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ROMAN {2}-BONDAGE NUMBER OF A GRAPH

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Abstract

For a given graph $G=(V,E)$, a Roman $\{2\}$-dominating function $f : V(G) \to \{0,1,2\}$ has the property that for every vertex u with $f(u) = 0$, either u is adjacent to a vertex assigned 2 under f, or is adjacent to at least two vertices assigned 1 under f. The Roman $\{2\}$-domination number of G, $\gamma_{\{R2\}}(G)$, is the minimum of $\sum_{u\in V(G)} f(u)$ over all such functions. In this paper, we initiate the study of the problem of finding Roman $\{2\}$-bondage number of G. The Roman $\{2\}$-bondage number of G, $b_{\{R2\}}$, is defined as the cardinality of a smallest edge set $E' \subseteq E$ for which $\gamma_{\{R2\}}(G - E') > \gamma_{\{R2\}}(G)$. We first demonstrate complexity status of the problem by proving that the problem is NP-Hard. Then, we derive useful parametric as well as fixed upper bounds on the Roman $\{2\}$-bondage number of G. Specifically, it is known that the Roman bondage number of every planar graph does not exceed 15 (see [S. Akbari, M. Khatirinejad and S. Qajar, *A note on the Roman bondage number of planar graphs*, Graphs Combin. 29 (2013) 327–331]). We show that same bound will be preserved while computing the Roman $\{2\}$-bondage number of such graphs. The paper is then concluded by computing exact value of the parameter for some classes of graphs.

Keywords: domination, Roman $\{2\}$-domination, Roman $\{2\}$-bondage number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Given a simple undirected graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V$, define $N_G(v)$, the *open* neighbor of v, as the set of all vertices adjacent to v. The *closed* neighbor of v, $N_G[v]$, is then defined by $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is simply defined as the cardinality of $N_G(v)$. The minimum and maximum degree of G are also denoted by δ_G and Δ_G, respectively. Recall that a *leaf* in G is a vertex of degree one and a *support* vertex is the one adjacent to a leaf.

In this work, we consider a class of labeling functions over G, namely Roman $\{2\}$-dominating functions, as recently defined and studied in [4]. Accordingly, a function $f : V \to \{0, 1, 2\}$, is called a *Roman* $\{2\}$*-dominating function* (R2DF) over G if it assigns labels to vertices in a way any vertex with label 0 is adjacent to a vertex with label 2 or at least two vertices with label 1. For an R2DF f and a label i, let $V_i(f)$ be the set of all vertices $v \in V$ with $f(v) = i$ where $i \in \{0, 1, 2\}$. If no ambiguity occurs, we simply drop the argument and write V_i. The partition (V_0, V_1, V_2) uniquely determines f and we could equivalently write $f = (V_0, V_1, V_2)$. The *weight* of an R2DF f, $w(f)$, is defined as $\sum_{v\in V} f(v)$. The minimum possible weight of an R2DF over G is called the *Roman* $\{2\}$*-dominating number* of G and is denoted by $\gamma_{\{R2\}}(G)$. A $\gamma_{\{R2\}}(G)$-function f is an R2DF over G with $w(f) = \gamma_{\{R2\}}(G)$. We define the *Roman* $\{2\}$*-bondage number* of G, $b_{\{R2\}}(G)$, as the cardinality of a smallest edge set $E' \subset E$ for which $\gamma_{\{R2\}}(G - E') > \gamma_{\{R2\}}(G)$. Note that, $b_{\{R2\}}(G)$ is only defined for a graph with $\Delta \geq 2$, as otherwise removing any set of edges will not increase the Roman $\{2\}$-dominating number. In the following, again if no ambiguity occurs, we drop argument G and simply write $\gamma_{\{R2\}}$ and $b_{\{R2\}}$ instead of $\gamma_{\{R2\}}(G)$ and $b_{\{R2\}}(G)$, respectively.

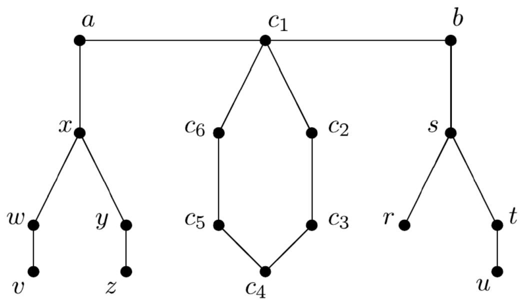

Figure 1. The graph G.

As an example, consider the graph $G = (V, E)$ illustrated in Figure 1 (originally introduced in [4]). For this graph, $\gamma_{\{R2\}}(G) = 9$ as mentioned by [4].

However, removing the edge $e = \{s, t\}$ will increase the Roman $\{2\}$-domination number to $\gamma_{\{R2\}}(G - e) = 10$ where an optimal R2DF of $G - e$ would set $V_1 = \{c_1, b, x, c_3, c_5, r, t, v, z, u\}$, $V_2 = \emptyset$, $V_0 = V \setminus V_1$. As a result, $b_{\{R2\}}(G) = 1$.

In a special case where an R2DF is restricted to support a vertex of label 0 only with an adjacent vertex of label 2, the labeling function is called a *Roman dominating function* (RDF) over G. Thereby, Roman dominating number and Roman bondage number of G are accordingly defined. Research community has been attracted to this special case much earlier than the more general case considered in this paper. For more information on the Roman dominating functions, the interested reader is referred to [3, 5, 9] and [2, 6, 10]. Several interesting works on bondage numbers could also be found in [11, 12].

Defining the Roman $\{2\}$-bondage number of a graph is simply motivated by the existing definition of Roman $\{2\}$-domination number of the graph. We are indeed interested to know how the Roman $\{2\}$-domination number increases in the given graph. To answer the question, we define and study the Roman $\{2\}$-bondage number as a similar measure, namely the Roman bondage number, was successfully studied before (see [10]). Roman $\{2\}$-bondage number is in fact a network vulnerability parameter. Finding the Roman $\{2\}$-domination number of a graph could be interpreted as a new network protection strategy as it is discussed in [4]. Accordingly, the Roman $\{2\}$-bondage number could be considered as a parameter measuring vulnerability of such defense strategy.

To the best of our knowledge, the Roman $\{2\}$-bondage number of a graph has not previously been considered or even defined in the literature. In this work, we first consider computational complexity of the problem and show that the problem is NP-Hard. Then, we prove a class of parametric as well as fixed upper bounds on the Roman $\{2\}$-bondage number. Specifically, we show that an existing bound on the Roman bondage number of planar graphs is preserved while computing the Roman $\{2\}$-bondage number of such graphs. The paper is then concluded by computing exact value of the Roman $\{2\}$-bondage number for some classes of graphs.

2. Computational Complexity

In this section, we show that finding the Roman $\{2\}$-bondage number of a graph is NP-Hard. Given a positive integer h and a graph G, the decision version of the problem is then to ask: Is $b_{\{R2\}}(G) \leq h$? We show that 3-Satisfiability (3-SAT), a well-known NP-Complete problem (see [7]), is polynomial time reducible to the decision version of finding Roman $\{2\}$-bondage number of the graph, thereby proving it NP-hardness. Henceforth, we adapt required terminology mainly taken from [8] and [2].

Let U be the set of Boolean variables $U = \{u_1, u_2, \ldots, u_n\}$ each of which can take values "true" or "false". A *term* over U, is simply a variable u_i or its negation $\bar{u}_i$. A *clause* is a disjunction of distinct terms. A clause of *length* l is the one consisting of exactly l terms. A *truth assignment* t for U is an assignment of the value "true" or "false" to each u_i. Note that, such a function t implicitly gives $\bar{u}_i$ the opposite truth value of u_i. A truth assignment t *satisfies* a clause, if, under the rules of Boolean logic, its truth values causes the clause to evaluate to "true". Now, given a set C of clauses over a given set of variables U where each clause is of length 3, the 3-SAT problem asks: Does there exists a truth assignment t satisfying all the clauses in C?

Theorem 1. *Given a graph G. The problem of finding $b_{\{R2\}}(G)$ is NP-Hard.*

Proof. Given an instance (U, C), of 3-SAT where $U = \{u_1, u_2, \ldots, u_n\}$ is the set of Boolean variables and $C = \{C_1, C_2, \ldots, C_m\}$ is the set of clauses each of length 3. In the following, we construct a graph G, in such a way that $b_{\{R2\}}(G) \leq 1$ if and only if C is satisfiable.

For each $i = 1, 2, \ldots, n$, add the sub-graph $H_i = (V_i, E_i)$ to G where

- $V_i = \{u_i, \bar{u}_i, x_i, y_i, p_i, q_i\}$,
- $E_i = \{p_i x_i, x_i u_i, u_i \bar{u}_i, \bar{u}_i y_i, y_i q_i, q_i u_i, p_i \bar{u}_i\}$.

For each $j = 1, 2, \ldots, m$, consider the clause $C_j = \{a_j^1, a_j^2, a_j^3\}$ and add a vertex c_j together with the edges $\{c_j a_j^1, c_j a_j^2, c_j a_j^3\}$ to G. Finally, add a vertex s together with the edges $\{sc_1, sc_2, \ldots, sc_m\}$ to G. An example of G is depicted in Figure 2. To prove the theorem, we state the following claims. Note that, throughout this proof, we apply i and j only to index over the variables in U and the clauses in C, respectively.

Claim 2. *We have $\gamma_{\{R2\}}(G) \geq 3n + 1$. In the case if $\gamma_{\{R2\}}(G) = 3n + 1$, then any $\gamma_{\{R2\}}$-function f on G would set*

(a) *for each i, let $f(H_i) = \sum_{v \in V_i} f(v)$, then $f(H_i) = 3$ and at most one of $f(u_i)$ and $f(\bar{u}_i)$ gets equal to one,*

(b) *for each j, $f(c_j) = 0$,*

(c) *$f(s) = 1$.*

Proof. Let f be a $\gamma_{\{R2\}}$-function of G. We need to show that for each i, $f(H_i) \geq 3$. It is only needed to consider all possible cases in which $f(u_i) + f(\bar{u}_i) \leq 2$. In case, $f(u_i) = f(\bar{u}_i) = 0$ or $f(u_i) = f(\bar{u}_i) = 1$, a simple inspection shows that $f(H_i) = 4 \geq 3$. In case, $f(u_i) \geq 1$ and $f(\bar{u}_i) = 0$, both p_i and y_i should receive non-zero label. As a result $f(H_i) \geq 3$. Similarly, in case $f(u_i) = 0$ and $f(\bar{u}_i) \geq 1$, both q_i and x_i should receive non-zero labels and we have $f(H_i) \geq 3$.

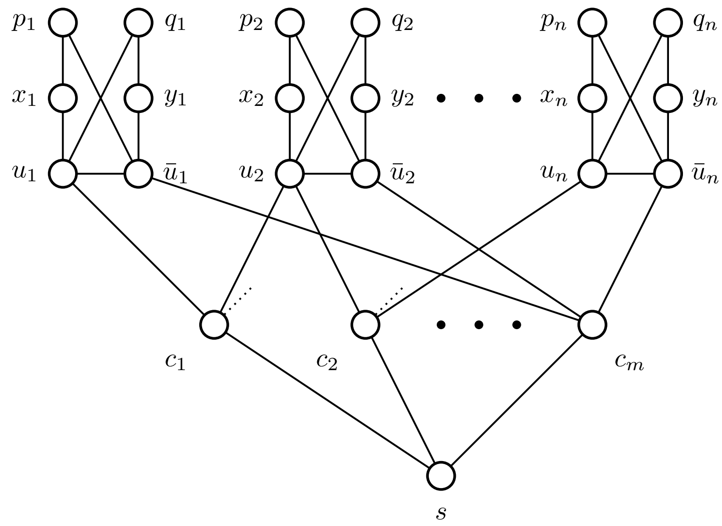

Figure 2. Graph G constructed over an instance (U, C) of 3-SAT with variable set $U = \{u_1, u_2, \dots, u_n\}$ and clause set $C = \{C_1, C_2, \dots, C_m\}$.

In the case $\gamma_{\{R2\}}(G) = 3n+1$, we get $f(H_i) = 3$ for each i, as otherwise $f(G) = \sum_i f(H_i) + f(N[s]) \geq (3n+1)+1$. This in turn implies that for each i, only $f(u_i) = 1$ or $f(\bar{u}_i) = 1$ as otherwise $f(H_i) = 4$. Other than that, any R2DF of G would set $f(N[s]) \geq 1$. The only possible way to get $f(N[s]) = 1$ is to set $f(s) = 1$ and $f(c_j) = 0$ for each j. □

Claim 3. $\gamma_{\{R2\}}(G) = 3n+1$ *if and only if* C *is satisfiable.*

Proof. First, let us assume that $\gamma_{\{R2\}}(G) = 3n+1$. By Claim 2, there exists a $\gamma_{\{R2\}}$-function f of G where, for each i, at most one of $f(u_i)$ and $f(\bar{u}_i)$ gets equal to one.

Define the truth assignment

$$t(u_i) = \begin{cases} \textit{true} & \text{if } f(u_i) = 1 \text{ and } f(\bar{u}_i) = 0, \\ \textit{false} & \text{if } f(u_i) = 0 \text{ and } f(\bar{u}_i) = 1, \end{cases} \qquad i = 1, 2, \dots, n,$$

since $f(s) = 1$ and for any j, $f(c_j) = 0$, there should be an index i where c_i is adjacent to u_i if $f(u_i) = 1$, or where c_j is adjacent to $\bar{u}_i$ if $f(\bar{u}_i) = 1$. In both cases, as the term itself belongs to C_j, the clause is satisfiable.

Now, let us assume that C is satisfiable. Then, it is not hard to build an R2DF f of G with $f(G) = 3n+1$. To that end, let t be a truth assignment that satisfies C. Define f as follows,

- if $t(u_i) = \textit{true}$, set $f(u_i) = f(y_i) = f(p_i) = 1$, $f(\bar{u}_i) = f(x_i) = f(q_i) = 0$,

- if $t(u_i) = false$, set $f(u_i) = f(y_i) = f(p_i) = 0$, $f(\bar{u}_i) = f(x_i) = f(q_i) = 1$,

also set $f(s) = 1$ and $f(c_j) = 0$ for each j. Clearly $f(G) = 3n + 1$. Since t is a satisfying truth assignment for C, for each $j = 1, \ldots, m$, at least one of the literals in C_j is true under the assignment t. By the construction of G it follows that the corresponding vertex c_j is adjacent to at least one vertex, without loss of generality, say u_i, with $f(u_i) = 1$. As c_j is also adjacent to s, f is an R2DF function of G. □

Claim 4. *For any edge $e \in E(G)$, we have $\gamma_{\{R2\}}(G - e) \leq 3n + 2$.*

Proof. Let f be a $\gamma_{\{R2\}}$-function of G. Even after removing an edge e, it is not hard to reconstruct an R2DF of $G - e$ using f as follows.

The case where e belongs to an H_i for some i. Any such edge has exactly one endpoint with zero label. Then after removing the edge one could increase the zero label to one and recover to an R2DF of $G - e$ with weight $3n + 2$.

The case where e is incident to a c_j for some j. Here, after removing the edge one could increase the label of c_j to one and again recover to an R2DF of $G - e$ with weight $3n + 2$. □

Claim 5. $\gamma_{\{R2\}}(G) = 3n + 1$ *if and only if* $b_{\{R2\}}(G) = 1$.

Proof. First, let us assume that $\gamma_{\{R2\}}(G) = 3n + 1$ and $e = x_1 p_1$. To prove the result by contradiction, suppose that $\gamma_{\{R2\}}(G) = \gamma_{\{R2\}}(G - e)$. Then, a $\gamma_{\{R2\}}$-function, f' on $G - e$ is also a $\gamma_{\{R2\}}$-function of G and according to Claim 2, $f'(c_j) = 0$ for any j and $f'(H_1 - e) = 3$. This is indeed a contradiction as any R2DF of $H_1 - e$ has weight more than or equal to 4.

Conversely, let us assume that $b_{\{R2\}}(G) = 1$. By Claim 2, $\gamma_{\{R2\}}(G) \geq 3n+1$. Now let e be an edge whose removal will increase the Roman $\{2\}$-dominating number, i.e. $\gamma_{\{R2\}}(G) < \gamma_{\{R2\}}(G - e)$. By Claim 4, $\gamma_{\{R2\}}(G - e) \leq 3n + 2$. As a result

$$3n + 1 \leq \gamma_{\{R2\}}(G) < \gamma_{\{R2\}}(G - e) \leq 3n + 2$$

and we have $\gamma_{\{R2\}}(G) = 3n + 1$. □

Claims 3 and 5 demonstrate that for the constructed graph G, $b_{\{R2\}}(G) = 1$ if and only if there exist a truth assignment for U that satisfies all the clauses in C. This ends the proof of Theorem 1. ■

3. Upper Bounds

Theorem 6. *Given a graph G and a path xyz of length 2 in G, then*

$$b_{\{R2\}}(G) \leq \deg_G(x) + \deg_G(y) + \deg_G(z) - 2 - \left|N_G(x) \cap N_G(z)\right|. \tag{1}$$

If x and z are adjacent, then

$$b_{\{R2\}}(G) \leq \deg_G(x) + \deg_G(y) + \deg_G(z) - 3 - |N_G(x) \cap N_G(z)|. \tag{2}$$

Proof. Consider the subgraph H of G obtained by removing all edges incident to x, y and z with the exception of the edges between z and $N(x) \cap N(z)$. Let $f = \big(V_0, V_1, V_2\big)$ be a $\gamma_{\{R2\}}$-function of H. As a result x gets isolated in H and $x \in V_1$. Without loss of generality, one could assume that $y \in V_0 \cup V_1$. Now consider the following possible cases.

If $y \in V_0$, then $z \in V_2$ and one could simply decrease $f(z)$ to 1 and still have an R2DF over G, like $\big(V_0, V_1 \cup \{z\}, V_2 \setminus \{z\}\big)$ of strictly less weight. Note also that, in this case, any vertex $p \in N(x) \cap N(z)$ with $f(p) = 0$ will have neighbors x and z in G both labeled 1.

If $y \in V_1$ and $z \in V_0$, then there exists a vertex $p \in N(x) \cap N(z)$, $p \neq y$ with $f(p) \geq 1$. Now decreasing $f(x)$ to 0 will result in $\big(V_0 \cup \{x\}, V_1 \setminus \{x\}, V_2\big)$, an R2DF over G of strictly less cost.

If $y \in V_1$ and $z \in V_1$, then decreasing $f(y)$ to 0 yields $\big(V_0 \cup \{y\}, V_1 \setminus \{y\}, V_2\big)$, again an R2DF over G of strictly less cost. ∎

Note that the inequalities (1) and (2) hold with equality if $G = P_3$ and $G = C_3$, respectively. With the aim of the above theorem we could prove the following important result on planar graphs. First, we mention a simple lemma.

Lemma 7 [1]**.** *Every planar graph with minimum degree* 5 *contains an edge xy with* $\deg(x) = 5$ *and* $\deg(y) \in \{5, 6\}$.

Theorem 8. *For every planar graph G with $\Delta_G \geq 2$, $b_{\{R2\}}(G) \leq 15$.*

Recall that this bound is also attained for the Roman bondage number of planar graphs (see [1]). One could apply Theorem 6, as an essential tool, to obtain the same bound on the Roman {2}-bondage number of planar graphs. Details are as follows.

Proof. First, note that $\Delta_G \geq 2$ as otherwise the Roman {2}-bondage number could not be defined. As a result there exists a vertex u of $\deg_G(u) \geq 2$. Now, if $\Delta_G \leq 5$, one could simply pick u and two vertices say $p, q \in N_G(u), p \neq q$ and apply Theorem 6 to get $b_{\{R2\}}(G) \leq \deg_G(p) + \deg_G(u) + \deg_G(q) - 2 - \big|N_G(p) \cap N_G(q)\big| = 5 + 5 + 5 - 2 - 1 = 12$. In the following we will assume that G is a planar graph of $\Delta_G \geq 6$.

Define $V^* = \big\{v \in V(G) : \deg_G(v) \leq 5\big\}$ and let $G^* = G - V^*$. As G^* is a planar graph, we have $\delta_{G^*} \leq 5$. Let v be a vertex of G^* with degree δ_{G^*}, and consider the following cases.

Case 1. v has at least three neighbors, say v_1, v_2 and v_3, in V^*. Let A be the set of all edges incident to the three neighbors, and consider the graph obtained by deleting the edge set A from G, namely $G - A$. As v_1, v_2 and v_3 are isolated in $G - A$, any optimal R2DF function f of $G - A$ would set $f\big(v_1\big) = f\big(v_2\big) = f\big(v_3\big) = 1$. Now the labeling function defined as $g\big(v_1\big) = g\big(v_2\big) = g\big(v_3\big) = 0$, $g(v) = 2$ and $g(p) = f(p)$ for every $p \in V(G) \setminus \big\{v, v_1, v_2, v_3\big\}$, is clearly an R2DF of G of strictly less weight. Then $\gamma_{\{R2\}}(G - A) < \gamma_{\{R2\}}(G)$ and as A has at most 15 edges, $b_{\{R2\}}(G) \leq 15$.

Case 2. Let v has two neighbors $v_1, v_2 \in V^*$. Then, applying Theorem 6 on the path $v_1 v v_2$ in G would imply that $b_{\{R2\}}(G) \leq \deg_G(v_1) + \deg_G(v) + \deg_G(v_2) - 2 - |N_G(v_1) \cap N_G(v_2)| \leq 5 + 7 + 5 - 2 - 1 = 14$.

Observation 1. Before discussing the last possible case, observe that if a vertex u out of V^* with $\deg_{G^*}(u) \leq 6$ has two neighbors $u_1, u_2 \in V^*$, then $\deg_G(u) \leq 8$ and applying Theorem 6 on the path $u_1 u u_2$ in G would imply that $b_{\{R2\}}(G) \leq 15$. Therefore, in the following we may assume that any vertex u out of V^* with $\deg_{G^*}(u) \leq 6$ has at most one neighbor in V^*. Accordingly, a vertex of degree 5 in G^*, has exactly one neighbor in V^* as otherwise, it belongs to V^*. Now, we have all the necessary tools to come into the last case, below.

Case 3. Let v has at most one neighbor in V^*. In this case $\delta_{G^*} = 5$ as any vertex of degree less than or equal to 4 does belong to V^*. Now, G^* met the condition of Lemma 7, according to which there exists an edge $xy \in E(G^*)$ with $\deg_{G^*}(x) = 5$ and $\deg_{G^*}(y) \leq 6$. As we have seen in Observation 1, x has exactly one neighbor, say v_1, in V^* and so $\deg_G(x) = 6$. Similarly vertex y has at most one neighbor in V^* and so $\deg_G(y) \leq 7$. Applying Theorem 6 on the path $v_1 x y$, we have $b_{\{R2\}}(G) \leq \deg_G\big(v_1\big) + \deg_G(x) + \deg_G(y) - 2 - \big|N_G(v_1) \cap N_G(y)\big| \leq 5 + 6 + 7 - 2 - 1 = 15$. ■

It remains open to show that the bound in Theorem 8 is sharp or not as it is still in question in the case of bounding the Roman bondage of planar graphs (see [1]). Akbari *et al.* in [1] constructed infinitely many planar graphs with Roman bondage number equal to 7 thereby conjecturing that the bound is at most 7 on planar graphs. In our case, one could take the same way and state a similar conjecture as a message of the following discussion (where we use the same terminology as used in [1]).

Given a graph G of order n, let $\widehat{G}$ be the graph of order $3n$ obtained from G by attaching a vertex of C_3 to each vertex of G (as depicted in Figure 3). Let us call the attached vertex as the *center* of C_3. Then, we have the following result.

Lemma 9. *For a given graph G of order n,*

(a) $\gamma_{\{R2\}}\big(\widehat{G}\big) = 2n$,

(b) $b_{\{R2\}}\big(\widehat{G}\big) = \delta_G + 2$.

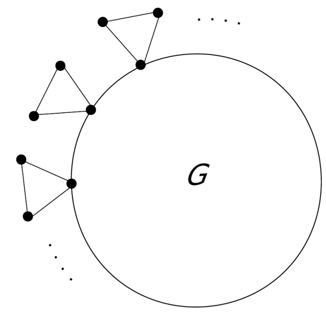

Figure 3. The graph $\widehat{G}$.

Proof. To prove part (a), first observe that giving Label 2 to every vertex of G yields an R2DF of $\widehat{G}$ of weight $2n$ and we get $\gamma_{\{R2\}}(\widehat{G}) \leq 2n$. On the other hand, any R2DF of $\widehat{G}$, say f, would set $f(C_3) \geq 2$ resulting in $\gamma_{\{R2\}}(\widehat{G}) \geq 2n$.

To prove part (b), first let u be a vertex of minimum degree δ_G. Also let E_u be the set of all edges incident to u in $\widehat{G}$. Clearly $|E_u| = \delta_G+2$. The graph $\widehat{G}-E_u$ is the union of $\widehat{G-u}$, P_2 and one isolated vertex. So by part (a), $\gamma_{\{R2\}}(\widehat{G}-E_u) = 2(n-1)+2+1 = 2n+1 > \gamma_{\{R2\}}(\widehat{G})$ resulting in $b_{\{R2\}}(\widehat{G}) \leq \delta_G+2$. To prove the other side, i.e., $b_{\{R2\}}(\widehat{G}) \geq \delta_G + 2$, we show that removing any set of cardinality δ_G+1, say E, will not change the Roman $\{2\}$-dominating number. If an attached C_3 remained unchanged in $\widehat{G} - E$ we call it *safe*, otherwise, it is called *wounded*. Now, consider the following two cases.

Case 1. The center of each wounded C_3 is adjacent (in G) to the center of at least one safe C_3 in $\widehat{G} - E$. In this case, giving label 2 to the center of every safe C_3 and label 1 to the two non-center vertices of every wounded C_3 and label 0 to any other vertex will resulted in an R2DF of $\widehat{G} - E$ of weight $2n$.

Case 2. There exists a wounded C_3 whose center, say v, is not adjacent (in G) to the center of any safe C_3 in $\widehat{G} - E$. In this case, as v is a wounded center and $|E| = \delta_G + 1$, we conclude that $|E \cap N_G(v)| \leq \delta_G$. Then, $\deg_G(v) \leq \delta_G$ as a result v is a vertex of minimum degree in G, i.e., $\deg_G(v) = \delta_G$. This in turn implies that every wounded C_3 is exactly a P_3. Now, for any wounded C_3, give label 2 to the middle vertex of the corresponding P_3. Also give label 2 to the center of any safe C_3 and label 0 to any other vertex. This defines an R2DF of $\widehat{G} - E$ of weight $2n$. ■

Corollary 10. *There exists infinitely many planar graphs with Roman* $\{2\}$-*bondage number equal to* 7.

Proof. Applying Lemma 9 to any planar graph G of $\delta_G = 5$ (like a icosahedron graph) will prove the result. ■

Conjecture 11. *The Roman $\{2\}$-bondage number of every planar graph is at most* 7.

Lemma 12. *Given a graph G with a support vertex v of* $\deg_G(v) \geq 3$. *If neighbors of v are all leaf, save one, then* $b_{\{R2\}}(G) \leq 2$.

Proof. Let $v_1, v_2, v_3, \ldots$ be neighbors of v where v_1 is the non-leaf one. In case $\deg_G(v) = 3$, one could simply apply Theorem 6 on the path $v_2 v v_3$ to obtain the result. Otherwise, when $\deg_G(v) \geq 4$, one could increase the Roman $\{2\}$-dominating number of G by removing the edge $v_2 v$. To see this, let $f = \big(V_0, V_1, V_2\big)$ be a $\gamma_{\{R2\}}$-function of $G - v_2 v$. As v_2 is an isolated vertex, $v_2 \in V_1$. Furthermore, without loss of generality, one would assume $v \in V_2$ as v is adjacent to at least two leaf vertices in $G - v_2 v$. Now, introduce $g = \big(V_0 \cup \{v_2\}, V_1 \setminus \{v_2\}, V_2\big)$ as a Roman $\{2\}$-dominating function of G. Clearly $w(g) < w(f)$ and the result follows. ∎

Theorem 13. *For any tree T, with at least three vertices, we have* $b_{\{R2\}}(T) \leq 2$.

Proof. If T contains a support vertex satisfying the condition stated in Lemma 12, the result clearly follows. Let us assume any support vertex of T either is of degree two or has at least two non-leaf neighbors. Now, consider a longest path of T, say $P = v_1 v_2 \cdots v_k$. By the assumption, we get $\deg_T(v_2) = 2$, as otherwise, P could not be a longest path. If $\deg_T(v_3) \leq 2$, one could apply Theorem 6 to the path $v_1 v_2 v_3$ and conclude that $b_{\{R2\}}(T) \leq 2$. In the case where $\deg_T(v_3) \geq 3$, removing $\big\{v_2 v_3, v_3 v_4\big\}$ is always enough to strictly increase the Roman $\{2\}$-dominating number. To see this, define $H = T - \big\{v_2 v_3, v_3 v_4\big\}$ and let $f = \big(V_0, V_1, V_2\big)$ be a $\gamma_{\{R2\}}$-function of H. We have the following cases.

Case 1. $v_3 \in V_1 \cup V_2$. It is not hard to see that the labeling function $g = \big(\big(V_0 \setminus \{v_1, v_2\}\big) \cup v_2, \big(V_1 \setminus \{v_1, v_2\}\big) \cup v_1, V_2\big)$ is an R2DF of T with $w(g) < w(f)$.

Case 2. $v_3 \in V_0$ and there exists $u \in N_H(v_3)$ with $f(u) = 2$. Here, if u is a leaf, one could simply swap labels of u and v_3. This way, f remains a $\gamma_{\{R2\}}$-function of H and the proof reduces to Case 1. Otherwise, u is a support vertex of degree two. Let the only leaf vertex adjacent to u be denoted as w. Then, one could set $f(u) = 0$ and $f(w) = f(v_3) = 1$, thereby reducing to case 1.

Case 3. $v_3 \in V_0$ and there exists $u_1, u_2 \in N_H(v_3)$ with $f(u_1) = f(u_2) = 1$. Here, if both u_1 and u_2 are leaf vertices, one could set $f(u_1) = f(u_2) = 0$ and $f(v_3) = 2$, and reduce to Case 1. Otherwise, let us suppose u_1 is a support vertex of degree two and u_3 is the leaf vertex adjacent to it. It follows that $f(u_3) = 1$. Now, swapping labels of v_3 and u_1, the labeling remains a $\gamma_{\{R2\}}$-function of H and we reduce to Case 1. ∎

Theorem 14. *For any unicyclic graph G, we have* $b_{\{R2\}}(G) \leq 3$.

Proof. Removing an edge on the unique cycle of G, produces a tree. Thereby, Theorem 13 shows the result. ■

Lemma 15. *For a connected graph G of order $n \geq 3$, we have $\gamma_{\{R2\}}(G) = 2$ if and only if $\Delta_G = n-1$ or G has two non-adjacent vertices of degree $n-2$.*

Proof. Let us first assume that $\gamma_{\{R2\}}(G) = 2$. Knowing that any $\gamma_{\{R2\}}$-function $f = (V_0, V_1, V_2)$ would set $f(V_1) = 2, f(V_2) = 0$ or $f(V_1) = 0, f(V_2) = 2$. In either case, the result follows. The other side of the result is also immediate. ■

Theorem 16. *For a graph G of order $n \geq 3$ with exactly k vertices of degree $n-1$ and l non-adjacent pair vertices of degree $n-2$ where $n > k + 2l$, we have*

$$b_{\{R2\}}(G) \leq \begin{cases} \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor, & \text{both } k \text{ and } \left\lfloor \frac{k}{2} \right\rfloor + l \text{ are even,} \\ \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

Proof. Lemma 15 simply shows that $\gamma_{\{R2\}}(G) = 2$. In the following, we try to find least number of edges whose removal leaves neither a vertex of degree $n-1$ nor a non-adjacent vertex pair of degree $n-2$. Put all vertices of degree $n-1$ in a set, say K and all of the non-adjacent vertex pairs of degree $n-2$ in a set, say L. Observe that any two different pairs $\{u, v\}, \{u', v'\} \in L$ are disjoint i.e. $\{u, v\} \cap \{u', v'\} = \emptyset$.

Let us assume both k and $\lfloor \frac{k}{2} \rfloor + l$ are even. It is not hard to see that, removing any $\lfloor \frac{k}{2} \rfloor$ independent edges whose endpoints are in K leaves no vertex of degree $n-1$ but increases the number of non-adjacent vertex pairs of degree $n-2$ to $\lfloor \frac{k}{2} \rfloor + l$. Now, for a given pair $\{u, v\}, \{u', v'\} \in T$, remove an edge with one endpoint in $\{u, v\}$ and the other endpoint in $\{u', v'\}$. This way, both pairs $\{u, v\}$ and $\{u', v'\}$ leave L out. As $\lfloor \frac{k}{2} \rfloor + l$ is even, after removing $\left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor$ number of such edges, T gets empty and Lemma 15 gives $b_{\{R2\}}(G) \leq \lfloor \frac{k}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor$.

Otherwise, three cases could occur. In case k is even but $\lfloor \frac{k}{2} \rfloor + l$ is odd, the same $\lfloor \frac{k}{2} \rfloor$ number of independent edges with both endpoints in K could be removed to reduce K to empty set. However this, in turn, increases size of L to $\lfloor \frac{k}{2} \rfloor + l$ which is odd. Here removing $\left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor$ number of the edges between non-adjacent pairs of L, as described above, reduces size of L to one where L still contains a non-adjacent vertex pair $\{u, v\}$ both of degree $n-1$. Here one extra edge with an endpoint in $\{u, v\}$ should be removed to make L empty. Thereby proving $b_{\{R2\}}(G) \leq \lfloor \frac{k}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor + 1$. Other two cases occur when k

is an odd number. Here again, the same $\lfloor \frac{k}{2} \rfloor$ number of independent edges with both endpoints in K could be removed to reduce $|K|$ to one and at the same time increases $|L|$ to $\lfloor \frac{k}{2} \rfloor + l$. In case, $\lfloor \frac{k}{2} \rfloor + l$ is even, one would remove the same $\left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor$ number of the edges between non-adjacent pairs of L, as described above, reducing $|L|$ to zero and then any edge incident to the last vertex in K. Thereby proving $b_{\{R2\}}(G) \leq \lfloor \frac{k}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor + 1$. In case $\lfloor \frac{k}{2} \rfloor + l$ is even, after removing $\lfloor \frac{k}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor$ edges as above, both $|K|$ and $|L|$ reduce to one. Here an extra edge with one endpoint in K and the other endpoint in the only remained non-adjacent pair of L could be removed to reduce both $|K|$ and $|L|$ to zero. Thereby proving $b_{\{R2\}}(G) \leq \lfloor \frac{k}{2} \rfloor + \left\lfloor \frac{\lfloor \frac{k}{2} \rfloor + l}{2} \right\rfloor + 1$. ■

4. Exact Bounds

Lemma 17 [4]**.** *For the classes of paths P_n, and cycles C_n, we have*

$$\gamma_{\{R2\}}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil, \ \gamma_{\{R2\}}(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 18. *For $n \geq 3$, we have $b_{\{R2\}}(P_n) = 1$.*

Proof. Let us denote P_n as $v_1v_2v_3 \cdots v_n$. Removing v_2v_3 leaves two components v_1v_2 and P_{n-2}. Now, any $\gamma_{\{R2\}}$-function f on $P_n - v_1v_2$ would set $f(\{v_1, v_2\}) = 2$. Thereby, Lemma 17 implies that

$$\gamma_{\{R2\}}\left(P_n - v_1v_2\right) = 2 + \left\lceil \frac{n-1}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil + 1 = \gamma_{\{R2\}}(P_n) + 1$$

■

Theorem 19. *For $n \geq 3$, we have*

$$b_{\{R2\}}(C_n) = \begin{cases} 1 & n \text{ is even,} \\ 2 & n \text{ is odd.} \end{cases}$$

Proof. Let us denote C_n as $v_1v_2 \cdots v_nv_1$. Removing v_nv_1 leaves a path P_n. For an even n, Lemma 17 shows that

$$\gamma_{\{R2\}}(C_n - v_nv_1) = \left\lceil \frac{n+1}{2} \right\rceil > \left\lceil \frac{n}{2} \right\rceil = \gamma_{\{R2\}}(C_n).$$

However, for an odd n, $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n}{2} \rceil$ and removing an edge is not enough. Now let us remove the extra edge v_1v_2 and makes v_1 isolated. Lemma 17 gives

$$\gamma_{\{R2\}}(C_n - v_nv_1 - v_1v_2) = 1 + \left\lceil \frac{n}{2} \right\rceil > \left\lceil \frac{n}{2} \right\rceil = \gamma_{\{R2\}}(C_n).$$

■

Theorem 20. *For a wheel W_n, with $n \geq 4$, we have*

$$b_{\{R2\}}(W_n) = \begin{cases} 2 & n = 4, 5, \\ 1 & n \geq 6. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be vertices of W_n. We mention that $\gamma_{\{R2\}}(W_n) = 2$. Let also assume that v_1 is a vertex of maximum degree. For $n = 4, 5$, simple inspections prove the result. For $n \geq 6$, remove v_1v_2 and let f be a Roman $\{2\}$-dominating function of $W_n - v_1v_2$. The following two cases could occur.

In case $f(v_2) \geq 1$, there exists at least two vertices of W_n, not incident to v_2 in $W_n - v_1v_2$, implying $f\big(W_n - v_1v_2\big) \geq 3$. In case $f\big(v_2\big) = 0$, there exists a vertex, say v, other than v_1, with $f(v) \geq 1$ and we reduce to previous case. ■

Theorem 21. *For a complete bipartite graph $K_{p,q}$ with $p \geq 4$, $q \geq 5$, we have*

$$b_{\{R2\}}\big(K_{p,q}\big) = \min\{p, q\}.$$

Proof. Without loss of generality suppose $\min\{p, q\} = p \leq q$. Label the vertices in the first part as $i_1, \ldots, i_p$, and the vertices in the second part as $j_1, \ldots, j_q$. This way, the edge set of $K_{p,q}$ would be $E = \big\{i_sj_t : s = 1, \ldots, p, t = 1, \ldots, q\big\}$. A simple inspection shows that $\gamma_{\{R2\}}\big(K_{p,q}\big) = 4$. Removing all the edges incident to j_1 makes the vertex isolated and increases the Roman $\{2\}$-domination number to 5. Then $b_{\{R2\}}\big(K_{p,q}\big) \leq p$. Other than that, removing a subset of edges, $|S|$, with $|S| \leq p - 1$, leaves two (unaffected) vertices, say i_s and j_t with deg $\big(i_s\big) = q$ and deg $\big(j_t\big) = p$. Now simply giving label 2 to these vertices will provide a Roman $\{2\}$-dominating function of weight 4 and then $b_{\{R2\}}\big(K_{p,q}\big) \geq p$. ■

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