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ON FACTORABLE BIGRAPHIC PAIRS [1]

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Abstract

Let $S = (a_1, \ldots, a_m;\ b_1, \ldots, b_n)$, where $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are two sequences of nonnegative integers. We say that S is a *bigraphic pair* if there exists a simple bipartite graph G with partite sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ such that $d_G(x_i) = a_i$ for $1 \le i \le m$ and $d_G(y_j) = b_j$ for $1 \le j \le n$. In this case, we say that G is a *realization* of S. Analogous to Kundu's k-factor theorem, we show that if $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ and $(a_1 - e_1, a_2 - e_2, \ldots, a_m - e_m;\ b_1 - f_1, b_2 - f_2, \ldots, b_n - f_n)$ are two bigraphic pairs satisfying $k \le f_i \le k+1$, $1 \le i \le n$ (or $k \le e_i \le k+1$, $1 \le i \le m$), for some $0 \le k \le m-1$ (or $0 \le k \le n-1$), then $(a_1, a_2, \ldots, a_m;$ $b_1, b_2, \ldots, b_n)$ has a realization containing an $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$-factor. For $m = n$, we also give a necessary and sufficient condition for an $(k^n; k^n)$-factorable bigraphic pair to be connected $(k^n; k^n)$-factorable when $k \ge 2$. This implies a characterization of bigraphic pairs with a realization containing a Hamiltonian cycle.

Keywords: degree sequence, bigraphic pair, Hamiltonian cycle.

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1. Introduction

If there is no special explanation, graphs in this paper are simple graphs, i.e., finite undirected graphs without loops or multiple edges. Terms and notation not defined here are from [1]. A sequence $(d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be a *graphic sequence* if it is the degree sequence of a graph G on n vertices.

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In this case, G is referred to as a *realization* of $(d_1, d_2, \ldots, d_n)$. An $(k_1, k_2, \ldots, k_n)$-factor of G is a spanning subgraph of G whose degree sequence is $(k_1, k_2, \ldots, k_n)$. A graphic sequence $(d_1, d_2, \ldots, d_n)$ is called to be $(k_1, k_2, \ldots, k_n)$-*factorable* (*connected* $(k_1, k_2, \ldots, k_n)$-*factorable*) if $(d_1, d_2, \ldots, d_n)$ has a realization G containing an $(k_1, k_2, \ldots, k_n)$-factor (connected $(k_1, k_2, \ldots, k_n)$-factor). The following theorem was conjectured by Rao and Rao [7] for the case $k_i = k$ for all i, and was proved by Kundu by using an alternating chain approach.

Theorem 1 (Kundu [5]). *Let $(d_1, d_2, \ldots, d_n)$ and $(d_1 - k_1, d_2 - k_2, \ldots, d_n - k_n)$ be two graphic sequences satisfying $k \leq k_i \leq k + 1$, $1 \leq i \leq n$, for some $k \geq 0$. Then $(d_1, d_2, \ldots, d_n)$ is $(k_1, k_2, \ldots, k_n)$-factorable.*

Some generalizations of Theorem 1 were obtained by Kundu [6], Kleitman and Wang [4]. Chen [2] gave a very short proof of Theorem 1. We denote $(k_1, k_2, \ldots, k_n) = (k^n)$ if $k_i = k$ for $1 \leq i \leq n$. Rao and Rao [7] gave a necessary and sufficient condition for an (k^n)-factorable graphic sequence to be connected (k^n)-factorable when $k \geq 2$.

Theorem 2 (Rao and Rao [7]). *Let $k \geq 2$ and $(d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $(d_1, d_2, \ldots, d_n)$ is connected (k^n)-factorable if and only if $(d_1, d_2, \ldots, d_n)$ is (k^n)-factorable and $\sum_{i=1}^{s} d_i < s(n-s-1) + \sum_{i=n-s+1}^{n} d_i$ for all s with $s < \frac{n}{2}$.*

The following corollary is a direct consequence of Theorems 1 and 2.

Corollary 3 (Kundu [5]). *Let $(d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $(d_1, d_2, \ldots, d_n)$ has a realization G containing a Hamiltonian cycle if and only if $(d_1 - 2, d_2 - 2, \ldots, d_n - 2)$ is graphic and $\sum_{i=1}^{s} d_i < s(n-s-1) + \sum_{i=n-s+1}^{n} d_i$ for all s with $s < \frac{n}{2}$.*

For $n \geq r$, Yin [9] extended Corollary 3 and characterized all graphic sequences $\pi = (d_1, d_2, \ldots, d_n)$ such that π has a realization G containing C_r, a cycle on r vertices.

Analogous problems are also studied in this paper. Let G be a bipartite graph with partite sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$. Denote $a_i = d_G(x_i)$ for $1 \leq i \leq m$ and $b_j = d_G(y_j)$ for $1 \leq j \leq n$. Then $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ is called the *degree sequence pair* of G. Let $S = (a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ be a pair of sequences of nonnegative integers. We say that S is a *bigraphic pair* if there exists a bipartite graph G whose degree sequence pair is S. In this case, we say that G is a *realization* of S. One easy method to determine if S is a bigraphic pair is the Gale-Ryser characterization.

Theorem 4 (Gale [3], Ryser [8]). *Let $S = (a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ be a pair of sequences of nonnegative integers with $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq$*

$b_2 \geq \cdots \geq b_n$. *Then S is a bigraphic pair if and only if* $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$ *and* $\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{n} \min\{k, b_j\}$ *for all k with* $1 \leq k \leq m$.

Let G be a bipartite graph with partite sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$. An $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$-factor of G is a spanning subgraph F of G such that $d_F(x_i) = e_i$ for $1 \leq i \leq m$ and $d_F(y_j) = f_j$ for $1 \leq j \leq n$. Let $S = (a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ and $S' = (e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$ be two bigraphic pairs. Then S is called to be *S'-factorable* (*connected S'-factorable*) if S has a realization G containing an S'-factor (connected S'-factor).

In this paper, we obtain a theorem on factorable bigraphic pairs as follows.

Theorem 5. *Let* $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ *and* $(a_1 - e_1, a_2 - e_2, \ldots, a_m - e_m;\ b_1 - f_1, b_2 - f_2, \ldots, b_n - f_n)$ *be two bigraphic pairs satisfying* $k \leq f_i \leq k+1$, $1 \leq i \leq n$ *(or* $k \leq e_i \leq k+1$, $1 \leq i \leq m$*), for some* $0 \leq k \leq m-1$ *(or* $0 \leq k \leq n-1$*). Then* $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ *is* $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$*-factorable.*

For $m = n$, we give a necessary and sufficient condition for an $(k^n; k^n)$-factorable bigraphic pair to be connected $(k^n; k^n)$-factorable when $k \geq 2$.

Theorem 6. *Let $k \geq 2$ and* $(a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ *be a bigraphic pair with* $a_1 \geq a_2 \geq \cdots \geq a_n$ *and* $b_1 \geq b_2 \geq \cdots \geq b_n$. *Then* $(a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ *is connected* $(k^n; k^n)$*-factorable if and only if* $(a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ *is* $(k^n; k^n)$*-factorable and* $\sum_{i=1}^{s} a_i < s(n-s) + \sum_{i=n-s+1}^{n} b_i$ *for all s with $s < n$.*

The following corollary is a direct consequence of Theorems 5 and 6.

Corollary 7. *Let* $(a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ *be a bigraphic pair with* $a_1 \geq a_2 \geq \cdots \geq a_n$ *and* $b_1 \geq b_2 \geq \cdots \geq b_n$. *Then* $(a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ *has a realization G containing a Hamiltonian cycle if and only if* $(a_1 - 2, a_2 - 2, \ldots, a_n - 2;\ b_1 - 2, b_2 - 2, \ldots, b_n - 2)$ *is a bigraphic pair and* $\sum_{i=1}^{s} a_i < s(n-s) + \sum_{i=n-s+1}^{n} b_i$ *for all s with $s < n$.*

2. Proof of Theorem 5

Firstly, we give a lemma which ensures that the condition in Theorem 5 that $k \leq f_i \leq k+1$, $1 \leq i \leq n$ (or $k \leq e_i \leq k+1$, $1 \leq i \leq m$) implies that $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$ is a bigraphic pair.

Lemma 8. *Let* $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$ *be a pair of sequences of non-negative integers with* $e_i \leq n$ *for* $1 \leq i \leq m$, $f_i \leq m$ *for* $1 \leq i \leq n$ *and* $\sum_{i=1}^{m} e_i = \sum_{i=1}^{n} f_i$. *If* $k \leq f_i \leq k+1$, $1 \leq i \leq n$ *(or* $k \leq e_i \leq k+1$, $1 \leq i \leq m$*), for some* $0 \leq k \leq m-1$ *(or* $0 \leq k \leq n-1$*), then* $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$ *is a bigraphic pair.*

Proof. Without loss of generality, we may assume that $e_1 \geq e_2 \geq \cdots \geq e_m$ and $f_1 \geq f_2 \geq \cdots \geq f_n$. By Theorem 4, we only need to check that $\sum_{i=1}^{t} e_i \leq \sum_{j=1}^{n} \min\{t, f_j\}$ for all t with $1 \leq t \leq m$. If $1 \leq t \leq k$, then $\sum_{i=1}^{t} e_i \leq tn = \sum_{j=1}^{n} \min\{t, f_j\}$. If $k+1 \leq t \leq m$, then $\sum_{i=1}^{t} e_i \leq \sum_{i=1}^{m} e_i = \sum_{i=1}^{n} f_i = \sum_{j=1}^{n} \min\{t, f_j\}$. ■

Now, we give a lemma which is a version of Theorem 5.

Lemma 9. *Let $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_m;\ d_1, d_2, \ldots, d_n)$ be two bigraphic pairs satisfying $k \leq b_i \leq k+1$, $1 \leq i \leq n$, for some $0 \leq k \leq m-1$. If $(a_1 - c_1, a_2 - c_2, \ldots, a_m - c_m;\ b_1 - d_1, b_2 - d_2, \ldots, b_n - d_n)$ is a bigraphic pair, then $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ is $(c_1, c_2, \ldots, c_m;\ d_1, d_2, \ldots, d_n)$-factorable.*

Proof. Let F and H be realizations of $(c_1, c_2, \ldots, c_m;\ d_1, d_2, \ldots, d_n)$ and $(a_1 - c_1, a_2 - c_2, \ldots, a_m - c_m;\ b_1 - d_1, b_2 - d_2, \ldots, b_n - d_n)$ respectively with partite sets $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ such that $d_F(x_i) = c_i, d_F(y_j) = d_j, d_H(x_i) = a_i - c_i, d_H(y_j) = b_j - d_j$ for all i and j and the multigraph $F \cup H$ ($V(F \cup H) = V(F) = V(H)$, $E(F \cup H) = E(F) \cup E(H)$ and there are at most two edges between two vertices) has the minimum number of multiple edges. If $F \cup H$ has no multiple edges, the lemma is proved. Otherwise, suppose that $F \cup H$ has a multiple edge $x_t y_r$, i.e., there are two edges between x_t and y_r in $F \cup H$, where $x_t \in \{x_1, x_2, \ldots, x_m\}$ and $y_r \in \{y_1, y_2, \ldots, y_n\}$. Since $d_{F \cup H}(x_t) = d_F(x_t) + d_H(x_t) = c_t + (a_t - c_t) = a_t \leq n$, there exists a vertex $y_q \in \{y_1, y_2, \ldots, y_n\}$ with $q \neq r$ such that there is no any edge between x_t and y_q in $F \cup H$, that is, $x_t y_q \notin E(F \cup H)$. By $d_{F \cup H}(y_r) = b_r$, $d_{F \cup H}(y_q) = b_q$ and $k \leq b_i \leq k+1$ for all i, we can find a vertex $x_p \in \{x_1, x_2, \ldots, x_m\}$ with $p \neq t$ such that the number of edges joining y_r and x_p is less than the number of edges joining y_q and x_p. Without loss of generality, we may assume that $y_q x_p \in E(F)$ and $y_r x_p \notin E(F)$. Therefore we must have either $y_r x_p \notin E(H)$ or $y_r x_p, y_q x_p \in E(H)$. If $y_r x_p \notin E(H)$, then there is no any edge between x_p and y_r in $F \cup H$; let $F' = F - \{x_t y_r, y_q x_p\} + \{x_t y_q, y_r x_p\}$. Then F' is a realization of $(c_1, c_2, \ldots, c_m;\ d_1, d_2, \ldots, d_n)$. Clearly, $F' \cup H$ has fewer multiple edges than $F \cup H$, a contradiction. If $y_r x_p, y_q x_p \in E(H)$, then there are two edges between x_p and y_q in $F \cup H$, by $x_t y_q \notin E(F \cup H)$ and $y_r x_p \notin E(F)$; let $F' = F - \{x_t y_r, y_q x_p\} + \{x_t y_q, y_r x_p\}$. Then F' is a realization of $(c_1, c_2, \ldots, c_m;\ d_1, d_2, \ldots, d_n)$. However, $F' \cup H$ has fewer multiple edges than $F \cup H$, a contradiction. ■

***Proof of Theorem* 5.** Since $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ and $(a_1 - e_1, a_2 - e_2, \ldots, a_m - e_m;\ b_1 - f_1, b_2 - f_2, \ldots, b_n - f_n)$ are bigraphic, we have that $e_i \leq n$ for $1 \leq i \leq m$, $f_i \leq m$ for $1 \leq i \leq n$ and $\sum_{i=1}^{m} e_i = \sum_{i=1}^{n} f_i$. It follows from $k \leq f_i \leq k+1$ for each i and Lemma 8 that $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$ is bigraphic. Clearly, $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n)$ is $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$-factorable if and only if $(n - e_1, n - e_2, \ldots, n - e_m;\ m - f_1, m - f_2, \ldots, m - f_n)$

is $(n-a_1, n-a_2, \ldots, n-a_m;\ m-b_1, m-b_2, \ldots, m-b_n)$-factorable. Now by $k \le f_i \le k+1$, $1 \le i \le n$, for some $0 \le k \le m-1$, we have that $s \le m - f_i \le s+1$, $1 \le i \le n$, for $s = m-k-1$ with $0 \le s \le m-1$. Moreover, $((n-e_1)-(n-a_1), (n-e_2)-(n-a_2), \ldots, (n-e_m)-(n-a_m);\ (m-f_1)-(m-b_1), (m-f_2)-(m-b_2), \ldots, (m-f_n)-(m-b_n)) = (a_1-e_1, a_2-e_2, \ldots, a_m-e_m;\ b_1-f_1, b_2-f_2, \ldots, b_n-f_n)$ is a bigraphic pair. It follows from Lemma 9 that $(n-e_1, n-e_2, \ldots, n-e_m;\ m-f_1, m-f_2, \ldots, m-f_n)$ is $(n-a_1, n-a_2, \ldots, n-a_m;\ m-b_1, m-b_2, \ldots, m-b_n)$-factorable. Thus $(a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n)$ is $(e_1, e_2, \ldots, e_m;\ f_1, f_2, \ldots, f_n)$-factorable. The proof is completed. ∎

3. Proof of Theorem 6

In order to prove Theorem 6, we also need some lemmas. For a bipartite graph G with partite sets X and Y, we let G_1 to be a connected subgraph of G with partite sets X_1 and Y_1 and G_2 to be a subgraph of G with partite sets X_2 and Y_2 so that $X_i \subseteq X$ and $Y_i \subseteq Y$ for $i = 1, 2$ and $V(G_1) \cap V(G_2) = \emptyset$. If $xy \in E(G)$ for all $x \in X_1$ and $y \in Y_2$ and $uv \notin E(G)$ for all $u \in Y_1$ and $v \in X_2$, then we write $G_1 \to G_2$. We first give Lemma 10 as follows.

Lemma 10. *Let $k \ge 2$ and F be an k-regular bipartite graph with partite sets X and Y. If F is connected, then F is 2-edge-connected.*

Proof. To the contrary, we assume that F has a cut edge xy with $x \in X$ and $y \in Y$. Let F' be a component of $F - xy$ with $x \in V(F')$. Then F' is a bipartite graph with $d_{F'}(z) = k$ for all $z \in V(F') \setminus \{x\}$ and $d_{F'}(x) = k-1$. If X' and Y' are the partite sets of F' with $X' \subseteq X$ and $Y' \subseteq Y$, then $k|X'| - 1 = k|Y'|$, a contradiction since $k \ge 2$. ∎

We now prove the following Lemma 11.

Lemma 11. *Let $k \ge 2$ and $S = (a_1, a_2, \ldots, a_n;\ b_1, b_2, \ldots, b_n)$ be an $(k^n; k^n)$-factorable, but not connected $(k^n; k^n)$-factorable, bigraphic pair and let G be a realization of S with partite sets X and Y such that G contains an $(k^n; k^n)$-factor having the minimum possible number p of components, $F_1, \ldots, F_p$. Then either $F_i \to F_j$ or $F_j \to F_i$ for any two components F_i and F_j.*

Proof. By Lemma 10, F_i is 2-edge-connected for each i. Without loss of generality, we consider the components F_1 and F_2. For $i = 1, 2$, we let F_i have partite sets X_i and Y_i so that $X_i \subseteq X$ and $Y_i \subseteq Y$. For $x \in X_1$, we denote by $A(x, F_1)$ (respectively, $B(x, F_1)$) the set of all vertices of F_1 at even (respectively, odd) distance in F_1 from x. Clearly, $A(x, F_1) = X_1$ and $B(x, F_1) = Y_1$. Let $xy \in E(F_1)$ and $uv \in E(F_2)$ with $x \in X_1$, $u \in X_2$, $y \in Y_1$ and $v \in Y_2$. If $xv, yu \in E(G)$ or

$xv, yu \notin E(G)$, then F_1 and F_2 can be combined into a single component by a simple interchange of edges. So we may assume that either $xv \in E(G), yu \notin E(G)$ or $xv \notin E(G), yu \in E(G)$. By the symmetry, we let $xv \in E(G), yu \notin E(G)$. If y' is any vertex adjacent to x in F_1 and x' is any vertex adjacent to y' in F_1, then $y'u \notin E(G)$ and $x'v \in E(G)$. Proceeding further, we get that every vertex of $A(x, F_1)$ is adjacent to v in G and every vertex of $B(x, F_1)$ is not adjacent to u in G. If v' is any vertex adjacent to u in F_2 and u' is any vertex adjacent to v' in F_2, then by the same argument, every vertex of $A(x, F_1)$ is adjacent to v' in G and every vertex of $B(x, F_1)$ is not adjacent to u' in G. Proceeding further, we finally get that every vertex of $A(x, F_1)$ is adjacent to every vertex of Y_2 in G and every vertex of $B(x, F_1)$ is not adjacent to every vertex of X_2 in G. In other words, $F_1 \to F_2$. The proof is completed. ■

Lemma 12 (Corollary 10.2 of [1]). *A tournament contains a vertex from which every other vertex is reachable by a directed path of length at most two.*

***Proof of Theorem* 6.** Let G be any realization of $(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n)$ with partite sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ such that $a_i = d_G(x_i)$ and $b_i = d_G(y_i)$ for $1 \le i \le n$. Let $A = \{x_1, \ldots, x_s\}$ and $B = \{y_{n-s+1}, \ldots, y_n\}$. Then we can see that $\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} d_G(x_i) \le |A| \times |\{y_1, y_2, \ldots, y_n\} \setminus B| + \sum_{i=n-s+1}^{n} d_G(y_i) = s(n-s) + \sum_{i=n-s+1}^{n} b_i$. If $\sum_{i=1}^{s} a_i = s(n-s) + \sum_{i=n-s+1}^{n} b_i$, then every edge with one end vertex in B has the other end vertex in A. It follows from $|A| = |B| < n$ that G does not contain a connected $(k^n; k^n)$-factor. This proves the necessity.

To prove the sufficiency, let $(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n)$ be $(k^n; k^n)$-factorable and $\sum_{i=1}^{s} a_i < s(n-s) + \sum_{i=n-s+1}^{n} b_i$ for all s with $s < n$. Let G be a realization of $(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n)$ with partite sets X and Y such that G contains an $(k^n; k^n)$-factor having the minimum number of components. Let $F_1, \ldots, F_p$ be the components in this $(k^n; k^n)$-factor of G. By Lemma 10, F_i is 2-edge-connected for each i. Assume $p \ge 2$. Then by Lemma 11, either $F_i \to F_j$ or $F_j \to F_i$ for any two components F_i and F_j. Let F_i have partite sets X_i and Y_i with $X_i \subseteq X$ and $Y_i \subseteq Y$ for each i. Construct a directed graph D with $F_1, F_2, \ldots, F_p$ as its vertices, an arc going from F_i to F_j if $F_i \to F_j$ in G. Then D is a tournament. By Lemma 12, D contains a vertex from which every other vertex is reachable by a directed path of length at most two. Thus either there is a directed 3-cycle in D or there is a F_i such that $F_i \to F_j$ for all j with $j \ne i$. Without loss of generality, if $F_1 \to F_2 \to F_3 \to F_1$, let $x_i y_i \in E(F_i)$ with $x_i \in X_i$ and $y_i \in Y_i$ for $i = 1, 2, 3$, then $x_1y_2, x_2y_3, x_3y_1 \in E(G)$. Thus the components F_1, F_2 and F_3 can be combined into a single component by a simple interchange of edges, a contradiction to the definition of p. If $F_1 \to F_i$ for $i = 2, \ldots, p$, then $F_1 \to G - V(F_1)$, where $G - V(F_1)$ has partite sets $X \setminus X_1$ and $Y \setminus Y_1$. Denote $s = |X_1| = |Y_1|$. Then $s < n$, and we can see that $\sum_{i=1}^{s} a_i \ge \sum_{x \in X_1} d_G(x) =$

$|X_1| \times |Y \setminus Y_1| + \sum_{y \in Y_1} d_G(y) \geq s(n-s) + \sum_{i=n-s+1}^{n} b_i$, a contradiction. Therefore, $p = 1$. In other words, G contains a connected $(k^n; k^n)$-factor. The proof is completed. ■

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