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REGULAR COLORINGS IN REGULAR GRAPHS

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Abstract

An $(r-1,1)$-*coloring* of an r-regular graph G is an edge coloring (with arbitrarily many colors) such that each vertex is incident to $r-1$ edges of one color and 1 edge of a different color. In this paper, we completely characterize all 4-regular pseudographs (graphs that may contain parallel edges and loops) which do not have a $(3,1)$-coloring. Also, for each $r \geq 6$ we construct graphs that are not $(r-1,1)$-colorable and, more generally, are not $(r-t,t)$-colorable for small t.

Keywords: edge coloring, graph factors, regular graphs.

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1. Introduction

A graph with no loops or multiple edges is called *simple*; a graph in which both multiple edges and loops are allowed is called a *pseudograph*. Unless specified otherwise, the word "graph" in this paper is reserved for pseudographs. All (pseudo)graphs considered here are undirected and finite. Note that we count a loop twice in the degree of a vertex.

The famous Berge-Sauer conjecture asserts that every 4-regular simple graph contains a 3-regular subgraph [6]. This conjecture was settled by Tashkinov in 1982 [12]. In fact, he proved that every connected 4-regular pseudograph with either at most two pairs of multiple edges and no loops or at most one pair of multiple edges and at most one loop contains a 3-regular subgraph. Observe that this cannot hold for all 4-regular pseudographs, because the graph consisting of a single vertex with two loops contains no 3-regular subgraph. The following question remains open.

Question 1. *Which* 4*-regular pseudographs contain* 3*-regular subgraphs?*

Note that in 1988, Tashkinov [13] determined the values of t and r for which every r-regular pseudograph contains a t-regular subgraph. Beyond finding regular subgraphs in regular graphs, finding factors—that is, regular spanning subgraphs—in regular graphs is also of special interest. As early as 1891, Petersen [10] studied the existence of factors in regular graphs. Since then numerous results on factors have appeared—see, for example, [2, 5, 7, 11]. The concept of factors can be generalized as follows: for any set of integers S, an S-*factor* of a graph is a spanning subgraph in which the degree of each vertex is in S [8]. Several authors [1, 3, 9] have recently studied $\{a,b\}$-factors in r-regular graphs with $a+b=r$. In particular, Akbari and Kano [1] made the following conjecture:

Conjecture 1. *If r is odd and $0 \leq t \leq r$, then every r-regular graph has an $\{r-t,t\}$-factor.*

However, Axenovich and Rollin [3] disproved this conjecture. The following theorem summarizes what is known about $\{r-t,t\}$-factors of r-regular graphs. (Note that although intended for simple graphs, the result of Petersen [10] applies to pseudographs as well.)

Theorem 2. *Let t and r be positive integers with $t \leq \frac{r}{2}$.*

(a) *When r is even.*

- *If t is even, then every r-regular graph has a t-factor, and thus has an $\{r-t,t\}$-factor* (*Petersen* [10]).
- *Every r-regular graph of even order has an $\left\{\frac{r}{2}+1, \frac{r}{2}-1\right\}$-factor* (*Lu, Wang, and Yu* [9]).
- *If t is odd and $t \leq \frac{r}{2}-2$, then there exists a connected r-regular graph of even order that has no $\{r-t,t\}$-factor* [9].
- *If t is odd and $t = \frac{r}{2}$, then every r-regular subgraph of even order has an $\{r-t,t\}$-factor* [9].
- *If t is odd, then trivially, no r-regular graph of odd order has an $\{r-t,t\}$-factor.*

(b) *When r is odd and $r \geq 5$.*

- *If t is even, then every r-regular graph has an $\{r-t,t\}$-factor* (*Akbari and Kano* [1]).
- *If t is odd and $\frac{r}{3} \leq t$, then every r-regular graph has an $\{r-t,t\}$-factor* [1].
- *If t is odd and $(t+1)(t+2) \leq r$, then there exists an r-regular graph that has no $\{r-t,t\}$-factor* (*Axenovich and Rollin* [3]).

(c) *Every 3-regular graph has a $\{2,1\}$-factor* (*Tutte* [14]).

An $(r-t,t)$-*coloring* of an r-regular graph G is an edge-coloring (with at least two colors) such that each vertex is incident to $r-t$ edges of one color and t edges of a different color. An *ordered* $(r-t,t)$-*coloring* of G is an $(r-t,t)$-coloring using integers as colors such that each vertex is incident to $r-t$ edges of some color i and t edges of some color j with $i<j$. Thus, in a graph with an ordered $(r-t,t)$-coloring, regardless of how many colors are used, the set of edges colored with the minimum integer induces an $(r-t)$-regular subgraph, and the set of edges colored with the maximum integer induces a t-regular subgraph.

Bernshteyn [4] introduced $(3,1)$-colorings as an approach to answer Question 1. A possible advantage of working with $(3,1)$-colorings is that this is a locally-defined notion. Bernshteyn proved the following.

Theorem 3 (Bernshteyn [4])**.** *A connected 4-regular graph contains a 3-regular subgraph if and only if it admits an ordered $(3,1)$-coloring.*

We observe that the notion of an $(r-t,t)$-coloring of an r-regular graph generalizes that of an $\{r-t,t\}$-factor. Indeed, an r-regular graph G has an

$(r-t,t)$-coloring with two colors if and only if G has an $\{r-t,t\}$-factor: the two color classes are precisely the $\{r-t,t\}$-factor and its complement, which is another $\{r-t,t\}$-factor. Thus, $(r-t,t)$-colorings provide a common approach to attacking Question 1 as well as any unresolved cases from Conjecture 1, specifically, when r and t are both odd and $3t < r < (t+1)(t+2)$. As an $(r-t,t)$-coloring with more than two colors can exist when there is no $\{r-t,t\}$-factor, we consider the following general question.

Question 2. *For which r and t does every r-regular graph have an $(r-t,t)$-coloring?*

For $r \geq 6$, we resolve Question 2 for various values of t, including $t = 1$ (see Section 3). However, the question remains open for $r = 5$ and $t = 1$.

There are trivial examples of 4-regular graphs without $(3,1)$-colorings, such as a single vertex with two loops. However, Theorem 3 motivates the following weaker version of Question 1.

Question 3. *Which 4-regular graphs have $(3,1)$-colorings?*

The arrows in Figure 1 indicate the relationships among t-factors, $\{r-t,t\}$-factors, ordered $(r-t,t)$-colorings, $(r-t,t)$-colorings, and t-regular subgraphs of r-regular graphs that hold for arbitrary r and t.

$$\begin{tikzcd}
G \text{ has an } (r-t,t)\text{-coloring.} & & G \text{ has a } t\text{-regular subgraph.} \\
G \text{ has an } \{r-t,t\}\text{-factor.} \arrow[u, Rightarrow] & & G \text{ has an ordered } (r-t,t)\text{-coloring.} \arrow[u, Rightarrow] \arrow[ull, Rightarrow] \\
& G \text{ has a } t\text{-factor.} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Figure 1. Implications that hold for every r-regular graph G and for all integers $0 < t < r$.

Now we are ready to describe our main results. First, in Section 2, we characterize all 4-regular graphs which are not $(3,1)$-colorable, which settles Question 3. Because the statement of the result requires additional definitions, we postpone it until then (see Theorem 4). Then, in Section 3, we construct relevant examples of r-regular graphs for $r \geq 6$ and various t: some with no $(r-t,t)$-coloring, others with an $(r-t,t)$-coloring but no $\{r-t,t\}$-factor.

2. $(3,1)$-Colorings in 4-Regular Graphs

In this section we characterize 4-regular graphs that do not admit $(3,1)$-colorings. Let us first establish some terminology. Let G_1 and G_2 be vertex-disjoint graphs with (possibly loop) edges $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. The disjoint union of X and Y is denoted by $X \dot\cup Y$. The *edge adhesion* of G_1 and G_2 at e_1 and e_2 is the graph $G = (G_1, e_1) + (G_2, e_2)$ obtained by subdividing edges e_1 and e_2 and identifying the two new vertices. (See Figure 2.) That is,

$$\begin{aligned} V(G) &= V(G_1) \dot\cup V(G_2) \dot\cup \{w\}; \\ E(G) &= (E(G_1) \setminus \{e_1\}) \dot\cup (E(G_2) \setminus \{e_2\}) \dot\cup \{u_1w, v_1w, u_2w, v_2w\}. \end{aligned}$$

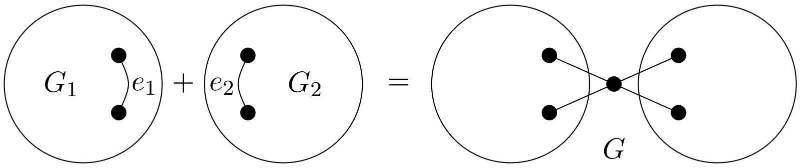

Figure 2. Edge adhesion of two graphs, $G = (G_1, e_1) + (G_2, e_2)$.

The *adhesion of a loop* to graph H at edge $e = uv \in E(H)$ is the graph $H' = (H, e) + O$ obtained by subdividing e and adding a loop at the new vertex. (See Figure 3.) That is,

$$\begin{aligned} V(H') &= V(H) \dot\cup \{x\}; \\ E(H') &= (E(H) \setminus \{e\}) \dot\cup \{ux, vx, xx\}. \end{aligned}$$

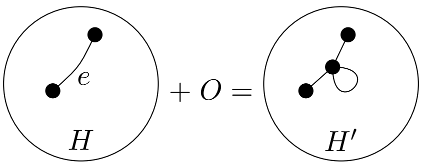

Figure 3. Adhesion of a loop at an edge, $H' = (H, e) + O$.

Let C be a cycle, which has $|E(C)| = |V(C)|$ (allowing for a degenerate cycle on 1 or 2 vertices). A *double cycle* is obtained from C by doubling each edge. We say a double cycle is even (respectively, odd) if it has an even (respectively, odd) number of vertices. (See Figure 4.)

Clearly, double cycles and graphs resulting from edge adhesion of two 4-regular graphs or from the adhesion of a loop to a 4-regular graph are all 4-regular. We are now ready to give the main result of this section.

Theorem 4. *A connected* 4*-regular graph is not* $(3, 1)$*-colorable if and only if it can be constructed from odd double cycles via a sequence of edge adhesions.*

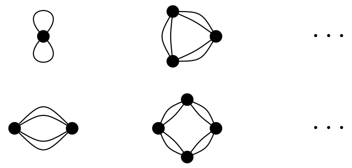

Figure 4. Double cycles (odd on top, even on bottom).

From Theorem 4 we see that any 4-regular graph that is not $(3,1)$-colorable has an odd number of vertices. Indeed, any 4-regular graph with an even number of vertices has a $\{3,1\}$-factor by Theorem 2 and hence a $(3,1)$-coloring using two colors.

Remark 5. Theorem 4 naturally lends itself to a proof by induction. In particular, an equivalent statement is that a connected 4-regular graph is not $(3,1)$-colorable if and only if it is an odd double cycle or obtained from two 4-regular, non-$(3,1)$-colorable graphs by a sequence of edge adhesions.

Before we prove Theorem 4, we need to develop a few lemmas.

Lemma 6. *A double cycle with $n \geq 1$ vertices is $(3,1)$-colorable if and only if n is even.*

Proof. Even double cycles have perfect matchings and are thus $(3,1)$-colorable.

Assume that there is a $(3,1)$-coloring c of an odd double cycle G. Let G' denote the cycle obtained by removing one of the parallel edges between any two adjacent vertices in G. Color an edge in G' red if its corresponding parallel edges in G are of the same color under c and blue otherwise. Observe that the edges incident to any vertex in G' are of different colors, since c is a $(3,1)$-coloring of G. This is a contradiction since G' is an odd cycle. ∎

Lemma 7 (Bernshteyn [4])**.** *If G is a 4-regular graph and there exists a non-double edge uv in G with $u \neq v$ such that $G - \{u,v\}$ is connected, then G is $(3,1)$-colorable.*

Lemma 8 (Bernshteyn [4])**.** *If G is a 4-regular graph and $G' = (G,e) + O$ for some edge $e \in E(G)$, then either G or G' has a 3-regular subgraph.*

Lemma 9. *Let G_1 and G_2 be $(3,1)$-colorable 4-regular graphs and let G_2 have a loop vv. Construct G by subdividing an edge uw in G_1, identifying the new vertex with v, and removing the loop vv, so*

$$\begin{aligned} V(G) &= V(G_1) \dot\cup V(G_2); \\ E(G) &= (E(G_1) \setminus \{uw\}) \dot\cup (E(G_2) \setminus \{vv\}) \dot\cup \{uv, wv\}. \end{aligned}$$

(*See Figure* 5.) *Then G is $(3,1)$-colorable.*

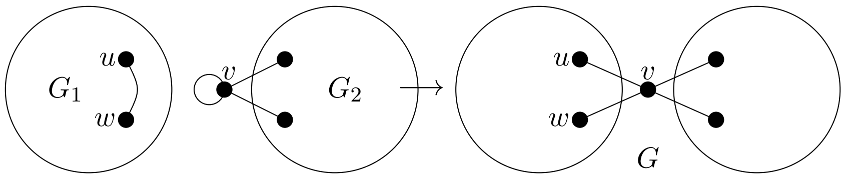

Figure 5. Joining G_2 to G_1 at a loop, as in Lemma 9.

Proof. Fix $(3,1)$-colorings c_i of G_i for $i \in \{1,2\}$. Note that v in G_2 is incident to only one loop and that the two non-loop edges incident to v have different colors under c_2. Without loss of generality, assume that $c_1(uw)$ is equal to the color of one of the non-loop edges incident to v. Therefore the colorings c_1 and c_2 extend to a $(3,1)$-coloring of G by coloring the edges uv and uw with color $c_1(uw)$. ∎

Corollary 10. *Suppose exactly one of the connected* 4*-regular graphs* G_1 *and* G_2 *is* $(3,1)$*-colorable. Then for any* $e_1 \in E(G_1)$ *and* $e_2 \in E(G_2)$*,* $(G_1, e_1) + (G_2, e_2)$ *is* $(3,1)$*-colorable.*

Proof. Without loss of generality, we assume that G_1 is $(3,1)$-colorable and G_2 is not. Let $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$. By Theorem 3 and Lemma 8, the graph $G_2' = (G_2, e_2) + O$ is $(3,1)$-colorable. Applying Lemma 9 to G_1 and G_2', we see that $(G_1, e_1) + (G_2, e_2)$ is $(3,1)$-colorable. ∎

Lemma 11. *Let* G *be a* 4*-regular graph that is not* $(3,1)$*-colorable. If* G *has a non-double, non-loop edge, then* G *is not* 2*-connected.*

Proof. Let uv be a non-double, non-loop edge, and suppose for contradiction that G is 2-connected. By Lemma 7, since G is not $(3,1)$-colorable, $G' = G - \{u, v\}$ is disconnected. Since G is 2-connected, neither u nor v is a cut-vertex. Therefore, every component of G' must contain at least one vertex from $N_G(u)$ and at least one vertex from $N_G(v)$. Since the sum of the degrees of the vertices must be even in each component, the 4-regularity of G implies that each component of G' must have been connected to $\{u, v\}$ by an even number of edges. Let $N_G(u) \setminus \{v\} = \{u_1, u_2, u_3\}$ and $N_G(v) \setminus \{u\} = \{v_1, v_2, v_3\}$. Without loss of generality, G' is the disjoint union of a component G_1 containing u_1 and v_1 and a subgraph G_2 (of one or two components) containing u_2, u_3, v_2, and v_3.

Let $G_1' = (G_1 + u_1v_1, u_1v_1) + O$ and $G_2' = ((G - G_1) + uv, uv) + O$. (See Figure 6.) That is,

$$\begin{aligned} V(G_1') &= V(G_1) \,\dot\cup\, \{w_1\}; \\ E(G_1') &= E(G_1) \,\dot\cup\, \{u_1w_1, v_1w_1, w_1w_1\}; \\ V(G_2') &= V(G_2) \,\dot\cup\, \{u, v, w_2\}; \\ E(G_2') &= E(G_2) \,\dot\cup\, \{uu_2, uu_3, uv, vv_2, vv_3, uw_2, vw_2, w_2w_2\}. \end{aligned}$$

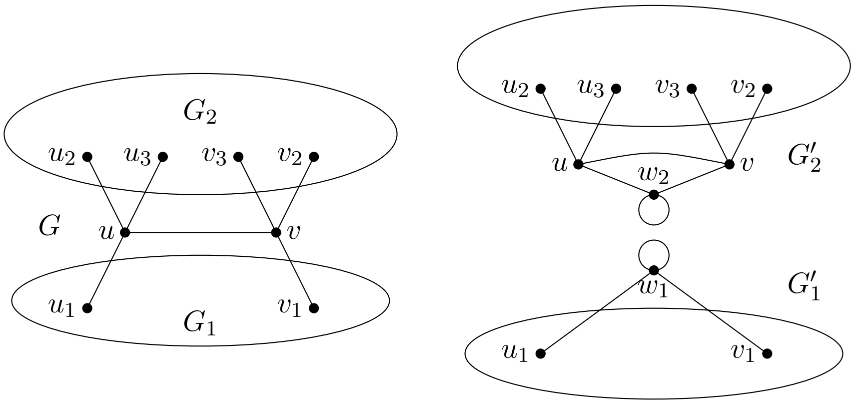

Figure 6. Splitting a 2-connected graph into two $(3,1)$-colorable graphs, from the proof of Lemma 11.

By the assumption of 2-connectedness, the vertex u_1 is not a cut-vertex of G. If $u_1 = v_1$, then the vertex also has a loop (so as not to be a cut vertex) and then G'_1 is trivially $(3,1)$-colorable. Otherwise, $u_1 \neq v_1$ and $G'_1 - \{u_1, w_1\}$ is connected. Thus by Lemma 7, G'_1 is $(3,1)$-colorable. Likewise, $G'_2 - \{u, w_2\}$ is connected, so G'_2 is $(3,1)$-colorable. Select $(3,1)$-coloring c_i of G'_i for $i \in \{1,2\}$. Note that because of the loops, $c_1(u_1w_1) \neq c_1(v_1w_1)$ and $c_2(uw_2) \neq c_2(vw_2)$. We can assume that $c_1(u_1w_1) = c_2(uw_2)$ and $c_1(v_1w_1) = c_2(vw_2)$. Therefore, the colorings c_1 and c_2 easily extend to a $(3,1)$-coloring c of G, which is a contradiction. ■

Lemma 12. *Let G be a connected 4-regular graph that is not 2-connected. Then $G = (G_1, e_1) + (G_2, e_2)$ for some 4-regular graphs G_1, G_2 and edges $e_1 \in E(G_1)$, $e_2 \in E(G_2)$.*

Proof. Indeed, let $w \in V(G)$ be a cut-vertex. Now the lemma is implied by the following observation. Since the number of vertices with odd degrees in a graph is always even, $G - w$ consists of exactly two components and each of these components receives exactly two of the edges incident to w. ■

***Proof of Theorem* 4.** Consider 4-regular graphs G_1 and G_2 and edges e_1 in G_1, e_2 in G_2. Any $(3,1)$-coloring of $(G_1, e_1) + (G_2, e_2)$ yields a $(3,1)$-coloring of G_1 or G_2, since the edges obtained by subdividing e_1 or e_2 are of the same color. Therefore every graph that is obtained from odd double cycles via edge adhesion is not $(3,1)$-colorable due to Lemma 6.

Now let G be a connected 4-regular graph that is not $(3,1)$-colorable. We use induction on $|V(G)|$ to prove that G is constructed from odd double cycles via edge adhesion. If $|V(G)| = 1$, then G is a double cycle of one vertex and the theorem trivially holds. Assume that $|V(G)| \geq 2$. We may also assume that

G contains a non-double edge. Otherwise, if every edge is double, then G is a double cycle, and by Lemma 6, G is an odd double cycle, and thus we are done.

If each non-double edge is a loop, then one can easily check that G is not 2-connected. If G has a non-double non-loop edge, Lemma 11 implies that it is not 2-connected. By Lemma 12, $G = (G_1, e_1) + (G_2, e_2)$ for some 4-regular graphs G_1, G_2 and edges $e_1 \in E(G_1)$, $e_2 \in E(G_2)$. Corollary 10 implies that either both G_1 and G_2 are $(3, 1)$-colorable or neither of them is $(3, 1)$-colorable. In the latter case, by the inductive hypothesis, we are done.

Assume that both G_1 and G_2 are $(3, 1)$-colorable. Let $G'_1 = (G_1, e_1) + O$ and observe that G is obtained from G'_1 and G_2 as in the statement of Lemma 9. Since G_2 is $(3, 1)$-colorable, but G is not, Lemma 9 implies that G'_1 is not $(3, 1)$-colorable. Therefore, by the inductive hypothesis, G'_1 is obtained from odd double cycles via edge adhesion. Since G'_1 contains a loop and at least two vertices, it is not a double cycle. Thus, $G'_1 = (G'_{11}, e'_{11}) + (G'_{12}, e'_{12})$, where neither G'_{11} nor G'_{12} is $(3, 1)$-colorable. Note that, without loss of generality, G'_{11} does not contain the subdivided edge e_1, and so $G = (G'_{11}, e'_{11}) + (H, f)$ for some graph H and edge f in H. Since both G and G'_{11} are not $(3, 1)$-colorable, neither is H by Corollary 10. We have shown that G is obtained from two graphs that are not $(3, 1)$-colorable via edge adhesion, and so the inductive step is complete. ■

3. r-Regular Graphs for $r \geq 5$

Question 2 for $r = 5$ remains open at this time. However, in this section we demonstrate that there are r-regular graphs with no $(1, r-1)$-coloring for each $r \geq 6$. More generally, for each odd t and each even r, as well as for each odd t and each odd $r \geq (t+2)(t+1)$, we construct an r-regular graph with no $(r-t, t)$-coloring. Note that for even t, every r-regular graph has an $(r-t, t)$-coloring and for odd $t \geq \frac{r}{3}$ and even r every r-regular graph has a $(r-t, t)$-coloring due to Theorem 2.

Theorem 13. *Let r and t be positive integers with $t \leq \frac{r}{2}$ odd. If r is even or $r \geq (t+2)(t+1)$, then there exists a connected r-regular graph that is not $(r-t, t)$-colorable.*

Observe that this is the same upper bound on odd r as in Theorem 2(b) (due to [3]) for the existence of r-regular graphs without $\{r-t, t\}$-factors.

Proof. First, if r is even, then the r-regular graph with one vertex and $\frac{r}{2}$ loops has no $(r-t, t)$-coloring, since t is odd.

Now suppose that $r \geq (t+2)(t+1) \geq 6$ is odd. Let G be a graph on vertices $v, u, u_1, \ldots, u_{t+1}$ with $t+2$ edges between v and u_i and $\frac{r-t-2}{2}$ loops incident to u_i, $1 \leq i \leq t+1$, and $r - (t+2)(t+1) \geq 0$ edges between v and u and $\frac{(t+2)(t+1)}{2}$

loops incident to u. Observe that G is r-regular. Suppose that G admits an $(r-t,t)$-coloring. Then there is an i such that all $t+2$ edges between v and u_i are of the same color. However, this is a contradiction, because there is no coloring of the loops incident to this u_i such that there are exactly t edges of another color incident to u_i, as t is odd. ■

Now we will exhibit r-regular graphs of even order that have $(r-1,1)$-colorings but not $\{r-1,1\}$-factors. The constructions are similar to constructions in [9].

Theorem 14. *For every even $r \geq 6$ there exists a connected $(r-1,1)$-colorable r-regular graph of even order without an $\{r-1,1\}$-factor.*

Proof. Note that K_{r+1} has an odd number of vertices and thus does not have an $\{r-1,1\}$-factor, as $r-1$ is odd. However, there is an $(r-1,1)$-coloring with 3 colors. Indeed color a copy of K_r red, $r-1$ of the remaining edges blue, and the last edge green.

If $\frac{r}{2}$ is odd, then let $G_1,\dots,G_{\frac{r}{2}}$ be vertex-disjoint copies of $K_{r+1}-e$. Form a graph G from the union of G_i by connecting all vertices of degree $r-1$ in the G_i to a new vertex u. Then G has an even number of vertices and is r-regular. Moreover there is an $(r-1,1)$-coloring with 3 colors. Indeed, start by coloring $r-1$ of the edges incident to u green, and the other blue. For each of the $\frac{r}{2}-1$ copies of $K_{r+1}-e$ with two incoming green edges, color red a copy of K_r that contains exactly one of the neighbors of u, and color the other $r-1$ edges (incident to the other neighbor of u) blue. In the final copy of $K_{r+1}-e$, do the same, making sure that the K_r contains the neighbor of u with the incoming blue end. Now that we have shown G to be $(r-1,1)$-colorable, assume that G has an $\{r-1,1\}$-factor, i.e., an $(r-1,1)$-coloring in two colors. Then there is an i, $1 \leq i \leq \frac{r}{2}$, such that both edges between G_i and u are of the same color. This yields an $(r-1,1)$-coloring of K_{r+1} in two colors, a contradiction.

If $\frac{r}{2}$ is even, then let $t = 3(\frac{r}{2}-1)$. Let $G_1,\dots,G_t$ be vertex-disjoint copies of $K_{r+1}-e$. Form a graph G from the union of the G_i and a disjoint copy of K_3 with vertex set $\{u_0,u_1,u_2\}$ by connecting both vertices of degree $r-1$ in G_i to u_j if $j(\frac{r}{2}-1) < i \leq (j+1)(\frac{r}{2}-1)$. Then G has an even number of vertices and is r-regular. One can show that G has an $(r-1,1)$-coloring but no $\{r-1,1\}$-factor with arguments similar to those given above. ■

4. Concluding Remarks

Here we state a number of open problems related to our work. Recall from the Introduction that Tashkinov [12] showed that every 4-regular graph with no multiple edges and at most one loop contains a 3-regular subgraph. It is not known whether the restriction on the number of loops is necessary.

Question 4. *Does every* 4*-regular graph with no multiple edges have a* 3*-regular subgraph?*

Let us note that Question 4 is open even for the class of 4-regular graphs with no multiple edges and at most two loops. (Note that we regard two loops at a single vertex as a pair of multiple edges.)

Most of our unanswered questions concern 5-regular graphs. The first case of Conjecture 1 that Theorem 2 does not address is when $r = 5$ and $t = 1$.

Conjecture 15. *Every* 5*-regular graph has a* $\{4, 1\}$*-factor.*

Weakening this, we have the following unresolved case of Question 2.

Question 5. *Does every* 5*-regular graph have a* $(4, 1)$*-coloring?*

Another variation of this question concerns colorings with a bounded number of colors. Bernshteyn [4] showed that if G is a 4-regular graph that has a $(3, 1)$-coloring, then G has a $(3, 1)$-coloring that uses at most three colors.

Question 6. *Is there a positive integer* K *such that every* 5*-regular graph has a* $(4, 1)$*-coloring using at most* K *colors?*

Question 6 lies "between" Conjecture 15 and Question 5 in the following sense. An affirmative answer to Question 6 clearly gives an affirmative answer to Question 5. On the other hand, as observed in the Introduction, Conjecture 15 implies an affirmative answer to Question 6 with $K = 2$.

Our final question concerns ordered $(r - 1, 1)$-colorings.

Question 7. *For* $r \geq 5$*, if* G *is an* r*-regular graph with an* $(r-1)$*-regular subgraph, does* G *admit an ordered* $(r - 1, 1)$*-coloring?*

As observed in the Introduction, the converse to this statement always holds (see Figure 1). Also, Theorem 3 implies that the corresponding statement is true for $r = 4$.

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