Discussiones Mathematicae
Graph Theory 40 (2020) 771–785
doi:10.7151/dmgt.2142

EXTREMAL GRAPHS FOR A BOUND ON THE ROMAN DOMINATION NUMBER

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Abstract

A Roman dominating function on a graph $G = (V, E)$ is a function $f{:}V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u\in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G, denoted by $\gamma_R(G)$. In 2009, Chambers, Kinnersley, Prince and West proved that for any graph G with n vertices and maximum degree Δ, $\gamma_R(G) \leq n + 1 - \Delta$. In this paper, we give a characterization of graphs attaining the previous bound including trees, regular and semiregular graphs. Moreover, we prove that the problem of deciding whether $\gamma_R(G) = n + 1 - \Delta$ is *co*-$\mathcal{NP}$-complete. Finally, we provide a characterization of extremal graphs of a Nordhaus–Gaddum bound for $\gamma_R(G) + \gamma_R(\overline{G})$, where $\overline{G}$ is the complement graph of G.

Keywords: Roman domination, Roman domination number, Nordhaus-Gaddum inequalities.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = \big(V(G), E(G)\big)$ be a simple graph of order $|V(G)| = |V| = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\big\{u \in V(G) : uv \in E(G)\big\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v of G is $\deg_G(v) = \big|N(v)\big|$. By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum degree* and the *minimum degree* of the graph G, respectively. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A set $S \subset V$ is *independent* if no two vertices in S are adjacent. For any $S \subseteq V$, we denote the subgraph of G induced by S as $G[S]$.

We write K_n for the *complete graph* of order n, P_n for the *path* of order n, C_n for the *cycle* of order n, and $K_{1,n}$ with $n \geq 1$, for the *star* of order $n+1$. A *tree* is a connected graph with no cycles. A graph G of order at least two is called *regular* if its vertices have the same degree and *semiregular* if $\Delta(G) - \delta(G) = 1$. For simplicity, a regular graph each of whose vertices has degree r is called *r-regular*.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V \setminus S$ has a neighbor in S, that is, $\big|N[v] \cap S\big| \geq 1$ for all $v \in V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The *weight* of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. A Roman dominating function of minimum weight is called a *γ_R-function*. In the whole paper, the function f will be denoted $f = \big(V_0, V_1, V_2\big)$, where $V_i = \big\{v \in V : f(v) = i\big\}$ for $i \in \{0, 1, 2\}$. The Roman domination number was introduced by Cockayne *et al.* [3] in 2004 and was inspired by the works of ReVelle and Rosing [7] and Stewart [8]. Since its introduction, more than a hundred papers have been published on various aspects of Roman domination in graphs (for examples, see list of references).

The following upper bound on the Roman domination number provided by Chambers *et al.* [2] will be the focus of our work.

Proposition 1 (Chambers *et al.* [2])**.** *If G is a graph of order n with maximum degree $\Delta(G)$, then $\gamma_R(G) + \Delta(G) \leq n + 1$.*

In this paper, we examine classes of extremal graphs for the inequality $\gamma_R(G) + \Delta(G) \leq n + 1$. We give a characterization of trees, regular and semiregular graphs that achieve equality in the inequality. Moreover, we prove that the problem of deciding whether $\gamma_R(G) = n + 1 - \Delta(G)$ is *co-$\mathcal{NP}$*-complete. Finally, we provide a characterization of extremal graphs of a Nordhaus–Gaddum bound for $\gamma_R(G) + \gamma_R\big(\overline{G}\big)$, where $\overline{G}$ is the complement graph of G. Such a bound will subsequently be substantially improved for graphs G of order $n \geq 160$.

2. Preliminary Results

We begin by recalling some important results that will be useful in our investigations.

Proposition 2 (Cockayne *et al.* [3]). *For every graph G, $\gamma_R(G) \leq 2\gamma(G)$.*

Proposition 3 (Cockayne *et al.* [3]). *If G is a path P_n or a cycle C_n, then $\gamma_R(G) = \lceil \frac{2n}{3} \rceil$.*

Theorem 4 (Chambers *et al.* [2]). *If G is a connected graph of order $n \geq 3$, then $\gamma_R(G) \leq \frac{4n}{5}$, with equality if and only if G is C_5 or is obtained from $\frac{n}{5}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{5}P_5$.*

Restricted to graphs with minimum degree at least two, Chambers *et al.* [2] improved the upper bound of Theorem 4. Consider a cycle C_8 whose vertices are labeled in order $x_1, x_2, \ldots, x_8, x_1$. Let F_1 be the graph obtained from the cycle C_8 by adding the edge x_1x_5, and F_2 the graph obtained from the cycle C_8 by adding the edges x_1x_5 and x_2x_6. Let $\mathcal{B} = \{C_4, C_5, C_8, F_1, F_2\}$.

Theorem 5 (Chambers *et al.* [2]). *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \mathcal{B}$, then $\gamma_R(G) \leq \frac{8n}{11}$.*

Consider a cycle C_7 whose vertices are labeled in order $x_1, x_2, \ldots, x_7, x_1$. Let M_1, M_2, M_3 and M_4 be four graphs obtained from the cycle C_7 as follows: M_1 is obtained by adding the edge x_1x_4; M_2 is obtained by adding the edges x_1x_4 and x_2x_5; M_3 is obtained by adding the edges x_1x_4, x_2x_5 and x_1x_5; M_4 is obtained by adding edges x_3x_6, x_3x_7. Let M_5 be the graph of order 7 obtained from two disjoint cycles C_4 sharing the same vertex. Let $\mathcal{A} = \{C_4, C_7, M_1, M_2, M_3, M_4, M_5\}$.

Theorem 6 (McCuaig and Shepherd [5]). *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \mathcal{A}$, then $\gamma(G) \leq \frac{2n}{5}$.*

Observation 7. *Let G be a graph of minimum degree $\delta(G) \geq 2$. If $\gamma_R(G) + \Delta(G) = n + 1$, then G is connected.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G. Without loss of generality, let $\Delta(G) = \Delta(G_k)$. Clearly $n - \Delta(G) + 1 = \gamma_R(G) = \sum_{i=1}^{k} \gamma_R(G_i)$, and each G_i satisfies $\gamma_R(G_i) \leq n(G_i) - \Delta(G_i) + 1$ (by Proposition 1). Using the fact that $\Delta(G_i) \geq \delta(G) \geq 2$ for each i, it follows that

$$\begin{aligned} n - \Delta(G) + 1 = \gamma_R(G) &= \textstyle\sum_{i=1}^{k} \gamma_R(G_i) \leq \sum_{i=1}^{k} \left(n(G_i) - \Delta(G_i) + 1\right) \\ &\leq n - \Delta(G) - 2(k-1) + k = n - \Delta(G) - k + 2, \end{aligned}$$

and thus $k = 1$, that is G is connected. ∎

In the next, we give a necessary condition for connected graphs G with $\gamma_R(G) + \Delta(G) = n + 1$. For any vertex vertex $v \in V(G)$, we write $\overline{N}(v) = V(G) - N[v]$. We also denote by $m\big(N(v), \overline{N}(v)\big)$ the number of edges having an endvertex in $N(v)$ and the other endvertex in $\overline{N}(v)$.

Proposition 8. *Let G be a graph of order n with maximum degree $\Delta(G)$. If $\gamma_R(G) + \Delta(G) = n + 1$, then for every vertex v of maximum degree we have*

(1) *Every vertex of $N(v)$ is adjacent to at most two vertices in $\overline{N}(v)$.*

(2) *Each component of $G\left[\overline{N}(v)\right]$ is either K_1 or K_2.*

Proof. Let G be a graph with $\gamma_R(G) + \Delta(G) = n + 1$ and let v a vertex of maximum degree. Consider an RDF f that assigns the value 2 to v, 0 to every neighbor of v and 1 to the remaining vertices. Clearly $w(f) = n + 1 - \Delta(G)$ and thus f is a $\gamma_R(G)$-function.

Now suppose to the contrary that v has a neighbor w having at least three neighbors in $\overline{N}(v)$. Then reassigning w a 2 instead of 0 and each vertex of $N(w) \cap \overline{N}(v)$ a 0 instead of 1 produces an RDF with smaller weight than $\gamma_R(G)$, a contradiction. Hence (1) follows. Moreover, if a vertex $x \in \overline{N}(v)$ has two neighbors in $\overline{N}(v)$, say y and z, then reassigning x a 2 instead of 1, and reassigning y and z a 0 instead of 1 produces an RDF with smaller weight than $\gamma_R(G)$, a contradiction. Hence (2) follows. ■

We note that the converse of Proposition 8 is not true as can be seen by the tree T obtained from a star $K_{1,3}$ by subdividing each edge of the star twice. Then $\gamma_R(T) = 7 < n + 1 - \Delta(T) = 8$.

In the next we show that if a graph G has a vertex with maximum degree satisfying items (1) and (2) of Proposition 8, then G has a Roman domination number bounded below by $n - 1 - \Delta(G)$.

Proposition 9. *Let G be a connected graph of order n and let v be a vertex of degree $\Delta(G)$ such that every vertex in $N(v)$ is adjacent to at most two vertices in $\overline{N}(v)$ and each component of $G\left[\overline{N}(v)\right]$ is either K_1 or K_2. Then $\gamma_R(G) + \Delta(G) \geq n - 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. Let $A = N(v) \cap V_2$ and let B be the set of vertices of $\overline{N}(v) \cap V_0$ that have no neighbor in $\overline{N}(v) \cap V_2$. Let $C = \overline{N}(v) - B$. Clearly, each vertex of B has a neighbor in A, and $f(C) = |C|$. Since each vertex of A has at most two neighbors in $\overline{N}(v)$, $|B| \leq 2|A|$. Therefore, $w(f) \geq |C| + 2|A| \geq |C| + |B| = \left|\overline{N}(v)\right| = n - 1 - \Delta(G)$. ■

3. Graphs G of Order n Satisfying $\gamma_R(G) + \Delta = n + 1$

In this section we provide a characterization of some classes of graphs G with $\gamma_R(G) + \Delta(G) = n + 1$, including regular graphs, trees and semiregular graphs. Using Proposition 3, one can easily check that the only paths and cycles attaining equality in the upper bound of Proposition 1 are $P_2, P_3, P_4, P_5, C_3, C_4$ and C_5. Trivially, graphs G with $\Delta(G) = 1$ satisfy $\gamma_R(G) + \Delta(G) = n + 1$. For graphs G with $\Delta(G) = 2$ we have the following straightforward observation.

Observation 10. *If G is a graph of order n with maximum degree $\Delta(G) = 2$, then $\gamma_R(G) + \Delta(G) = n + 1$ if and only if $G = pK_1 \cup qK_2 \cup H$, where $H \in \{P_3, P_4, P_5, C_3, C_4, C_5\}$ and $p + 2q + |V(H)| = n$.*

Moreover, the following observation shows that equality is attained in the upper bound of Proposition 1 for every graph G of order n with $\Delta(G) \geq n - 3$. We omit the details of the proof.

Observation 11. *Let G be a graph of order n and maximum degree $\Delta(G)$. If $\Delta(G) \geq n - 3$, then $\gamma_R(G) + \Delta(G) = n + 1$.*

We now consider regular graphs.

Theorem 12. *Let G be a Δ-regular graph of order n and degree $\Delta(G) \geq 1$. Then $\gamma_R(G) + \Delta(G) = n + 1$ if and only if $\Delta(G) \in \{1, n-3, n-2, n-1\}$.*

Proof. Let G be a regular graph with $\gamma_R(G) + \Delta(G) = n + 1$. If $\Delta(G) = 1$, then we are done. Hence assume that $\Delta(G) \geq 2$. Note that G is connected (by Observation 7). Now, let v be a vertex of G. According to Proposition 8 and the fact that G is regular, we have $2\,|N(v)| \geq m\big(N(v), \overline{N}(v)\big) \geq (\Delta(G) - 1)\,\big|\overline{N}(v)\big|$, which provides $n \leq \Delta(G) + 3 + \frac{2}{\Delta(G)-1}$.

If $\Delta(G) = 2$, then $n \leq 7$ and by Observation 10, we obtain $G \in \{C_3, C_4, C_5\}$, that is $\Delta(G) \in \{n-1, n-2, n-3\}$. If $\Delta(G) = 3$, then $n \leq 7$, and since cubic graphs have an even order we deduce that $n \in \{4, 6\}$, that is G is either K_4, $K_{3,3}$ or the complement of C_6. Clearly, all these cubic graphs have $\Delta(G) \in \{n-1, n-2, n-3\}$. Finally, if $\Delta(G) \geq 4$, then $n \leq \Delta(G) + 3 + \frac{2}{\Delta(G)-1}$ leads to $\Delta(G) \geq n - 3$.

The converse follows from Observations 10 and 11. ∎

In the aim to characterize all trees T of order n for which $\gamma_R(T) + \Delta(T) = n + 1$, we give some additional definitions and notations.

Let T be a tree with a unique vertex of maximum degree $\Delta \geq 3$, say x, where each leaf of T is at distance at most three from x and such that all components in $T - x$ are paths of order at most 5 (for example, see Figure 1). Let H_i be a component of $T - x$ of order i, where $i \in \{1, 2, \ldots, 5\}$. Clearly, since T is a tree,

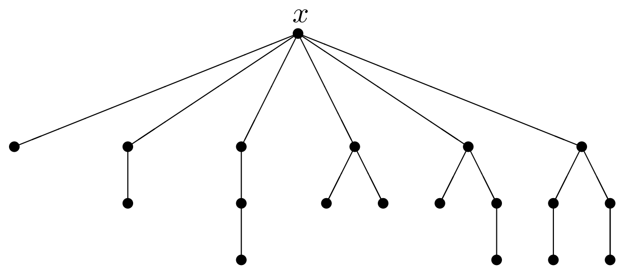

Figure 1. Example of a tree T, where each component of $T - x$ is a path of order at most 5.

H_i contains exactly one vertex of $N(x)$. Let n_1 be the number of components H_1, n_2 the number of components H_2, n_3 the number of components H_3 having a leaf belonging to $N(x)$, n_4 the number of components H_3 whose center vertices belong to $N(x)$, n_5 the number of components H_4 having a support vertex belonging to $N(x)$ and n_6 be the number of components H_5 whose center vertices belong to $N(x)$. Note that $\sum_{i=1}^{6} n_i = \Delta(T)$.

Let $\mathcal{T}$ be the family of trees T with only two adjacent vertices of maximum degree three such that every leaf of T is at distance at most two from a vertex of maximum degree. Note that any tree $T \in \mathcal{T}$ has order $n \in \{6, 7, 8, 9, 10\}$.

Now, we are ready to characterize the class of trees T for which $\gamma_R(T) = n - \Delta + 1$.

Theorem 13. *Let T be a nontrivial tree of order n and maximum degree $\Delta(T)$. Then $\gamma_R(T) + \Delta(T) = n + 1$ if and only if $T \in \left\{P_2, P_3, P_4, P_5\right\}$ or T is one of the trees defined above such that either $n_4 + n_5 + n_6 = 0$ and ($n_1 \geq 1$ or $n_2 \geq 2$) or $n_4 + n_5 + n_6 \neq 0$ and $n_1 + n_2 \geq 2$.*

Proof. Let T be a nontrivial tree with $\gamma_R(T) = n - \Delta(T) + 1$. If $\Delta(T) \in \{1, 2\}$, then by Proposition 3, we have $T \in \left\{P_2, P_3, P_4, P_5\right\}$. Hence let $\Delta(T) \geq 3$. We note that if v is a vertex of degree $\Delta(T) \geq 4$, then, since T is a tree, and every vertex of $N(v)$ has at most two neighbors in $\overline{N}(v)$ (by Proposition 8), vertex v is the unique vertex of maximum degree. Suppose now that $\Delta(T) = 3$. It is easy to see that if T has two non-adjacent vertices of degree 3, then $\gamma_R(T) + \Delta(T) < n+1$. Hence T has either one vertex of degree 3 or two adjacent vertices of degree 3. Assume that T has two adjacent vertices of degree 3, say x and y. Since each of x and y satisfies conditions (1) and (2) of Proposition 8, we deduce that each leaf of T is at distance at most 2 from either x or y. Hence $T \in \mathcal{T}$.

From now on, we may assume that T has a unique vertex of degree $\Delta(T) \geq 3$, say v. By Proposition 8, each component in $T - v$ is a path of order at most 5 containing exactly one vertex of $N(v)$. Also, if $x \in N(v)$ and H_i is a nontrivial component of $T-v$ of order i, then x is either a leaf of H_i if $i \in \{2,3\}$ or a center vertex of H_i if $i \in \{3,5\}$ or a support vertex of H_i if $i = 4$. We also note that $n(T) = 5n_6 + 4n_5 + 3n_4 + 3n_3 + 2n_2 + n_1 + 1$, $\Delta(T) = \sum_{i=1}^{6} n_i$ and $\gamma_R(T) = 4n_6 + 3n_5 + 2n_4 + 2n_3 + n_2 + 2$. Consider the following two cases.

Case 1. $n_4 + n_5 + n_6 = 0$. Suppose that $n_1 = 0$ and $n_2 = 0$. Then all components in $T-v$ are H_3 having a leaf belonging to $N(v)$. Define the function f on T by assigning a 2 to every support vertex of H_3, a 1 to v and a 0 to the remaining vertices of T. It follows that $f\big(V(T)\big) = 2n_3 + 1 < \gamma_R(T)$, a contradiction. Now suppose that $n_1 = 0$ and $n_2 = 1$. Then all components in $T-v$ but one are H_3 having a leaf belonging to $N(v)$. Define the function f on T by assigning a 2 to every support vertex of H_3, a 2 to $N(v) \cap V(H_2)$ and a 0 to the remaining vertices of T. It follows that $f\big(V(T)\big) = 2n_3 + 2 < \gamma_R(T)$, a contradiction too. Hence either $n_1 \geq 1$ or $n_2 \geq 2$.

Case 2. $n_4+n_5+n_6 \neq 0$. Suppose that $n_1+n_2 \leq 1$. Define a function f on T by assigning a 0 to v and to any component H_i so that $f\big(V(H_i)\big) = \gamma_R\big(H_i\big)$ and the vertex of a H_i belonging to $N(v)$ received a 2. Note that such an assignment is possible for any component H_i with $i \in \{3,4,5\}$. It follows that $f\big(V(T)\big) = 4n_6+3n_5+2n_4+2n_3+2n_2+n_1$ and since $n_1+n_2 \leq 1$, we have $f\big(V(T)\big) < \gamma_R(T)$, a contradiction. Hence $n_1 + n_2 \geq 2$.

Conversely, if $T \in \big\{P_2, P_3, P_4, P_5\big\}$, then clearly $\gamma_R(T) + \Delta(T) = n + 1$. Suppose that $T \in \mathcal{T}$. Then T has order $n = 6 + t$, where t is the number of leaves at distance two from a vertex of degree three. One can easily check that $\gamma_R(T) = 4 + t = n + 1 - \Delta(T)$. Finally, let T be a tree with a unique vertex of degree $\Delta(T) \geq 3$, say v, such that either ($n_4+n_5+n_6 = 0$ and $n_1 \geq 1$ or $n_2 \geq 2$) or ($n_4 + n_5 + n_6 \neq 0$ and $n_1 + n_2 \geq 2$). Observe that for every $\gamma_R(T)$-function f, $f\big(V(H_i)\big) = \gamma_R\big(H_i\big)$ for every $i \in \{3,4,5\}$. Also vertex v can be assigned a 2 under f in each of the two situations that T can have. Therefore a simple calculation shows that $\gamma_R(T) = 2 + n_2 + 2n_3 + 2n_4 + 3n_5 + 4n_6 = n + 1 - \Delta(T)$. ■

Theorem 14. *Let G be a semiregular graph of order n and maximum degree $\Delta(G) = \Delta$. Then $\gamma_R(G) + \Delta = n + 1$ if and only if G fulfills one of the following:*

(a) $\Delta \geq n - 3$,

(b) $G = pK_1 \cup qK_2$ *with* $p \geq 1$, $q \geq 1$ *and* $p + 2q = n$,

(c) $G = qK_2 \cup H$ *with* $2q + |V(H)| = n$, *where* $H \in \big\{P_3, P_4, P_5, C_3, C_4, C_5\big\}$ *if* $q \neq 0$ *and* $H \in \big\{P_3, P_4, P_5\big\}$ *if* $q = 0$,

(d) G *is isomorphic to one of the nine graphs in Figure* 2.

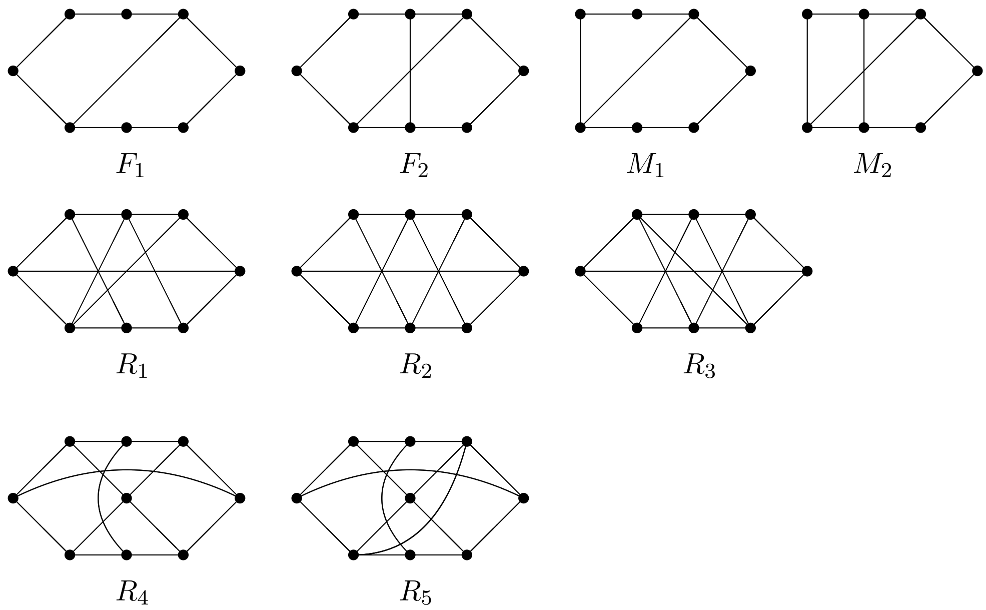

Figure 2. Connected semiregular extremal graphs of order $n \in \{7, 8, 9\}$.

Proof. Let G be a semiregular graph of order n such that $\gamma_R(G) + \Delta = n + 1$. Note that every vertex of G has degree Δ or $\Delta - 1$. If $\Delta = 1$, then clearly $G = pK_1 \cup qK_2$ with $p \geq 1$ and $q \geq 1$ (since G is semiregular). If $\Delta = 2$, then by Observation 10 and since $\delta(G) = 1$, $G = qK_2 \cup H$, where $H \in \{P_3, P_4, P_5\}$ if $q = 0$, and $H \in \{P_3, P_4, P_5, C_3, C_4, C_5\}$ if $q \neq 0$. For the next we can assume that $\Delta \geq 3$. Since $\delta(G) \geq 2$, by Observation 7, G is connected. Let v be a vertex of degree maximum. According to Proposition 8, $m(N(v), \overline{N}(v))$ satisfies $(\Delta - 2)\left|\overline{N}(v)\right| \leq m(N(v), \overline{N}(v)) \leq 2\,|N(v)|$. Therefore $(\Delta - 2)(n - \Delta - 1) \leq 2\Delta$, and thus

$$(1) \qquad \Delta + 1 \leq n \leq \Delta + 3 + \frac{4}{\Delta - 2}.$$

Clearly, for $\Delta \geq 7$ we have $\Delta \geq n - 3$ since $\frac{4}{\Delta - 2} < 1$. Assume now that $\Delta = 6$. Thus by (1) we have $n \in \{7, 8, 9, 10\}$. If $n \in \{7, 8, 9\}$, then $\Delta \geq n - 3$ and we are done. If $n = 10$, then $G\left[\overline{N}(v)\right]$ has order 3 and contains an isolated vertex whose neighbors are all in $N(v)$. Hence $m(N(v), \overline{N}(v)) \geq 2\,(\Delta - 2) + (\Delta - 1) = 13 > 2\,|N(v)|$, implying that such a graph does not exist. Therefore it remains to examine the cases $\Delta \in \{3, 4, 5\}$. Consider each case separately.

Case 1. $\Delta = 3$. By (1) we have $n \in \{4, 5, \ldots, 10\}$. If $n \in \{4, 5, 6\}$, then $\Delta \geq n - 3$. Assume that $n = 7$. Then $\gamma_R(G) = 5$, and by Proposition 2, $\gamma(G) \geq 3$.

So by Theorem 6, $\gamma(G) > \frac{2n}{5}$ and since M_3, M_4, M_5 are not semiregular, we deduce that $G \in \{M_1, M_2\}$. Assume now that $n = 8$. Hence $\gamma_R(G) = 6 > \frac{8n}{11}$. By Theorem 5, $G \in \mathcal{B}$, and clearly $G \in \{F_1, F_2\}$. The remaining cases ($n = 9$ and $n = 10$) are excluded by using Theorem 5.

Case 2. $\Delta = 4$. By (1) we have $n \in \{5, 6, 7, 8, 9\}$. If $n \in \{5, 6, 7\}$, then $\Delta \geq n - 3$. The remaining two situations are considered separately. Let $N(v) = \{a, b, c, d\}$.

Subcase 2.1. $n = 8$. Then $\gamma_R(G) = 5$. Since $G\left[\overline{N}(v)\right]$ has order 3 and contains an isolated vertex, $2\Delta = 8 \geq m\big(N(v), \overline{N}(v)\big) \geq 2\,(\Delta - 2) + (\Delta - 1) = 7$. Let $\overline{N}(v) = \{x, y, z\}$. Clearly, since $m\big(N(v), \overline{N}(v)\big) \leq 8$, $\overline{N}(v)$ is not independent. Without loss of generality, assume that $xy \in E$, and thus $xz, yz \notin E$ (by Proposition 8). Since $|N(z) \cap N(v)| \geq 3$, we may assume that $\{a, b, c\} \subseteq N(z)$. By Proposition 8, $|N(t) \cap \{x, y\}| \leq 1$ for $t \in \{a, b, c\}$. Moreover, since $\deg_G(x) \geq 3$ and $\deg_G(y) \geq 3$, we have, without loss of generality, ax, bx, cy and $dy \in E$.

If $m\big(N(v), \overline{N}(v)\big) = 7$, then $N(d) \cap \{a, b, c\} \neq \emptyset$ (since G is semiregular). If $cd \in E$, then $\big(\{v, a, b, d, y, z\}, \emptyset, \{c, x\}\big)$ is an RDF of G with weight 4, a contradiction. Thus $cd \notin E$. Up to isomorphism, assume that $da \in E$. Since $\deg_G(a) = 4$, ab and $ac \notin E$. Now clearly, if $bc \in E$, then $\big(\{v, b, d, x, y, z\}, \emptyset, \{a, c\}\big)$ is an RDF of G with weight 4, a contradiction. Hence $bc \notin E$, and we conclude that $G = R_1$.

If $m\big(N(v), \overline{N}(v)\big) = 8$, then we must have $zd \in E$. Let us examine the existence or not of the edges between vertices of $N(v)$. As G is semiregular of maximum degree 4, $G[N(v)]$ contains at most two independent edges. If $N(v)$ is independent, then $G = R_2$. Now assume that $G[N(v)]$ contains exactly one edge. If $ab \in E$, then $\big(\{v, b, c, d, x, z\}, \emptyset, \{a, y\}\big)$ is an RDF of G with weight 4. If $cd \in E$, then $\big(\{v, a, b, d, y, z\}, \emptyset, \{c, x\}\big)$ is an RDF of G with weight 4. Clearly, whatever the edge among bc, ac, ad or bd we obtain $G = R_3$. Suppose now that $G[N(v)]$ contains two independent edges. Seeing the previous situations, $ab \notin E$ and $bc \notin E$. Thus the only possibilities are either (bc and $ad \in E$) or (ac and $bd \in E$). For both situations, one can easily construct an RDF of G with weight 4.

Subcase 2.2. $n = 9$. Then $\gamma_R(G) = 6$. Since $\big|\overline{N}(v)\big| = 4$, $m\big(N(v), \overline{N}(v)\big) = 8$. It follows that $G\left[\overline{N}(v)\right] = 2P_2$, where each vertex of $\overline{N}(v)$ has exactly two neighbors in $N(v)$ and every vertex of $N(v)$ has two neighbors in $\overline{N}(v)$. Let $\overline{N}(v) = \{x, y, w, z\}$. Without loss of generality, we assume that $xy, wz \in E$. Consider the subgraph H, where $V(H) = V(G)\backslash\{v\}$ and $E(H) = m\big(N(v), \overline{N}(v)\big)$. Since H is a 2-regular bipartite graph, $H = 2C_4$ or C_8.

Assume first that $H = 2C_4$. If x and y belong to the same cycle C_4, then $\big(\{y, a, b, c, d, z\}, \{v\}, \{x, w\}\big)$ is an RDF of G with weight 5, a contradiction. Hence we may assume, without loss of generality, that the vertices of one of the two cycles are in order a, x, c, w, and b, y, d, z are on the other cycle. But then $\big(\{y, a, b, c, d, w\}, \{v\}, \{x, z\}\big)$ is an RDF of G with weight 5, a contradiction.

Assume now that $H = C_8$. Up to isomorphism, there are two possible situations, H_1 : a-x-d-z-c-w-b-y-a or H_2 : a-x-b-w-d-y-c-z-a. Note that by taking into account the edges xy and wz, H_1 has two disjoint triangles and H_2 has no triangle. If H_1 occurs, then $\big(\{a, b, c, d, y, w\}, \{v\}, \{x, z\}\big)$ is an RDF of G with weight 5, a contradiction. Thus H_2 occurs. Now if $N(v)$ is independent, then $G = R_4$. Hence assume that $N(v)$ is not independent. If $ab \in E$, then $\big(\{v, b, c, d, z, x\}, \{w\}, \{a, y\}\big)$ is an RDF of G with weight 5, a contradiction. Hence $ab \notin E$, and likewise ac, cd and $bd \notin E$. Now if ad and $bc \in E$, then $\big(\{v, c, d, w, z, x\}, \{y\}, \{a, b\}\big)$ is an RDF of G with weight 5, a contradiction. Hence either $ad \in E$ or $bc \in E$. Whatever the case, $G = R_5$.

Case 3. $\Delta = 5$. By (1) we have $n \in \{6, 7, 8, 9\}$. If $n \in \{6, 7, 8\}$, then $\Delta \geq n-3$. Hence we assume that $n = 9$, and so $\gamma_R(G) = 5$. Since $G\left[\,\overline{N}(v)\right]$ has order 3 and contains an isolated vertex, we have $2\Delta = 10 \geq m\big(N(v), \overline{N}(v)\big) \geq 2\,(\Delta - 2) + (\Delta - 1) = 10$, and so $m\big(N(v), \overline{N}(v)\big) = 10$. Let $N(v) = \{a, b, c, d, e\}$ and $\overline{N}(v) = \{x, y, z\}$. Without loss of generality, we assume that $xy \in E$ and z is isolated in $G\left[\,\overline{N}(v)\right]$. Since $m\big(N(v), \overline{N}(v)\big) = 10$, $|N(x) \cap N(v)| = |N(y) \cap N(v)| = 3$, and $|N(z) \cap N(v)| = 4$. Let $N(z) = \{a, b, c, d\}$. Clearly, x and y have at least one common neighbor in $N(v)$. Since each vertex of $N(v)$ has exactly two neighbors in $\overline{N}(v)$ and $N(z) = \{a, b, c, d\}$, we deduce that $e \in N(x) \cap N(y)$. But then $\big(\{x, y, a, b, c, d, v\}, \emptyset, \{z, e\}\big)$ is an RDF of G with weight 4, a contradiction.

The converse is easy to show by examining each graph separately. ■

4. Complexity Result

In this section we consider the complexity of the problem of deciding whether a graph G has $\gamma_R(G) = n - \Delta(G) + 1$, to which we shall refer as **MRDF($n-\Delta+1$)**.

MRDF($n - \Delta + 1$)
INSTANCE: A graph $G = (V, E)$.
QUESTION: Does G have a minimum RDF f with $f(V) = n - \Delta(G) + 1$?

We show that this problem is *co*-$\mathcal{NP}$-complete by reducing the 3-satisfiability problem (3-SAT) to the problem of deciding whether $\gamma_R(G) \leq n - \Delta(G)$ is $\mathcal{NP}$-complete. Recall that the 3-SAT problem is a well known NP-complete problem [4].

3-SAT
INSTANCE: A collection $\mathcal{C} = \big\{C_1, C_2, \ldots, C_q\big\}$ of clauses over a finite set X of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, q$.
QUESTION: Is there a truth assignment for X that satisfies all the clauses in $\mathcal{C}$?

RDF$(n-\Delta)$
INSTANCE: A graph $G=(V,E)$.
QUESTION: Does G have an RDF f with $f(V)\le n-\Delta(G)$?

Theorem 15. *Problem* RDF$(n-\Delta)$ *is* $\mathcal{NP}$*-complete.*

Proof. RDF$(n-\Delta)$ is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f:V\to\{0,1,2\}$ has weight at most $n-\Delta(G)$ and is a Roman dominating function. Now let us show how to transform any instance of 3-SAT into an instance G of RDF$(n-\Delta)$ so that one of them has a solution if and only if the other one has a solution.

Let I be an arbitrary instance of 3-SAT for the set of clauses $\mathcal{C}=\{C_1,C_2,\dots,C_q\}$ on the set of variables $X=\{x_1,x_2,\dots,x_p\}$. We construct a graph $G(I)$ such that I has a satisfying truth assignment if and only if $G(I)$ has an RDF $f=(V_0,V_1,V_2)$ with $f(V)\le n-\Delta(G)$.

Corresponding to each clause C_i, we create a vertex labeled c_i. Corresponding to each variable $x_i\in X$ we build a copy of the graph K_4-e whose vertices are labeled $x_i,\overline{x_i},u_i,v_i$, where x_i and $\overline{x_i}$ are the vertices of degree 3 of K_4-e. Next add a path $P_2: y$-w and join y to every vertex of $\{x_i,\overline{x_i}:1\le i\le p\}\cup\{c_j:1\le j\le q\}$. Finally, add three edges from each clause vertex c_i to the three vertices corresponding to the three literals in clause C_i (for example, see Figure 3). Clearly, $G(I)$ has order $4p+q+2$ and y is a vertex of maximum degree $\Delta(G(I))=2p+q+1$. We shall show that the given instance of 3-SAT has a satisfying truth assignment if and only if the graph $G(I)$ has an RDF f with $f(V)\le n-\Delta(G)$.

Let the given instance of 3-SAT have a satisfying truth assignment A. For each variable x_i, if $A(x_i)=$TRUE, then let $f(x_i)=2$. Otherwise, let $f(\overline{x_i})=2$. Also, assign a 1 to w and a 0 to all the remaining vertices. It is easy to see that the function f is an RDF of the graph $G(I)$ of weight $f(V)=2p+1=n-\Delta(G)$, since exactly one of x_i and $\overline{x_i}$ is in V_2, and dominates all vertices assigned a 0. In particular, every vertex c_i has at least one neighbor in V_2 because A assigns at least one literal to TRUE in every clause C_i.

Conversely, assume that $G(I)$ has an RDF $f=(V_0,V_1,V_2)$ of weight $f(V)\le n-\Delta(G)$. If $f(y)=2$, then for every i, we need two legions to defend u_i and v_i, implying that $f(V)=2p+2>n-\Delta(G)$. Hence $f(y)\in\{0,1\}$. If $f(y)=1$, then $f(w)=1$ and as previously we need $2p$ legions to defend all u_i and v_i which leads to a contradiction too. Therefore $f(y)=0$ and thus $f(w)>0$. Since $f(V)\le n-\Delta(G)$, it is easy to see that for each subgraph induced by $\{x_i,\overline{x_i},u_i,v_i\}$, we must have either $f(x_i)=2$ or $f(\overline{x_i})=2$. Now since $\sum_{i=1}^{p} f(r_i)+f(w)\ge 2p+1=n-\Delta(G)$ with $r_i\in\{x_i,\overline{x_i}\}$, we deduce that $f(w)=1$ and $f(c_i)=0$ for every i. It follows that the set V_2 dominates the set of clause vertices. Therefore, the given instance of 3-SAT has a satisfying truth assignment A, where $A(x_i)=$TRUE if and only if $f(r_i)=2$. ■

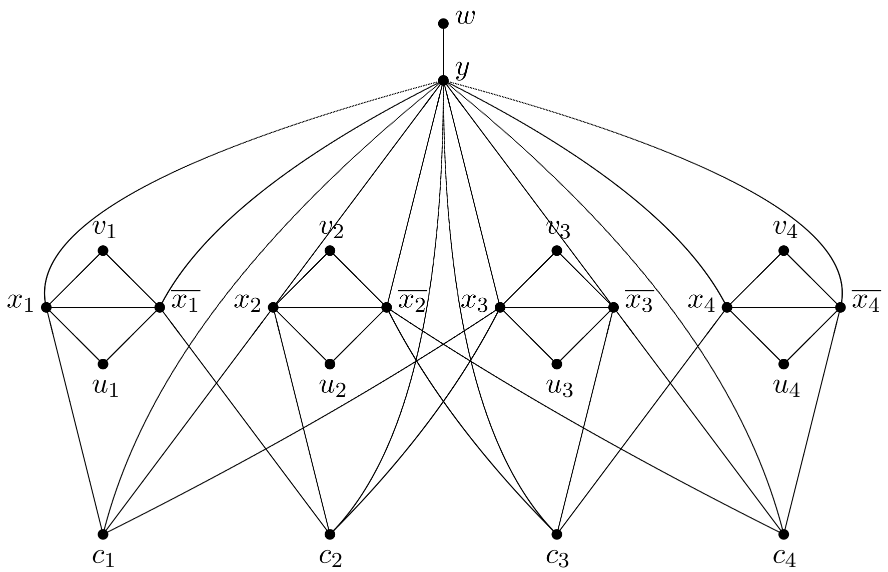

Figure 3. A construction of $G(I)$ for $\left(x_1 \vee x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(\overline{x_2} \vee \overline{x_3} \vee x_4\right) \wedge \left(\overline{x_2} \vee \overline{x_3} \vee \overline{x_4}\right)$.

Corollary 16. *Problem* MRDF$(n-\Delta+1)$ *is* co-$\mathcal{NP}$*-complete.*

Proof. Given any instance G, problem MRDF$(n-\Delta+1)$ has an answer YES if and only if Problem RDF$(n-\Delta)$ has an answer NO. Since Problem RDF$(n-\Delta)$ is $\mathcal{NP}$-complete, by Theorem 15, we conclude that Problem MRDF$(n-\Delta+1)$ is *co-*$\mathcal{NP}$-complete. ■

5. Nordhaus-Gaddum Inequalities

In [2], Chambers *et al.* gave the following Nordhaus-Gaddum bound for $\gamma_R(G)+\gamma_R(\overline{G})$ in terms of the order of the graph G.

Theorem 17 (Chambers *et al.* [2])**.** *If G is graph of order $n \geq 3$, then*

$$\gamma_R(G)+\gamma_R(\overline{G}) \leq n+3.$$

Furthermore, equality holds only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

According to Theorem 17, if G is a graph different from $C_5, \frac{n}{2}K_2$ or $\overline{\frac{n}{2}K_2}$, then $\gamma_R(G)+\gamma_R(\overline{G}) \leq n+2$. In the sequel, we provide a characterization of graphs G of order n for which $\gamma_R(G)+\gamma_R(\overline{G}) = n+2$. For this purpose, we define the following family of graphs.

- $\mathcal{H}_0 = \{C_6, C_7, C_8, C_i \cup C_j, \text{where } i, j \in \{3,4,5\}\}$.
- $\mathcal{H}_1 = \{pK_1 \cup qK_2 : p \geq 1, q \geq 1 \text{ and } p + 2q = n\}$,
- $\mathcal{H}_2 = \{qK_2 \cup H$ with $2q + |V(H)| = n$, where $H \in \{P_3, P_4, P_5, C_3, C_4, C_5\}$ if $q \neq 0$ and $H \in \{P_3, P_4, P_5\}$ if $q = 0\}$.
- $\mathcal{H}_3 = \{F_1, F_2, M_1, M_2\}$.

Theorem 18. *Let G be a graph of order $n \geq 3$ such that $G \notin \{C_5, \frac{n}{2}K_2, \overline{\frac{n}{2}K_2}\}$. Then $\gamma_R(G) + \gamma_R(\overline{G}) \leq n+2$, with equality if and only if G or $\overline{G} \in \{K_n\} \cup \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$.*

Proof. Clearly, the upper bound follows from Theorem 17.

Assume now that $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$. By Proposition 1, we have

$$\begin{aligned} n + 2 = \gamma_R(G) + \gamma_R(\overline{G}) &\leq n - \Delta(G) + 1 + n - \Delta(\overline{G}) + 1 \\ &= n - \Delta(G) + \delta(G) + 3. \end{aligned}$$

Hence $\Delta(G) - \delta(G) \leq 1$. Therefore G is either regular or semiregular. Consider the two cases:

Case 1. G is a regular graph. Hence $\overline{G}$ is also regular. Clearly, $\gamma_R(G) + \gamma_R(\overline{G}) = n+2$, implies that either $\gamma_R(G) = n - \Delta(G) + 1$ and $\gamma_R(\overline{G}) = \Delta(G)+1$ or $\gamma_R(G) = n - \Delta(G)$ and $\gamma_R(\overline{G}) = \Delta(G) + 2$. Without loss of generality, assume that $\gamma_R(G) = n - \Delta(G) + 1$ and $\gamma_R(\overline{G}) = \Delta(G) + 1$. By Theorem 12, and since $G \notin \{C_5, \frac{n}{2}K_2, \overline{\frac{n}{2}K_2}\}$, we obtain that $\Delta(G) \in \{n-3, n-1\}$. Clearly, if $\Delta(G) = n - 1$, then $G = K_n$. Thus assume that $\Delta(G) = n - 3$. Then each component of $\overline{G}$ is a cycle. Now using the fact that $\gamma_R(\overline{G}) = \Delta(G)+1 = n-2$ and whether $\overline{G}$ is connected or not, we deduce that $\overline{G} = C_6, C_7, C_8$ or $\overline{G} = C_i \cup C_j$, where $i, j \in \{3,4,5\}$. Hence $\overline{G} \in \mathcal{H}_0$.

Case 2. G is a semiregular graph. Note that $\overline{G}$ is also semiregular. In this case $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$ implies that $\gamma_R(G) = n - \Delta(G) + 1$ and $\gamma_R(\overline{G}) = n - \Delta(\overline{G}) + 1$. By Theorem 14, G and $\overline{G}$ fulfill one of the four items of that theorem. Since $\Delta(G) + \Delta(\overline{G}) = n$, we may assume, without loss of generality, that $\Delta(G) \leq \frac{n}{2}$. By Theorem 14, $\Delta(G) \leq 4$. Now if $\Delta(G) = 1$, then $G = pK_1 \cup qK_2$ with $p, q \geq 1$ and $p + 2q = n$. Hence $G \in \mathcal{H}_1$. If $\Delta(G) = 2$, then $G = qK_2 \cup H$ with $2q + |V(H)| = n$, where $H \in \{P_3, P_4, P_5\}$ if $q = 0$, and $H \in \{P_3, P_4, P_5, C_3, C_4, C_5\}$ if $q \neq 0$. Hence $G \in \mathcal{H}_2$. If $\Delta(G) = 3$, then $G \in \{F_1, F_2, M_1, M_2\} = \mathcal{H}_3$. Finally, if $\Delta(G) = 4$, then $G \in \{R_1, R_2, \ldots, R_5\}$. But since $\overline{G} \notin \{R_1, R_2, \ldots, R_5\}$ we have $\gamma_R(\overline{G}) < n - \Delta(\overline{G}) + 1$, which leads to a contradiction.

The converse is easy to see and we omit the details. ■

We close this section by improving the bound of Theorem 17 for all graphs G when $n(G) \geq 160$. To do this, we need the use of the following Nordhaus-Gaddum bound for $\gamma_R(G)\gamma_R(\overline{G})$ given in [2].

Theorem 19 (Chambers *et al.* [2])**.** *If G is a graph of order $n \geq 160$, then*

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \frac{16n}{5},$$

with equality only when G or $\overline{G}$ is $\frac{n}{5}C_5$.

Theorem 20. *If G is a graph of order $n \geq 160$ such that every component of G or $\overline{G}$ is of order at least 3, then*

$$\gamma_R(G) + \gamma_R(\overline{G}) \leq \frac{4n}{5} + 4,$$

with equality only when G or $\overline{G}$ is $\frac{n}{5}C_5$.

Proof. By Theorem 4, $\gamma_R(G) \leq \frac{4n}{5}$ and $\gamma_R(\overline{G}) \leq \frac{4n}{5}$. If $\gamma_R(G) \leq 4$ or $\gamma_R(\overline{G}) \leq 4$, then $\gamma_R(G) + \gamma_R(\overline{G}) \leq \frac{4n}{5} + 4$. If $\gamma_R(G) \geq 8$ and $\gamma_R(\overline{G}) \geq 8$, then by Theorem 19, we have $\gamma_R(G) \leq \frac{16n}{5\gamma_R(\overline{G})} \leq \frac{2n}{5}$ and $\gamma_R(\overline{G}) \leq \frac{16n}{5\gamma_R(G)} \leq \frac{2n}{5}$. Hence $\gamma_R(G) + \gamma_R(\overline{G}) \leq \frac{4n}{5} < \frac{4n}{5} + 4$. For the next we can assume that $5 \leq \gamma_R(G) \leq 7$ or $5 \leq \gamma_R(\overline{G}) \leq 7$. It follows by Theorem 19 that $\gamma_R(G) \leq \frac{16n}{5\gamma_R(\overline{G})} \leq \frac{16n}{25}$, and thus $\gamma_R(G) + \gamma_R(\overline{G}) \leq \frac{16n}{25} + 7 < \frac{4n}{5} + 4$, since $n \geq 160$.

Moreover, based on the previous proof, $\gamma_R(G) + \gamma_R(\overline{G}) = \frac{4n}{5} + 4$ is only possible when $\{\gamma_R(G), \gamma_R(\overline{G})\} = \{\frac{4n}{5}, 4\}$. By Theorem 4, $G = \frac{n}{5}C_5$ and the other graph is excluded since $\gamma_R(\overline{G}) = 3$. ■

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Received 11 December 2017
Revised 24 April 2018
Accepted 24 April 2018