graphs G, H the symbol $G \cup H$ denotes a disjoint sum of graphs and $G \setminus H$ denotes a graph obtained from G by removing a subgraph H (with all incident edges). The independence number of a graph G, i.e. the size of the largest set of mutually nonadjacent vertices, we denote by $\alpha(G)$, the clique number, i.e., the size of the largest clique, by $\omega(G)$ and the chromatic number, i.e., the smallest number of colors needed to color the vertices of G so that no pair of adjacent vertices have the same color by $\chi(G)$. The symbols P_n, C_n, K_n stand for a path, a cycle and a complete graph on n vertices, respectively and S_k for a star with k rays.

3. Lower Bounds for an Induced Ramsey Number

As we mentioned in the introduction, little is known about lower bounds for the induced Ramsey numbers and a natural lower bound is the usual Ramsey number. To prove a lower bound for the induced Ramsey number we should show that we can color every graph F with a prescribed number of vertices without induced monochromatic copies of given graphs (G, H). Thus we have to examine not only the number of vertices of the graph F but also its structure. It turns out that if we consider the independence number of the graph G and the clique number of the graph H, as somehow opposite notions, it is not so difficult to deduce something about the structure of an arrowing graph F. Then we can construct appropriate colorings. The constructions described below mainly arose from the fact that since $\omega(H) = \omega$, K_ω is contained in F and a subgraph and an induced subgraph in case of cliques is the same. Therefore if we avoid a blue K_ω, we avoid an induced H as well. The colorings avoid connected red graphs with the independence number α simply by taking red subgraphs with the independance number at most $\alpha - 1$.

Theorem 1. *Let G be an arbitrary connected graph with $\alpha(G) = \alpha \geq 2$ and H be an arbitrary graph with $\omega(H) = \omega$. Then*

$$IR(G, H) \;\geq\; (\alpha - 1)\frac{\omega(\omega - 1)}{2} + \omega.$$

Proof. Let F be an arbitrary graph on $(\alpha - 1)\frac{\omega(\omega-1)}{2} + \omega - 1$ vertices. We shall show that F can be 2-colored with no red induced G and no blue induced H.

The proof will be conducted by induction on ω. Note that is enough to prove the theorem for $H = K_\omega$. It is trivial for $\omega = 2$. Note that certainly F contains a clique K_ω otherwise it could be colored blue. Let us denote this clique K^0 and color it red. It is easy to observe that $F \setminus K^0$ must contain a clique $K_{\omega-1}$ otherwise we could color the remaining edges of F blue. Denote this clique K^1 and color $F_1 = F[V(K^0) \cup V(K^1)]$ red. Similarly $F \setminus F_1$ must contain a clique $K_{\omega-1}$ which we denote by K^2. Repeating the above consideration we conclude that apart from K^0 the graph F contains $\alpha - 2$ disjoint cliques $K_{\omega-1}$ denoted by $K^1, K^2, \ldots, K^{\alpha-2}$. Let all edges of $F[V(\bigcup_{i=0}^{\alpha-2} K^i)]$ be red. Let $F' = F \setminus \bigcup_{i=0}^{\alpha-2} K^i$. Note that

$$\begin{aligned}
|V(F')| &= |V(F)| - [(\alpha-1)(\omega-1) + 1] \\
&= (\alpha-1)\frac{\omega(\omega-1)}{2} + \omega - 1 - [(\alpha-1)(\omega-1) + 1] \\
&= (\alpha-1)[\frac{\omega(\omega-1)}{2} - (\omega-1)] + (\omega-1) - 1 \\
&= (\alpha-1)\frac{(\omega-1)(\omega-2)}{2} + (\omega-1) - 1,
\end{aligned}$$

so F' fulfills the inductive assumption. Therefore it can be 2-colored with no red induced G and no blue $K_{\omega-1}$. Let all not so far colored edges of F be blue. In such a coloring there is no red induced G and no blue induced H. Indeed each connected red subgraph has the independence number at most $\alpha - 1$. Moreover, we can take at most one vertex from $V(\bigcup_{i=0}^{\alpha-2} K^i)$ to widen a blue clique from F', so the order of the largest blue clique is at most $\omega - 1$. Therefore $IR(G, H) > (\alpha - 1)\frac{\omega(\omega-1)}{2} + \omega - 1$. ■

It follows from the above inductive proof that the strongly arrowing graph F must contain a number of disjoint cliques. Precisely $\bigcup_{j=0}^{\omega-2}(K_{\omega-j} \cup (\alpha-2)K_{\omega-j-1})$ is a subgraph of F.

As we mentioned the lower bound from Theorem 1 is sharp. Gorgol [9] showed the exact value of the induced Ramsey number for stars versus complete graphs.

Theorem 2 [9]. *For arbitrary $k \geq 1$ and $n \geq 2$ holds*

$$IR(S_k, K_n) = (k-1)\frac{n(n-1)}{2} + n.$$

On the other hand if we take a path instead of the star we obtain $IR(P_t, K_n) \geq \left(\lceil \frac{t}{2} \rceil - 1\right) \frac{n(n-1)}{2} + n$. Comparing this with the well known result of Chvàtal [4] $R(T, K_n) = (t-1)(n-1) + 1$, where T denotes a tree on t vertices, we obtain a better result for big complete graphs obtaining a quadratic instead of a linear bound. We can claim similarly if we take a cycle instead of a path, assuming that the conjecture of Erdős ($R(C_t, K_n) = (t-1)(n-1) + 1$) is true.

Moreover Kohayakawa, Prömel and Rödl [13] showed that the induced Ramsey number of a tree T and any graph H grows polynomially with $|T| = t$ and $|H| = n$

$$IR(T, H) \leq ct^2n^4 \left(\frac{\log(tn^2)}{\log\log\log(tn^2)}\right).$$

Some more effort is needed to prove the analogous lower bound if we allow the graph G not necessarily to be connected. However we assume that it does not contain isolates. The proof is inductive again, but now the first step requires a little bit more attention.

Lemma 3. *Let G be an arbitrary isolates-free graph with $\alpha(G) = 2$ and H be an arbitrary graph with $\omega(H) = \omega \geq 3$. Then*

$$IR(G, H) \geq 2\omega.$$

Proof. Let F be an arbitrary graph on $2\omega - 1$ vertices. We will show that it can be colored without red induced graph with $\alpha = 2$ and blue induced graph with clique number ω. Obviously F is not complete. Assume that x, y are such that $xy \notin E(F)$. While constructing our coloring we take into account the following fact.

(i) If there exist two independent red edges xu and yv, then at least one the edges uv, xv, yu must exist and be blue.

Note that if $K_t \subset F$ and $t > \omega$, then we color this K_t red, all the remaining edges blue and we are done. Hence we assume that there is no $K_{\omega+1}$ in F. On the other hand F must contain $K^1 = K_\omega$, otherwise it could be colored blue. Similary, like in the proof of Theorem 1 coloring this K_ω red we conclude that the remaining $\omega - 1$ vertices form a clique $K^2 = K_{\omega-1}$.

If K^2 does not form K_ω with some vertices from K^1, then we color K^1 red and all remaining edges blue.

Therefore assume that there exists $A \subset V(K^1)$ and $C \subset V(K^2)$ such that $F[A \cup B] = K_\omega = K^3$. Let $s = |A|$. Certainly $1 \leq s \leq \omega - 1$. Let us take A with maximum s. Let $B = V(K^1) \setminus A$ and $D = V(K^2) \setminus C$.

If $s = 1$ then we have two cliques K_ω sharing one vertex, say a. Then we color with red two edges: ab for arbitrary $b \in B$ and c_1c_2 for c_1, $c_2 \in C$ and the remaining edges blue. This coloring fulfils (i).

Let now $s \geq 2$. Note that apart from K^1 and K^3 the graph F may contain at most two more cliques K_ω. There exists at most one $a \in A$ such that $F[B \cup D \cup \{a\}] = K_\omega$ and at most one $c \in C$ such that $F[B \cup D \cup \{c\}] = K_\omega$. For more than one such vertices we would obtain a larger clique. By the choice of s, there is no K_ω with vertices in all four sets A, B, C and D.

We color the edges aa_1, ab, cd, where $a_1 \in A$, $b \in B$, $d \in D$ are chosen arbitrarily, red. If any of these additional cliques does not exist, adequate a and c we can also choose arbitrarily. This coloring also fulfills (i). ∎

Theorem 4. *Let G be an arbitrary isolates-free graph with $\alpha(G) = \alpha \geq 2$ and H be an arbitrary graph with $\omega(H) = \omega \geq 3$. Then*

$$IR(G, H) \geq \alpha\omega.$$

Proof. Let F be an arbitrary graph on $\alpha\omega - 1$ vertices. We apply induction on α to prove that F can be 2-colored with no red induced G and no blue induced H.

The assertion for $\alpha = 2$ follows from Lemma 3. Thus, let $\alpha > 2$. We may assume that G contains a clique K_ω, otherwise we color all edges of F blue. Colour this clique red. A graph induced by the remaining vertices fulfills the inductive assumption so it can be colored with no red induced graph with independence number $(\alpha - 1)$ and no blue induced H. Now, color red all edges of F which have not been colored so far. ∎

It is worth noticing that if we allow the graph G to be disconnected this lower bound is sharp. Gorgol and Łuczak [10] have shown the exact value of the induced Ramsey number for a matching and a complete graph.

Theorem 5 [10]**.** *For arbitrary $k \geq 1$ and $n \geq 2$*

$$IR(kK_2, K_n) = kn.$$

References

[1] J. Beck, *On the size Ramsey number of paths, trees and circuits* I, J. Graph Theory **7** (1983) 115–129.
doi:10.1002/jgt.3190070115

[2] D. Conlon, J. Fox and B. Sudakov, *On two problems in graph Ramsey theory*, Combinatorica **32** (2012) 513–535.
doi:10.1007/s00493-012-2710-3

[3] V. Chvátal and F. Harary, *Generalized Ramsey theory for graphs.* III. *Small off-diagonal numbers*, Pacific J. Math. **41** (1972) 335–345. doi:10.2140/pjm.1972.41.335

[4] V. Chvátal, *Tree-complete graph Ramsey numbers*, J. Graph Theory **1** (1977) 93. doi:10.1002/jgt.3190010118

[5] W. Deuber, *A generalization of Ramsey's theorem*, in: Infinite and Finite Sets, R. Rado A. Hajnal and V. Sós, (Eds.) **10** (North-Holland, Amsterdam, 1975) 323–332.

[6] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947) 292–294
doi:10.1090/S0002-9904-1947-08785-1

[7] P. Erdős, *On some problems in graph theory, combinatorial analysis and combinatorial number theory*, in: Proc. Conf. Hon. P. Erdös, Cambridge 1983, Graph Theory and Combinatorics (Academic Press, New York, 1984) 1–17.

[8] P. Erdős, A. Hajnal and L. Pósa, *Strong embeddings of graphs into colored graphs*, in: Infinite and Finite Sets, R. Rado A. Hajnal and V. Sós, (Eds.) **10** (North-Holland, 1975) 585–595.

[9] I. Gorgol, *A note on a triangle-free-complete graph induced Ramsey number*, Discrete Math. **235** (2001) 159–163.
doi:10.1016/S0012-365X(00)00269-7

[10] I. Gorgol and T. Łuczak, *On induced Ramsey numbers*, Discrete Math. **251** (2002) 87–96.
doi:10.1016/S0012-365X(01)00328-4

[11] F. Harary, J. Nešetřil and V. Rödl, *Generalized Ramsey theory for graphs.* XIV. *Induced Ramsey numbers*, in: Proceedings of the Third Czechoslovak Symposium on Graph Theory, Praque 1982, Graphs and Other Combinatorial Topics **59** (1983) 90–100.

[12] P. Haxell, Y. Kohayakawa and T. Łuczak, *The induced size-Ramsey number of cycles*, Combin. Probab. Comput. **4** (1995) 217–239.
doi:10.1017/S0963548300001619

[13] Y. Kohayakawa, H.J. Prömel and V. Rödl, *Induced Ramsey numbers*, Combinatorica **18** (1998) 373–404.
doi:10.10071PL00009828

[14] A. Kostochka and N. Sheikh, *On the induced Ramsey number $IR(P3, H)$*, Topics in Discrete Mathematics **26** (2006) 155–167.
doi:10.1007/3-540-33700-8_10

[15] T. Łuczak and V. Rödl, *On induced Ramsey numbers for graphs with bounded maximum degree*, J. Combin. Theory Ser. B **66** (1996) 324–333.
doi:10.1006/jctb.1996.0025

[16] V. Rödl, The Dimention of a Graph and Generalized Ramsey Theorems, Master's Thesis (Charles University, Prague, 1973).

[17] V. Rödl, *A generalization of Ramsey theorem*, in: Proc. Symp. Comb. Anal., Zielona Góra 1976, Graphs, Hypergraphs and Block Systems (1976) 211–219.

[18] M. Schaefer and P. Shah, *Induced graph Ramsey theory*, Ars Combin. **66** (2003) 3–21.

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