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NICHE HYPERGRAPHS OF PRODUCTS OF DIGRAPHS

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Abstract

If $D = (V, A)$ is a digraph, its *niche hypergraph* $N\mathcal{H}(D) = (V, \mathcal{E})$ has the edge set $\mathcal{E} = \{e \subseteq V \mid |e| \geq 2 \wedge \exists v \in V : e = N_D^-(v) \vee e = N_D^+(v)\}$. Niche hypergraphs generalize the well-known niche graphs and are closely related to competition hypergraphs as well as common enemy hypergraphs. For several products $D_1 \circ D_2$ of digraphs D_1 and D_2, we investigate the relations between the niche hypergraphs of the factors D_1, D_2 and the niche hypergraph of their product $D_1 \circ D_2$.

Keywords: niche hypergraph, product of digraphs, competition hypergraph.

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1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered in the following may have isolates but no multiple edges. Moreover, in digraphs loops are forbidden. With $N_D^-(v)$, $N_D^+(v)$, $d_D^-(v)$ and $d_D^+(v)$ we denote the in-neighborhood, the out-neighborhood, the in-degree and the out-degree of $v \in V(D)$, respectively. In standard terminology we follow Bang-Jensen and Gutin [1].

In 1968, Cohen [3] introduced the *competition graph* $C(D) = \big(V, E(C(D))\big)$ of a digraph $D = (V, A)$ representing a food web of an ecosystem. Here the vertices correspond to the species and different vertices v_1, v_2 are connected by an edge if and only if they compete for a common prey w, i.e.,

$$E\big(C(D)\big) = \big\{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists\, w \in V : v_1 \in N_D^-(w) \wedge v_2 \in N_D^-(w)\big\}.$$

Surveys of the large literature around competition graphs (and its variants) can be found in [5, 6, 11]; for (a selection of) recent results see [4, 7–10, 12–17, 21].

Meanwhile the following variants of $C(D)$ have been investigated.
The *common enemy graph* $CE(D)$ (cf. [11]) with the edge set

$$E\big(CE(D)\big) = \big\{\{v_1, v_2\} \mid v_1 \neq v_2 \wedge \exists\, w \in V : v_1 \in N_D^+(w) \wedge v_2 \in N_D^+(w)\big\},$$

the *double competition graph* or *competition-common enemy graph* $DC(D)$ with the edge set $E(DC(D)) = E(C(D)) \cap E(CE(D))$ (cf. [18]), and the *niche graph* $N(D)$ with $E\big(N(D)\big) = E\big(C(D)\big) \cup E\big(CE(D)\big)$ (cf. [2]).

In 2004, the concept of competition hypergraphs was introduced by Sonntag and Teichert [19]. The *competition hypergraph* $C\mathcal{H}(D)$ of a digraph $D = (V, A)$ has the vertex set V and the edge set

$$\mathcal{E}\big(C\mathcal{H}(D)\big) = \big\{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^-(v)\big\}.$$

As a second hypergraph generalization, recently Park and Sano [16] defined the *double competition hypergraph* $DC\mathcal{H}(D)$ of a digraph $D = (V, A)$, which has the vertex set V and the edge set

$$\mathcal{E}\big(DC\mathcal{H}(D)\big) = \big\{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v_1, v_2 \in V : e = N_D^-(v_1) \cap N_D^+(v_2)\big\}.$$

Our paper [5] was a third step in this direction; there we considered the *niche hypergraph* $N\mathcal{H}(D)$ of a digraph $D = (V, A)$, again with the vertex set V and the edge set

$$\mathcal{E}\big(N\mathcal{H}(D)\big) = \big\{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^-(v) \vee e = N_D^+(v)\big\}.$$

Note that $N\mathcal{H}(D) = N\mathcal{H}\left(\overleftarrow{D}\right)$ holds for any digraph D, if $\overleftarrow{D}$ denotes the digraph obtained from D by reversing all arcs.

In [5] we present results on several properties of niche hypergraphs and the so-called *niche number* $\hat{n}$ of hypergraphs. In most of the investigations in [5] the *generating digraph* D of $N\mathcal{H}(D)$ is assumed to be acyclic.

For technical reasons, we define another hypergraph generalization. The *common enemy hypergraph* $CE\mathcal{H}(D)$ of a digraph $D = (V, A)$ has the vertex set V and the edge set

$$\mathcal{E}\big(CE\mathcal{H}(D)\big) = \big\{e \subseteq V \mid |e| \geq 2 \wedge \exists\, v \in V : e = N_D^+(v)\big\}.$$

In the hypergraphs $C\mathcal{H}(D)$, $CE\mathcal{H}(D)$ and $N\mathcal{H}(D)$ no loops are allowed. Therefore, by definition the in-neighborhoods and out-neighborhoods of cardinality 1 in the digraph D play no role in the corresponding hypergraphs. This loss of information proved to be disadvantageous in the investigation of competition hypergraphs of products of digraphs (cf. [20]). So, considering niche hypergraphs of products of digraphs, it seems to be consequent to allow loops in niche hypergraphs, too. Therefore, we define the *l-competition hypergraph* $C\mathcal{H}^l(D)$, the *l-common enemy hypergraph* $CE\mathcal{H}^l(D)$ and the *l-niche hypergraph* $N\mathcal{H}^l(D)$ (*with loops*) having the edge sets

$$\begin{aligned}
\mathcal{E}\big(C\mathcal{H}^l(D)\big) \quad &= \{e \subseteq V \mid \exists\, v \in V : e = N_D^-(v) \neq \emptyset\},\\
\mathcal{E}\big(CE\mathcal{H}^l(D)\big) &= \big\{e \subseteq V \mid \exists\, v \in V : e = N_D^+(v) \neq \emptyset\big\} \quad \text{and}\\
\mathcal{E}\big(N\mathcal{H}^l(D)\big) \quad &= \big\{e \subseteq V \mid \exists\, v \in V : e = N_D^-(v) \neq \emptyset \vee e = N_D^+(v) \neq \emptyset\big\}\\
&= \ \mathcal{E}\big(C\mathcal{H}^l(D)\big) \ \cup \ \mathcal{E}\big(CE\mathcal{H}^l(D)\big).
\end{aligned}$$

For the sake of brevity, in the following we often use the term (l)-*competition hypergraph* (sometimes in connection with the notation $C\mathcal{H}^{(l)}(D)$) for the competition hypergraph $C\mathcal{H}(D)$ as well as for the l-competition hypergraph $C\mathcal{H}^l(D)$, analogously for (l)-common enemy and (l)-niche hypergraphs with the notations $CE\mathcal{H}^{(l)}(D)$ and $N\mathcal{H}^{(l)}(D)$, respectively.

For five products $D_1 \circ D_2$ (*Cartesian product* $D_1 \times D_2$, *Cartesian sum* $D_1 + D_2$, *normal product* $D_1 * D_2$, *lexicographic product* $D_1 \cdot D_2$ and *disjunction* $D_1 \vee D_2$) of digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ we investigate the construction of the (l)-niche hypergraph $N\mathcal{H}^{(l)}\big(D_1 \circ D_2\big) = \Big(V, \mathcal{E}_\circ^{(l)}\Big)$ from $N\mathcal{H}^{(l)}\big(D_1\big) = \Big(V_1, \mathcal{E}_1^{(l)}\Big)$, $N\mathcal{H}^{(l)}\big(D_2\big) = \Big(V_2, \mathcal{E}_2^{(l)}\Big)$ and vice versa.

The products considered here have always the vertex set $V := V_1 \times V_2$; using the notation $\widetilde{A} := \{((a, b), (a', b')) \mid a, a' \in V_1 \wedge b, b' \in V_2\}$ their arc sets are defined as follows:

$$\begin{aligned}
&A\big(D_1 \times D_2\big) := \big\{\, ((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \wedge (b, b') \in A_2\big\},\\
&A\big(D_1 + D_2\big) := \big\{((a, b), (a', b')) \in \widetilde{A} \mid ((a, a') \in A_1 \wedge b = b') \vee (a = a' \wedge (b, b') \in A_2)\big\},\\
&A\big(D_1 * D_2\big) := A\big(D_1 \times D_2\big) \cup A\big(D_1 + D_2\big),\\
&A\big(D_1 \cdot D_2\big) := \big\{((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \vee (a = a' \wedge (b, b') \in A_2)\big\},\\
&A\big(D_1 \vee D_2\big) := \big\{((a, b), (a', b')) \in \widetilde{A} \mid (a, a') \in A_1 \vee (b, b') \in A_2\big\}.
\end{aligned}$$

It follows immediately that $A\big(D_1 + D_2\big) \subseteq A\big(D_1 * D_2\big) \subseteq A\big(D_1 \cdot D_2\big) \subseteq A\big(D_1 \vee D_2\big)$ and $A\big(D_1 \times D_2\big) \subseteq A\big(D_1 * D_2\big)$. Except the lexicographic product all these products are commutative in the sense that $D_1 \circ D_2 \simeq D_2 \circ D_1$, where $\circ \in \{\times, +, *, \vee\}$.

Usually we number the vertices of D_1 and D_2 such that $V_1 = \{1, 2, \ldots, r\}$, $V_2 = \{1, 2, \ldots, s\}$ and arrange the vertices of $V = V_1 \times V_2$ according to the places of an (r, s)-matrix.

In analogy with the rows and the columns of the described (r, s)-matrix we call the set $Z_i = \{(i, j) \mid j \in V_2\}$ $(i \in V_1)$ and the set $S_j = \{(i, j) \mid i \in V_1\}$ $(j \in V_2)$ the *i-th row* and the *j-th column* of $D_1 \circ D_2$, respectively.

Then, for each $\circ \in \{+, *, \cdot, \vee\}$, the subdigraph $\langle S_j \rangle_{D_1 \circ D_2}$ of $D_1 \circ D_2$ induced by the vertices of a column S_j is isomorphic to D_1, and, analogously, the subdigraph $\langle Z_i \rangle_{D_1 \circ D_2}$ of $D_1 \circ D_2$ induced by the vertices of a row Z_i is isomorphic to D_2. Moreover, if an arc $a \in A\big(D_1 \circ D_2\big)$ consists only of vertices of one row Z_i $(i \in V_1)$, we refer to a as a *horizontal arc.* Analogously, an arc a containing only vertices of one column S_j $(j \in V_2)$ is called a *vertical arc.*

Considering (l)-niche hypergraphs, the question arises, whether or not $N\mathcal{H}^{(l)}\big(D_1 \circ D_2\big)$ can be obtained from $N\mathcal{H}^{(l)}(D_1)$ and $N\mathcal{H}^{(l)}(D_2)$ and vice versa.

As an instance for competition hypergraphs $C\mathcal{H}^{(l)}$, we cite two results from [20].

Theorem 1 [20]**.** *The l-competition hypergraph $\mathcal{CH}^l(D_1 \times D_2) = \big(V, \mathcal{E}^l_\times\big)$ of the Cartesian product can be obtained from the l-competition hypergraphs $\mathcal{CH}^l(D_1) = \big(V_1, \mathcal{E}^l_1\big)$ and $\mathcal{CH}^l(D_2) = \big(V_2, \mathcal{E}^l_2\big)$ of D_1 and D_2 : $\mathcal{E}^l_\times = \big\{e_1 \times e_2 \mid e_1 \in \mathcal{E}^l_1 \wedge e_2 \in \mathcal{E}^l_2\big\}$.*

Theorem 2 [20]**.** *The l-competition hypergraph $\mathcal{CH}^l(D_1 \vee D_2) = \big(V, \mathcal{E}^l_\vee\big)$ of the disjunction can be obtained from the l-competition hypergraphs $\mathcal{CH}^l(D_1) = \big(V_1, \mathcal{E}^l_1\big)$ and $\mathcal{CH}^l(D_2) = \big(V_2, \mathcal{E}^l_2\big)$ of D_1 and D_2, if for each of the following conditions is known whether it is true or not:*

(a) $\exists\, v_2 \in V_2 : N^-_2(v_2) = \emptyset$ *and* (b) $\exists\, v_1 \in V_1 : N^-_1(v_1) = \emptyset$.

In general, $\mathcal{CH}^l(D_1 \vee D_2)$ cannot be obtained from $\mathcal{CH}^l(D_1)$ and $\mathcal{CH}^l(D_2)$ without the extra information on points (a) *and* (b).

Note that in some cases under certain conditions $D_1 \circ D_2$ and even D_1 and D_2 can be reconstructed from $C\mathcal{H}^{(l)}(D_1 \circ D_2)$. For niche hypergraphs such strong results are not expectable.

The main reason why the reconstruction of D_1 and D_2 from $N\mathcal{H}^{(l)}(D_1 \circ D_2)$ is much more difficult is the following. In general, for any hyperedge $e \in \mathcal{E}\big(N\mathcal{H}^{(l)}(D)\big)$ it is not possible to see whether e is a set of predecessors $e = N^-_D(v)$ or a set of successors $e = N^+_D(v)$ of a certain vertex $v \in V(D)$.

It is interesting that, in general, for the same reason also the construction of $N\mathcal{H}(D_1 \circ D_2)$ from $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ is impossible.

2. Construction of $N\mathcal{H}^{(l)}(D_1 \circ D_2)$ from $N\mathcal{H}^{(l)}(D_1)$ and $N\mathcal{H}^{(l)}(D_2)$

The digraphs $D = (V, A)$ and $D' = (V, A')$ are (l)-*niche equivalent* if and only if D and D' have the same (l)-niche hypergraph, i.e., $N\mathcal{H}^{(l)}(D) = N\mathcal{H}^{(l)}(D')$.

Theorem 3. *Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be digraphs. In general, for $\circ \in \{\times, +, *, \cdot, \vee\}$, the niche hypergraph $N\mathcal{H}(D_1 \circ D_2) = (V, \mathcal{E}_\circ)$ of $D_1 \circ D_2$ cannot be obtained from the l-niche hypergraphs $N\mathcal{H}^l(D_1) = \big(V_1, \mathcal{E}_1^l\big)$ and $N\mathcal{H}^l(D_2) = \big(V_2, \mathcal{E}_2^l\big)$ of D_1 and D_2.*

Proof. It suffices to present digraphs $D_1 = (V_1, A_1)$, $D_1' = \big(V_1, A_1'\big)$, $D_2 = \big(V_2, A_2\big)$ such that D_1 and D_1' are l-niche equivalent, but the niche hypergraphs of $D_1 \circ D_2$ and $D_1' \circ D_2$ are distinct, i.e., $N\mathcal{H}\left(D_1 \circ D_2\right) \neq N\mathcal{H}\big(D_1' \circ D_2\big)$.

So let us consider the following digraphs and their niche hypergraphs:
$D_1 = \big(V_1, A_1\big)$ with $V_1 = \{1, 2, 3, 4, 5\}$ and $A_1 = \big\{(1, 2), (3, 2), (4, 5), (2, 4)\big\}$,
$D_1' = \big(V_1, A_1'\big)$ with $A_1' = \big\{(1, 2), (3, 2), (4, 5)\big\}$ and
$D_2 = (V_2, A_2)$ with $V_2 = \{1, 2, 3\}$ and $A_2 = \big\{(1, 3), (2, 3)\big\}$.

Obviously, D_1 and D_1' are l-niche equivalent, they have the l-niche hypergraph $N\mathcal{H}^l(D_1) = N\mathcal{H}^l(D_1') = \big(V_1, \mathcal{E}_1^l\big)$, where $\mathcal{E}_1^l = \big\{\{1, 3\}, \{2\}, \{4\}, \{5\}\big\}$.

In detail, looking at D_1 we have

$$\begin{aligned}\mathcal{E}_1^l = \mathcal{E}\left(N\mathcal{H}^l(D_1)\right) = \big\{&\{1, 3\} = N^-_{D_1}(2), \{2\} = N^-_{D_1}(4) = N^+_{D_1}(1) = N^+_{D_1}(3),\\ &\{4\} = N^-_{D_1}(5) = N^+_{D_1}(2), \{5\} = N^+_{D_1}(4)\big\};\end{aligned}$$

regarding D_1' we get

$$\begin{aligned}\mathcal{E}_1^l = \mathcal{E}\big(N\mathcal{H}^l(D_1')\big) = \big\{&\{1, 3\} = N^-_{D_1'}(2), \{2\} = N^+_{D_1'}(1) = N^+_{D_1'}(3), \{4\} = N^-_{D_1'}(5),\\ &\{5\} = N^+_{D_1'}(4)\big\}.\end{aligned}$$

Note that D_1 and D_1' — despite having one and the same l-niche hypergraph — are significantly different in the sense that $D_1' \neq \overleftarrow{D_1}$, $D_1 \not\simeq D_1'$, and, moreover, D_1 is connected but D_1' consists of two components. Of course, using D_1 and $\overleftarrow{D_1}$ instead of D_1 and D_1' could be an alternative approach for proving Theorem 3.

For the sake of completeness, we give the l-niche hypergraph $N\mathcal{H}^l(D_2) = \big(V_2, \mathcal{E}_2^l\big)$, with $\mathcal{E}_2^l = \big\{\{1, 2\} = N^-_{D_2}(3), \{3\} = N^+_{D_2}(1) = N^+_{D_2}(2)\big\}$.

Now we compare the niche hypergraphs of the products $D_1 \circ D_2$ and $D_1' \circ D_2$.

- *Cartesian product $D_1^{(\prime)} \times D_2$.*

Since the Cartesian product has not so many arcs and, consequently, its niche hypergraph $N\mathcal{H}\left(D_1^{(\prime)} \times D_2\right)$ includes only few hyperedges, we present the whole edge sets $\mathcal{E}\left(N\mathcal{H}\left(D_1^{(\prime)} \times D_2\right)\right)$ here (in case of the other four products the edge sets of $N\mathcal{H}\left(D_1^{(\prime)} \circ D_2\right)$ will be considerably larger, hence in these cases we will give up on writing down these sets completely).

$$\begin{aligned}\mathcal{E}(N\mathcal{H}(D_1 \times D_2)) = \big\{&\{(1,1),(1,2),(3,1),(3,2)\} = N^-_{D_1\times D_2}((2,3)),\\ &\{(2,1),(2,2)\} = N^-_{D_1\times D_2}((4,3)),\\ &\{(4,1),(4,2)\} = N^-_{D_1\times D_2}((5,3))\big\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(N\mathcal{H}(D_1' \times D_2)) = \big\{&\{(1,1),(1,2),(3,1),(3,2)\} = N^-_{D_1'\times D_2}((2,3)),\\ &\{(4,1),(4,2)\} = N^-_{D_1'\times D_2}((5,3))\big\}.\end{aligned}$$

• *Cartesian sum $D_1^{(\prime)} + D_2$, normal product $D_1^{(\prime)} * D_2$ and lexicographic product $D_1^{(\prime)} \cdot D_2$.*
Since D_1 is connected, the Cartesian sum $D_1 + D_2$, the normal product $D_1 * D_2$ as well as the lexicographic product $D_1 \cdot D_2$ are connected, too. Considering the (disconnected) digraph D_1', obviously $D_1' + D_2$, $D_1' * D_2$ and $D_1' \cdot D_2$ are disconnected. In detail, each of the products $D_1' \circ D_2$ ($\circ \in \{+, *, \cdot\}$) consists of the two components $\langle Z_1 \cup Z_2 \cup Z_3\rangle_{D_1'\circ D_2}$ and $\langle Z_4 \cup Z_5\rangle_{D_1'\circ D_2}$.

Therefore, in the niche hypergraph $N\mathcal{H}(D_1' \circ D_2)$ hyperedges containing vertices of both components cannot exist:

$$\forall e \in \mathcal{E}(N\mathcal{H}(D_1' \circ D_2)) : \ e \cap (Z_1 \cup Z_2 \cup Z_3) = \emptyset \ \vee \ e \cap (Z_4 \cup Z_5) = \emptyset.$$

Consequently, to show $N\mathcal{H}(D_1 \circ D_2) \neq N\mathcal{H}\big(D_1' \circ D_2\big)$, it suffices to find a hyperedge $e \in \mathcal{E}\big(N\mathcal{H}(D_1 \circ D_2)\big)$ such that both $e \cap \big(Z_1 \cup Z_2 \cup Z_3\big)$ and $e \cap \big(Z_4 \cup Z_5\big)$ are nonempty.

For each of the three products $D_1 \circ D_2$ we will obtain such a hyperedge by considering the set of the predecessors of the vertex $(4,3) \in V\big(D_1 \circ D_2\big)$, i.e., $e = N^-_{D_1\circ D_2}((4,3))$. Clearly, e results from $N^-_{D_1}(4) = \{2\}$ and $N^-_{D_2}(3) = \{1,2\}$.

For the Cartesian sum $D_1 + D_2$, we have

$$e = \big\{(2,3),(4,1),(4,2)\big\} = N^-_{D_1+D_2}((4,3)).$$

In case of the normal product $D_1 * D_2$, we obtain

$$e = \big\{(2,1),(2,2),(2,3),(4,1),(4,2)\big\} = N^-_{D_1*D_2}((4,3)).$$

It it easy to see that in the lexicographic product $D_1 \cdot D_2$ the vertex $(4,3)$ has the same predecessors as in the normal product, hence

$$e = N^-_{D_1\cdot D_2}((4,3)) = N^-_{D_1*D_2}((4,3)) = \big\{(2,1),(2,2),(2,3),(4,1),(4,2)\big\}.$$

• *Disjunction $D_1^{(\prime)} \vee D_2$.*
Now both $D_1 \vee D_2$ and $D_1' \vee D_2$ are connected. Nevertheless, as in the previous cases, we consider the predecessors of the vertex $(4,3)$ and get the hyperedge

$$\begin{aligned}e &= N^-_{D_1\vee D_2}((4,3))\\ &= \big\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(4,1),(4,2),(5,1),(5,2)\big\}\\ &= S_1 \cup S_2 \cup \{(2,3)\} = S_1 \cup S_2 \cup Z_2 \in \mathcal{E}(N\mathcal{H}(D_1 \vee D_2)).\end{aligned}$$

Note that $S_1 \cup S_2$ in e result from $N^-_{D_2}(3) = \{1, 2\}$ and Z_2 from $N^-_{D_1}(4) = \{2\}$.

We search for this hyperedge e in $N\mathcal{H}(D'_1 \vee D_2)$.

Assume $e = N^+_{D'_1 \vee D_2}((i, j))$ or $e = N^-_{D'_1 \vee D_2}((i, j))$. Since D'_1 and D_2 are loopless digraphs, we obtain $(i, j) \notin e$ and $(i, j) \in \{(1, 3), (3, 3), (4, 3), (5, 3)\}$, i.e., $j = 3$.

Let $e = N^+_{D'_1 \vee D_2}((i, 3))$. Because of $N^+_{D_2}(3) = \emptyset$ and $S_1 \subseteq e$, all vertices of S_1 have to be successors of $(i, 3)$ in $D'_1 \vee D_2$ and $\{1, 2, \ldots, 5\} = N^+_{D'_1}(i)$, where $i \in \{1, 2, \ldots, 5\}$. This contradicts the fact that D'_1 is loopless.

Consequently, $e = N^-_{D'_1 \vee D_2}((i, 3))$. Then, $S_1 \cup S_2 \subseteq e$ holds trivially. Owing to $(2, 3) \in e$ we get $(2, 3) \in N^-_{D'_1 \vee D_2}((i, 3))$, i.e., $2 \in N^-_{D'_1}(i)$ with $i \in \{1, 2, \ldots, 5\}$. This contradicts $N^+_{D'_1}(2) = \emptyset$.

Hence, $e \notin \mathcal{E}(N\mathcal{H}(D'_1 \vee D_2))$, thus $D_1 \vee D_2$ and $D'_1 \vee D_2$ are not niche equivalent. Therefore, the niche hypergraph of the disjunction $D_1 \vee D_2$ cannot be constructed from the niche hypergraphs of D_1 and D_2 in general. ■

Using Theorems 1 and 2, for the Cartesian product and the disjunction some positive construction results can be derived. For this end we have to make use of $\mathcal{E}\left(N\mathcal{H}^{(l)}(D)\right) = \mathcal{E}\left(C\mathcal{H}^{(l)}(D)\right) \cup \mathcal{E}\left(CE\mathcal{H}^{(l)}(D)\right)$ and $CE\mathcal{H}^{(l)}(D) = C\mathcal{H}^{(l)}\left(\overleftarrow{D}\right)$.

Remark 4. The l-niche hypergraph $N\mathcal{H}^l(D_1 \times D_2)$ of the Cartesian product can be obtained from the l-competition hypergraphs $C\mathcal{H}^l(D_1)$, $C\mathcal{H}^l(D_2)$ and the l-common enemy hypergraphs $CE\mathcal{H}^l(D_1)$, $CE\mathcal{H}^l(D_2)$:

$$\begin{aligned}\mathcal{E}\left(N\mathcal{H}^l(D_1 \times D_2)\right) &= \mathcal{E}\big(C\mathcal{H}^l(D_1 \times D_2)\big) \cup \mathcal{E}\big(CE\mathcal{H}^l(D_1 \times D_2)\big)\\ &= \big\{e_1 \times e_2 \,|\, e_1 \in \mathcal{E}\big(C\mathcal{H}^l(D_1)\big) \wedge e_2 \in \mathcal{E}\big(C\mathcal{H}^l(D_2)\big)\big\}\\ &\cup \big\{e_1 \times e_2 \,|\, e_1 \in \mathcal{E}(CE\mathcal{H}^l(D_1)) \wedge e_2 \in \mathcal{E}(CE\mathcal{H}^l(D_2))\big\}.\end{aligned}$$

Remark 5. The l-niche hypergraph $N\mathcal{H}^l\big(D_1 \vee D_2\big)$ of the disjunction can be obtained from the l-competition hypergraphs $C\mathcal{H}^l(D_1)$, $C\mathcal{H}^l(D_2)$ and the l-common enemy hypergraphs $CE\mathcal{H}^l(D_1)$, $CE\mathcal{H}^l(D_2)$ provided that each of the following conditions is known to be true or false:

(a) $\exists\, v_2 \in V_2 : N^-_{D_2}(v_2) = \emptyset$ and (b) $\exists\, v_1 \in V_1 : N^-_{D_1}(v_1) = \emptyset$ and

(c) $\exists\, v_2 \in V_2 : N^+_{D_2}(v_2) = \emptyset$ and (d) $\exists\, v_1 \in V_1 : N^+_{D_1}(v_1) = \emptyset$.

In general, $N\mathcal{H}^l\big(D_1 \vee D_2\big)$ cannot be obtained from $C\mathcal{H}^l(D_1)$, $C\mathcal{H}^l(D_2)$, $CE\mathcal{H}^l(D_1)$ and $CE\mathcal{H}^l(D_2)$ without the extra information on points (a)–(d).

3. Reconstruction of $N\mathcal{H}^{(l)}(D_1)$ and $N\mathcal{H}^{(l)}(D_2)$ from $N\mathcal{H}^{(l)}(D_1 \circ D_2)$

In the following, for a set $e = \big\{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\big\} \subseteq V_1 \times V_2$ we define $\pi_1(e) :=$

$\{i_1, \ldots, i_k\}$ and $\pi_2(e) := \{j_1, \ldots, j_k\}$, respectively, i.e., π_i denotes the projection of vertices of $N\mathcal{H}^{(l)}(D_1 \circ D_2)$ onto their i-th components, for $i \in \{1, 2\}$.

Theorem 6 (Cartesian product $D_1 \times D_2$)**.**

(a) *If* $\mathcal{E}\big(N\mathcal{H}\big(D_1 \times D_2\big)\big) \neq \emptyset$, *then* $N\mathcal{H}(D_1)$ *and* $N\mathcal{H}(D_2)$ *can be obtained from* $N\mathcal{H}\big(D_1 \times D_2\big)$.

(b) *If* $\mathcal{E}(N\mathcal{H}^l(D_1 \times D_2)) \neq \emptyset$, *then* $N\mathcal{H}^l(D_1)$ *and* $N\mathcal{H}^l(D_2)$ *can be obtained from* $N\mathcal{H}^l(D_1 \times D_2)$.

Proof. Note that $\mathcal{E}\left(N\mathcal{H}\big(D_1 \times D_2\big)\right) \neq \emptyset$ implies $A_1 \neq \emptyset \neq A_2$ and $\max\big(|A_1|, |A_2|\big) \geq 2$. Moreover, $\mathcal{E}\left(N\mathcal{H}^l\big(D_1 \times D_2\big)\right) \neq \emptyset$ is equivalent to $A_1 \neq \emptyset \neq A_2$ and, consequently, to $\mathcal{E}\left(N\mathcal{H}^l(D_1)\right) \neq \emptyset \neq \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

(b) Let $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 \times D_2)\right)$. This is equivalent to $e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 \times D_2)\right)$ or $e \in \mathcal{E}\left(CE\mathcal{H}^l(D_1 \times D_2)\right)$, i.e., $e = N^-_{D_1 \times D_2}((i,j))$ or $e = N^+_{D_1 \times D_2}((i,j))$, with a certain $(i,j) \in V_1 \times V_2$.

This holds if and only if there is a vertex $(i,j) \in V_1 \times V_2$ such that

$$\pi_1(e) = N^-_{D_1}(i) \text{ and } \pi_2(e) = N^-_{D_2}(j) \quad \text{or} \quad \pi_1(e) = N^+_{D_1}(i) \text{ and } \pi_2(e) = N^+_{D_2}(j),$$

which implies $\pi_1(e) \in \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $\pi_2(e) \in \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

Clearly, this way we can get all hyperedges $e_1 \in \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $e_2 \in \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

(a) An analog argumentation holds if we consider the niche hypergraphs $N\mathcal{H}$ instead of the l-niche hypergraphs $N\mathcal{H}^l$, since hyperedges $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 \times D_2)\right)$ of cardinality 1 can be omitted if we are interested only in hyperedges $e_i \in \mathcal{E}(N\mathcal{H}(D_i))$ (which have cardinality greater than 1), for $i = 1, 2$. ■

Theorem 7 (Cartesian sum $D_1 + D_2$)**.**

(a) $N\mathcal{H}(D_1)$ *and* $N\mathcal{H}(D_2)$ *can be obtained from* $N\mathcal{H}(D_1 + D_2)$.

(b) $N\mathcal{H}^l(D_1)$ *and* $N\mathcal{H}^l(D_2)$ *can be obtained from* $N\mathcal{H}^l(D_1 + D_2)$, *provided that one of the following conditions is true:*

(1) $\mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) = \emptyset$;

(2) $\forall\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_1(e)| = 1$ *and* $\exists\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_2(e)| \geq 2$;

(3) $\forall\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_2(e)| = 1$ *and* $\exists\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_1(e)| \geq 2$;

(4) $\exists\, (i,j) \in V_1 \times V_2\ \forall\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : (i,j) \notin e$.

Proof. (a) Let $e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2))$ and $(i,j) \in V_1 \times V_2$ with $e = N^-_{D_1+D_2}((i,j))$ or $e = N^+_{D_1+D_2}((i,j))$. Then $e = \{(i,j_1), \ldots, (i,j_k), (i_1,j), \ldots, (i_l,j)\}$, where $i, i_1, \ldots, i_l$ and $j, j_1, \ldots, j_k$ are pairwise distinct vertices in V_1 and V_2, respectively.

To construct $\mathcal{E}(N\mathcal{H}(D_1))$, we need only those hyperedges $e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2))$ which contain $l \geq 2$ vertices with one and the same second component:

$$\begin{aligned}\mathcal{E}(N\mathcal{H}(D_1)) = \Big\{&\pi_1(e) \setminus I \,|\, e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2)) \wedge \\ &e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\} \wedge l \geq 2 \wedge \\ &I = \left\{ \begin{array}{ll} \{i\}, & k \geq 1 \\ \emptyset\ , & k = 0 \end{array} \right.\Big\}.\end{aligned}$$

Analogously, we obtain $\mathcal{E}(N\mathcal{H}(D_2))$:

$$\begin{aligned}\mathcal{E}(N\mathcal{H}(D_2)) = \Big\{&\pi_2(e) \setminus J \,|\, e \in \mathcal{E}(N\mathcal{H}(D_1 + D_2)) \wedge \\ &e = \{(i, j_1), \dots, (i, j_k), (i_1, j), \dots, (i_l, j)\} \wedge k \geq 2 \wedge \\ &J = \left\{ \begin{array}{ll} \{j\}, & l \geq 1 \\ \emptyset\ , & l = 0 \end{array} \right.\Big\}.\end{aligned}$$

(b) The proof of (1)–(3) is similar to the proof of (1)–(3) of Proposition 2 in [20].

Case (1): $\mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) = \emptyset$. Obviously, $A(D_1 + D_2) = \emptyset = A(D_1) = A(D_2) = \mathcal{E}\left(N\mathcal{H}^l(D_1)\right) = \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

Case (2): $\forall\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_1(e)| = 1$ and $\exists\, e \in \mathcal{E}\big(N\mathcal{H}^l(D_1 + D_2)\big)$: $|\pi_2(e)| \geq 2$.

Let $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ with $|\pi_2(e)| \geq 2$, i.e., $e = \{(i, j_1), \dots, (i, j_k)\} = N^-_{D_1+D_2}((i, j))$ or $e = \{(i, j_1), \dots, (i, j_k)\} = N^+_{D_1+D_2}((i, j))$ with $k \geq 2$ and suitable $i \in V_1$, $j \in V_2$ and $j_1, \dots, j_k \in V_2$.

We discuss only the situation $e = N^-_{D_1+D_2}((i, j))$, since $e = N^+_{D_1+D_2}((i, j))$ can be proved analogously.

Clearly, $N^-_{D_2}(j) = \{j_1, \dots, j_k\} = \pi_2(e)$. The assumption that there are $i' \in V_1$, $l \geq 1$ and $i'_1, \dots, i'_l \in V_1$ with $N^-_{D_1}(i') = \{i'_1, \dots, i'_l\} \neq \emptyset$ would lead to $e' = N^-_{D_1+D_2}((i', j)) = \{(i'_1, j), \dots, (i'_l, j), (i', j_1), \dots, (i', j_k)\}$ with $|\pi_1(e')| \geq 2$, a contradiction.

Therefore, $\mathcal{E}\left(N\mathcal{H}^l(D_1)\right) = \emptyset$ and $\mathcal{E}\left(N\mathcal{H}^l(D_2)\right) = \big\{\pi_2(e) \,|\, e \in \mathcal{E}\big(N\mathcal{H}^l(D_1 + D_2)\big)\big\}$.

Case (3): $\forall \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_2(e)| = 1$ and $\exists\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$: $|\pi_1(e)| \geq 2$.

This can be treated in the same way as Case (2).

Case (4): $\exists\ (i, j) \in V_1 \times V_2\ \forall e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : (i, j) \notin e$. Since for every $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ we have $(i, j) \notin e$, the vertex $i \in V_1$ is an isolate in

$N\mathcal{H}^l(D_1)$ and in D_1. For the same reason, $j \in V_2$ is an isolate in $N\mathcal{H}^l(D_2)$ and in D_2. We discuss only the construction of $N\mathcal{H}^l(D_2)$, the rest follows analogously.

Since i is an isolate, in $D_1 + D_2$ there is no arc between the i-th row Z_i and any other row. Therefore, all arcs with an initial or a terminal vertex in Z_i result from arcs in D_2 and we have

$$\forall a \in A(D_1 + D_2) : V(a) \cap Z_i \neq \emptyset \Rightarrow V(a) \subseteq Z_i.$$

Hence, denoting by $\langle Z_i \rangle_{D_1+D_2}$ and by $\langle Z_i \rangle_{N\mathcal{H}^l(D_1+D_2)}$ the subdigraph of $D_1 + D_2$ and the subhypergraph of $N\mathcal{H}^l(D_1 + D_2)$ generated by the vertices of Z_i, respectively, we obtain

- $\langle Z_i \rangle_{D_1+D_2} \simeq D_2$,
- $\langle Z_i \rangle_{N\mathcal{H}^l(D_1+D_2)} \simeq N\mathcal{H}^l(D_2)$ and
- $\mathcal{E}\left(N\mathcal{H}^l(D_2)\right) = \left\{\pi_2(e) \mid e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) \wedge e \subseteq Z_i\right\}$. ■

Note that, being interested in l-niche hypergraphs, loops $e = \{(i, j)\} \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ could lead to the problem that $\{(i, j)\}$ can be a loop in $N\mathcal{H}^l(D_1 + D_2)$ either because of $\{i\} \in \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and j is an isolate in D_2 or because of i is an isolate in D_1 and $\{j\} \in \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$ — and without further information it cannot be decided which of theses cases occurs.

In comparison with Proposition 2(4) of our paper [20] we see that for the reconstruction of the l-competition graphs $C\mathcal{H}^l(D_1)$ and $C\mathcal{H}^l(D_2)$ from $C\mathcal{H}^l(D_1 + D_2)$ there is another sufficient condition, namely:

$$\exists\, e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 + D_2)\right) : |\pi_1(e)| \geq 3 \,\wedge\, |\pi_2(e)| \geq 3.$$

Remark 8. In general, for niche hypergraphs an analogous condition to Proposition 2(4) in [20], i.e.,

$$(\alpha) \qquad \exists\, e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right) : |\pi_1(e)| \geq 3 \,\wedge\, |\pi_2(e)| \geq 3$$

is unsuited to ensure that $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ can be reconstructed from $N\mathcal{H}^l(D_1 + D_2)$.

Proof. Without loss of generality, let $e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\}$ be a hyperedge in $N\mathcal{H}^l(D_1 + D_2)$ with $k \geq 2$ and $l \geq 2$.

There are two possibilities for the hyperedge e, namely $e = \left\{\begin{array}{l} N^-_{D_1+D_2}((i,j)) \\ N^+_{D_1+D_2}((i,j)) \end{array}\right.$, i.e.,

$$\pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = \left\{\begin{array}{l} N^-_{D_1}(i) \\ N^+_{D_1}(i) \end{array}\right., \text{ and}$$

$$\pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = \left\{\begin{array}{l} N^-_{D_2}(j) \\ N^+_{D_2}(j) \end{array}\right..$$

Then we have $e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 + D_2)\right)$, which is equivalent to $e = N^-_{D_1+D_2}((i,j))$, or otherwise $e \in \mathcal{E}\left(CE\mathcal{H}^l(D_1 + D_2)\right)$, i.e., $e = N^+_{D_1+D_2}((i,j))$. In the first case it follows $\pi_1(e) \setminus \{i\} = N^-_{D_1}(i)$ and $\pi_2(e) \setminus \{j\} = N^-_{D_2}(j)$, in the second case $\pi_1(e) \setminus \{i\} = N^+_{D_1}(i)$ and $\pi_2(e) \setminus \{j\} = N^+_{D_2}(j)$ is valid.

In both cases we obtain $\pi_1(e) \setminus \{i\} \in \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $\pi_2(e) \setminus \{j\} \in \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$ and both sets $\pi_1(e) \setminus \{i\}$ and $\pi_2(e) \setminus \{j\}$ are hyperedges in the corresponding competition hypergraph $C\mathcal{H}^l(D_\tau)$ ($\tau \in \{1,2\}$) or both are hyperedges in the common enemy hypergraph $CE\mathcal{H}^l(D_\tau)$ ($\tau \in \{1,2\}$).

Our argumentation is the following.

• The above implies that, in this sense, *"competition hyperedges"* $e \in \mathcal{E}\big(C\mathcal{H}^l(D_1 + D_2)\big) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ include only information on "competition hyperedges" in $\mathcal{E}\left(C\mathcal{H}^l(D_1)\right) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $\mathcal{E}\left(C\mathcal{H}^l(D_2)\right) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$, respectively. The same applies to *"common enemy hyperedges"* $e \in \mathcal{E}\big(CE\mathcal{H}^l(D_1 + D_2)\big) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ and "common enemy hyperedges" in $\mathcal{E}\left(CE\mathcal{H}^l(D_1)\right) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $\mathcal{E}\left(CE\mathcal{H}^l(D_2)\right) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

• Below, we will describe the reconstruction of the hyperedges of $C\mathcal{H}^l(D_1)$ and $C\mathcal{H}^l(D_2)$ from $C\mathcal{H}^l(D_1 + D_2)$ according to Case 4 of the proof of Proposition 2 in [20]. We will see that in this reconstruction procedure the conditions $|\pi_1(e)| \geq 3$ and $|\pi_2(e)| \geq 3$ (for a certain hyperedge $e \in \mathcal{E}(C\mathcal{H}^l(D_1+D_2))$) are essential. Obviously, an analog reconstruction procedure can be used to obtain $CE\mathcal{H}^l(D_1)$ and $CE\mathcal{H}^l(D_2)$ from $CE\mathcal{H}^l(D_1+D_2)$, if there is a hyperedge $e \in \mathcal{E}\left(CE\mathcal{H}^l(D_1 + D_2)\right)$ with $|\pi_1(e)| \geq 3$ and $|\pi_2(e)| \geq 3$. Clearly, the described reconstruction will fail if there is no such hyperedge e with the required properties.

• Now let D_1 and D_2 be digraphs fulfilling (α). Note that, in general, for an arbitrarily chosen hyperedge e in $N\mathcal{H}^l(D_1+D_2)$ it cannot be found out whether e is a "competition hyperedge", i.e., $e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 + D_2)\right)$, or a "common enemy hyperedge", i.e., $e \in \mathcal{E}\left(CE\mathcal{H}^l(D_1 + D_2)\right)$.

• We additionally assume that in $N\mathcal{H}^l(D_1 + D_2)$ all hyperedges fulfilling (α) are edges of the competition hypergraph $C\mathcal{H}^l(D_1 + D_2)$ but not edges of the common enemy hypergraph $CE\mathcal{H}^l(D_1 + D_2)$. Then, clearly, the reconstruction method from Proposition 2 in [20] has to fail for hyperedges in $\mathcal{E}\left(CE\mathcal{H}^l(D_2)\right) \setminus \mathcal{E}\left(C\mathcal{H}^l(D_2)\right) \subseteq \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$.

It remains to describe the reconstruction method from Case 4 of the proof of Proposition 2 in [20].

Under the assumptions given above, let $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 + D_2)\right)$ be a hyperedge with (α), i.e., $e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 + D_2)\right)$. Because of $|\pi_1(e)| \geq 3$ and $|\pi_2(e)| \geq 3$, there are vertices $i \in V_1$ and $j \in V_2$ with $k := |\{(i,j') \,|\, j' \in V_2\} \cap e| \geq 2$ and $l := |\{(i',j) \,|\, i' \in V_1\} \cap e| \geq 2$.

Then $e = \{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j)\} = N^-_{D_1+D_2}((i, j))$ and therefore $N^-_{D_1}(i) = \{i_1, \ldots, i_l\} = \pi_1(e) \setminus \{i\}$ and $N^-_{D_2}(j) = \{j_1, \ldots, j_k\} = \pi_2(e) \setminus \{j\}$.

For each $x \in V_1$ let $e^x := \{(x, j_1), \ldots, (x, j_k), (x_1, j), \ldots, (x_{l_x}, j)\} \in \mathcal{E}\big(C\mathcal{H}^l(D_1 + D_2)\big)$ with $l_x \geq 0$. Obviously, $e^x = N^-_{D_1+D_2}((x, j))$ and $N^-_{D_1}(x) = \{x_1, \ldots, x_{l_x}\} = \pi_1(e^x) \setminus \{x\}$. This way we obtain $D_1 = (V_1, A_1)$ as well as $\mathcal{E}\left(C\mathcal{H}^l(D_1)\right) = \Big\{N^-_{D_1}(x) \,|\, x \in V_1 \wedge N^-_{D_1}(x) \neq \emptyset\Big\}$.

Analogously, for each $y \in V_2$ let $e^y := \{(i_1, y), \ldots, (i_l, y), (i, y_1), \ldots, (i, y_{k_y})\} \in \mathcal{E}\left(C\mathcal{H}^l(D_1 + D_2)\right)$ with $k_y \geq 0$. Then $e^y = N^-_{D_1+D_2}((i, y))$ and $N^-_{D_2}(y) = \{y_1, \ldots, y_{k_y}\} = \pi_2(e^y) \setminus \{y\}$. ∎

Theorem 9 (Normal product $D_1 * D_2$)**.**

(a) *$N\mathcal{H}(D_1)$ and $N\mathcal{H}(D_2)$ can be obtained from $N\mathcal{H}(D_1 * D_2)$.*

(b) *If there is a hyperedge $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$, then $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ can be obtained from $N\mathcal{H}(D_1 * D_2)$.*

Proof. (b) The existence of a hyperedge $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$ is equivalent to $A_1 \neq \emptyset \neq A_2$. Let

$$\begin{aligned} e = \ &\big\{(i, j_1), \ldots, (i, j_k), (i_1, j), \ldots, (i_l, j), (i_1, j_1), (i_1, j_2), \ldots, (i_1, j_k), \ldots, (i_l, j_1), \\ &\quad (i_l, j_2), \ldots, (i_l, j_k)\big\} \\ &\in \mathcal{E}(N\mathcal{H}(D_1 * D_2)) = \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \cup \mathcal{E}(CE\mathcal{H}(D_1 * D_2)), \end{aligned}$$

with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$.

We will follow the idea of the proof of Case 2 of Corollary 2 in our paper [20], where a similar result for competition hypergraphs was given.

But by contrast to Corollary 2 in [20], in the case of niche hypergraphs it is impossible to reconstruct the digraphs D_1 and D_2 themselves in general. The reason is the same as mentioned before for the Cartesian sum (see the proof of Remark 8). Although for a hyperedge $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ we can find out the vertex (i, j) with $e = N^-_{D_1 * D_2}((i, j))$ or $e = N^+_{D_1 * D_2}((i, j))$, in general it will be impossible to determine whether e is the set of predecessors (e is a "competition hyperedge") or the set of successors (e is a "common enemy hyperedge") of the vertex (i, j) in $D_1 * D_2$.

Note that, in spite of the distinction of cases below, it is unnecessary to know for the actual hyperedge $e \in \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ under investigation whether or not it is a "competition hyperedge" ($e \in \mathcal{E}(C\mathcal{H}(D_1 * D_2))$) or it is an "common enemy hyperedge" ($e \in \mathcal{E}(CE\mathcal{H}(D_1 * D_2))$). This will become clear by the remarks to Case (2) below.

Case (1): $e \in \mathcal{E}(C\mathcal{H}(D_1 * D_2))$. With some modifications of the proof of Case 2 of Corollary 2 in [20] we get the following.

(a) Because of $l = |\pi_1(e)| - 1 \geq 1$ and $k = |\pi_2(e)| - 1 \geq 1$, the vertices $i \in V_1$ and $j \in V_2$ with $N^-_{D_1 * D_2}((i,j)) = e$ can be identified as the only vertices which occur exactly k and l times in $\pi_1(e)$ and $\pi_2(e)$, respectively. Moreover, $\pi_1(e) \setminus \{i\} = \{i_1, \ldots, i_l\} = N^-_{D_1}(i)$ and $\pi_2(e) \setminus \{j\} = \{j_1, \ldots, j_k\} = N^-_{D_2}(j)$.

(b) Obviously, for every $x \in V_1$ with $N^-_{D_1}(x) \neq \emptyset$ in $N^-_{D_1*D_2}((x,j))$ there are at least 3 vertices: $(x,j_1),(x',j),(x',j_1)$, where $x' \in N^-_{D_1}(x)$. Therefore $N^-_{D_1*D_2}((x,j)) \in \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \subseteq \mathcal{E}(N\mathcal{H}(D_1 * D_2))$. Analogously, for each $y \in V_2$ with $N^-_{D_2}(y) \neq \emptyset$ we get $N^-_{D_1*D_2}((i,y)) \in \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \subseteq \mathcal{E}(N\mathcal{H}(D_1 * D_2))$.

(c) Note that if $x \in V_1$ with $N^-_{D_1}(x) = \emptyset$, then $N^-_{D_1*D_2}((x,j)) = \{(x,j_1), \ldots, (x,j_k)\}$; i.e., $N^-_{D_1*D_2}((x,j)) \in \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \subseteq \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ if and only if $k \geq 2$. Analogously, for every $y \in V_2$ with $N^-_{D_2}(y) = \emptyset$ it follows $N^-_{D_1*D_2}((i,y)) \in \mathcal{E}(C\mathcal{H}(D_1 * D_2)) \subseteq \mathcal{E}(N\mathcal{H}(D_1 * D_2))$ if and only if $l \geq 2$.

Because of (b), for all vertices of D_1 and D_2, respectively, with positive indegree we get their sets of predecessors applying the procedure described in (a) to all hyperedges $e \in \mathcal{E}(C\mathcal{H}(D_1 * D_2))$ with $|\pi_1(e)| \geq 2$ and $|\pi_2(e)| \geq 2$. (In general, for a vertex $v_1 \in V_1$ and $v_2 \in V_2$, respectively, with positive indegree, procedure (a) will produce its set of predecessors more than once.) Trivially, each vertex for which (a) does not provide a set of predecessors has indegree 0 (cf. (c)).

Thus we obtain the edge set $\mathcal{E}\left(C\mathcal{H}^l(D_1)\right)$ and $\mathcal{E}\left(C\mathcal{H}^l(D_2)\right)$ of the l-competition hypergraph $C\mathcal{H}^l(D_1)$ and $C\mathcal{H}^l(D_2)$, respectively.

Note that we did not need hyperedges $e \in \mathcal{E}\left(C\mathcal{H}^l(D_1 * D_2)\right) \setminus \mathcal{E}\big(C\mathcal{H}(D_1 * D_2)\big)$, i.e., hyperedges of cardinality 1.

Case (2): $e \in \mathcal{E}\left(CE\mathcal{H}(D_1 * D_2)\right)$. Note that $C\mathcal{H}(D) = CE\mathcal{H}\left(\overleftarrow{D}\right)$, for any digraph D. Applying the following substitutions to the proof of Case (1), word-for-word we obtain the verification of Case (2):

$$
\begin{array}{llll}
C\mathcal{H} & \hookrightarrow & CE\mathcal{H}, & \\
N^- & \hookrightarrow & N^+, & \\
\text{indegree} & \hookrightarrow & \text{outdegree} & \text{and} \\
\text{predecessor} & \hookrightarrow & \text{successor}. &
\end{array}
$$

(a) Because of (b) it suffices to consider the case when $A_1 = \emptyset$ or $A_2 = \emptyset$ holds. Replacing "+" by "$*$" in (1)–(3) of Theorem 7, we see that the occurrence of (1), (2) or (3) is equivalent to $A_1 = \emptyset$ or $A_2 = \emptyset$ and we can use an analog argumentation as in the corresponding part of the proof of Theorem 7. So using (2) we obtain $\mathcal{E}\left(N\mathcal{H}^l(D_2)\right) = \left\{\pi_2(e) \mid e \in \mathcal{E}(N\mathcal{H}^l(D_1 * D_2))\right\}$ and $\mathcal{E}\left(N\mathcal{H}(D_2)\right) = \left\{\pi_2(e) \mid e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) \wedge |\pi_2(e)| \geq 2\right\}$, respectively. ■

Note that $A_1 = \emptyset$ or $A_2 = \emptyset$ implies $D_1 * D_2 = D_1 + D_2$. Therefore, the last part of the above proof in connection with Theorem 7 lead to the following consequence.

Corollary 10. *$N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ can be obtained from $N\mathcal{H}^l(D_1 * D_2)$, provided that one of the following conditions is true:*
(1) $\mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) = \emptyset$;
(2) $\forall e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) : |\pi_1(e)| = 1$ *and* $\exists e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) : |\pi_2(e)| \geq 2$;
(3) $\forall e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) : |\pi_2(e)| = 1$ *and* $\exists e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 * D_2)\right) : |\pi_1(e)| \geq 2$.

Theorem 11 (Lexicographic product $D_1 \cdot D_2$)**.**
(a) *$N\mathcal{H}(D_1)$ and $N\mathcal{H}(D_2)$ can be obtained from $N\mathcal{H}(D_1 \cdot D_2)$.*
(b) *If $|V_2| \geq 2$, then $N\mathcal{H}^l(D_1)$ can be obtained from $N\mathcal{H}(D_1 \cdot D_2)$.*
(c) *$N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ can be obtained from $N\mathcal{H}^l(D_1 \cdot D_2)$.*

Proof. First we will show (c), i.e., $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ can be reconstructed from $N\mathcal{H}^l(D_1 \cdot D_2)$. Then we obtain (b) and (a) as follows:

Since for $|V_2| \geq 2$ every loop $e_1 = \{i\}$ in $N\mathcal{H}^l(D_1)$ leads to a non-loop e in $N\mathcal{H}^l(D_1 \cdot D_2)$ (containing at least all vertices of the row Z_i), we will see that we need no loops of $N\mathcal{H}^l(D_1 \cdot D_2)$ in order to obtain $N\mathcal{H}^l(D_1)$, this includes (b).

Analogously, it is obvious that non-loops e_i of $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$, respectively, result in non-loops in $N\mathcal{H}^l(D_1 \cdot D_2)$. In our considerations it will become clear that for the reconstruction of $N\mathcal{H}(D_1)$ and $N\mathcal{H}(D_2)$ we do not need the loops in $N\mathcal{H}^l(D_1 \cdot D_2)$, so we get (a).

In order to prove (c), we consider a hyperedge $e \in \mathcal{E}\left(N\mathcal{H}^l(D_1 \cdot D_2)\right)$. Then there is a vertex $(i,j) \in V_1 \times V_2$ such that $e = N^-_{D_1 \cdot D_2}((i,j))$ or $e = N^+_{D_1 \cdot D_2}((i,j))$. In order to simplify our depictions, we write down the considerations only for the case $e = N^-_{D_1 \cdot D_2}((i,j)) \in \mathcal{E}\left(C\mathcal{H}^l(D_1 \cdot D_2)\right)$; the hyperedges $e = N^+_{D_1 \cdot D_2}((i,j)) \in \mathcal{E}\left(CE\mathcal{H}^l(D_1 \cdot D_2)\right)$ can be treated analogously.

In $N\mathcal{H}^l(D_1 \cdot D_2)$ there are two possibilities for the hyperedge e.

Case 1. $\exists\, l \geq 1\; \exists\, i_1, \ldots, i_l \in V_1 : e = Z_{i_1} \cup \cdots \cup Z_{i_l}$. Without loss of generality let $i_1, \ldots, i_l$ be pairwise distinct.

Hence, e is the union of the complete rows $Z_{i_1}, \ldots, Z_{i_l}$ of $D_1 \cdot D_2$ and from the definition of $D_1 \cdot D_2$ it follows $i \notin \{i_1, \ldots, i_l\}$, $N^-_{D_1}(i) = \{i_1, \ldots, i_l\}$ and $N^-_{D_2}(j) = \emptyset$.

Therefore, Case 1 does not provide any hyperedges of $N\mathcal{H}^l(D_2)$ but with $\pi_1(e) = \{i_1, \ldots, i_l\} = N^-_{D_1}(i) \in \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ we obtain a hyperedge of $N\mathcal{H}^l(D_1)$.

Note that the vertex $i \in V_1$ is unknown if $l < |V_1| - 1$. Moreover, Case 1 occurs if and only if there exists a vertex $j \in V_2$ with $N^-_{D_2}(j) = \emptyset$.

Case 2. $\exists\, l \geq 0\; \exists\, i_1, \ldots, i_l, i' \in V_1\; \exists\, Z' \subset Z_{i'} : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \cup Z' \wedge Z' \neq \emptyset$. We get $i = i' \in V_1 \setminus \{i_1, \ldots, i_l\}$ as well as $N^-_{D_1}(i') = \{i_1, \ldots, i_l\} = \pi_1(e) \setminus \{i'\} \in$

$\mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$ and $N^-_{D_2}(j) = \pi_2(e \cap Z') = \pi_2(Z') \in \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$ with a certain $j \in V_2$. In general, if $|Z'| < |V_2| - 1$ holds, the vertex j cannot be determined.

Again, for any hyperedge $e \in \mathcal{E}(N\mathcal{H}^l(D_1 \cdot D_2))$ it cannot be found out whether e is a *competition hyperedge* (i.e., $e \in \mathcal{E}(C\mathcal{H}^l(D_1 \cdot D_2))$) or e is a *common enemy hyperedge* (i.e., $e \in \mathcal{E}(CE\mathcal{H}^l(D_1 \cdot D_2))$) in general. But for the reconstruction of $N\mathcal{H}^l(D_1)$ and $N\mathcal{H}^l(D_2)$ this plays no role, since the considerations of Case 1 and Case 2 are valid for competition hyperedges (i.e., sets of predecessors) as well as, analogously, for common enemy hyperedges (i.e., sets of successors).

Moreover, we remark that Cases 1 and 2 (together with their analogs for the common enemy hyperedges) provide all hyperedges of the (l)-niche hypergraphs $N\mathcal{H}^{(l)}(D_1)$ and $N\mathcal{H}^{(l)}(D_2)$. ∎

Now we discuss the disjunction $D_1 \vee D_2$. The case $|V_1| = 1$ or $|V_2| = 1$ implies $D_1 \vee D_2 = D_1 \cdot D_2$. Therefore, because of Theorem 11 it suffices to investigate the case $|V_1|, |V_2| \geq 2$.

Theorem 12 (Disjunction $D_1 \vee D_2$)**.** *If* $|V_1|, |V_2| \geq 2$, *then* $N\mathcal{H}^l(D_1)$ *and* $N\mathcal{H}^l(D_2)$ *can be obtained from* $N\mathcal{H}(D_1 \vee D_2)$.

Proof. Since both V_1 and V_2 contain at least two vertices, in $N\mathcal{H}^l(D_1 \vee D_2)$ there are no loops and $N\mathcal{H}^l(D_1 \vee D_2) = N\mathcal{H}(D_1 \vee D_2)$.

Moreover, for every hyperedge $e \in \mathcal{E}(N\mathcal{H}(D_1 \vee D_2))$ it holds

$$\exists\, l \geq 0\, \exists\, i_1, \ldots, i_l \in V_1\, \exists\, k \geq 0\, \exists\, j_1, \ldots, j_k \in V_2 : e = Z_{i_1} \cup \cdots \cup Z_{i_l} \cup S_{j_1} \cup \cdots \cup S_{j_k}$$

and, clearly, $\min(l, k) > 0$.

By analogy with the proof of Theorem 11 let $(i, j) \in V_1 \times V_2$ be a vertex such that $e = N^-_{D_1 \vee D_2}((i, j))$ or $e = N^+_{D_1 \vee D_2}((i, j))$. Now we follow the idea of the proof of Proposition 2 in [20], subsection 3.5, and use the abbreviations $\mathcal{E}^l_1 := \mathcal{E}\left(N\mathcal{H}^l(D_1)\right)$, $\mathcal{E}^l_2 := \mathcal{E}\left(N\mathcal{H}^l(D_2)\right)$ and $\mathcal{E}_\vee := \mathcal{E}(N\mathcal{H}(D_1 \vee D_2))$.

In case of $\mathcal{E}_\vee = \emptyset$ both $\mathcal{E}^l_1$ and $\mathcal{E}^l_2$ are empty, too.

So let $\mathcal{E}_\vee \neq \emptyset$. Additionally, for an arbitrary hyperedge $e \in \mathcal{E}_\vee$ we define $\pi^j_1(e) := \{i \,|\, (i, j) \in e\}$ (for $j \in \pi_2(e)$) and $\pi^i_2(e) := \{j \,|\, (i, j) \in e\}$ (for $i \in \pi_1(e)$).

In $N\mathcal{H}(D_1 \vee D_2)$ we have three types of hyperedges:
$\mathcal{A} := \{e \in \mathcal{E}_\vee \,|\, \pi_1(e) \subset V_1\}$,
$\mathcal{B} := \{e \in \mathcal{E}_\vee \,|\, \pi_2(e) \subset V_2\}$ and
$\mathcal{C} := \{e \in \mathcal{E}_\vee \,|\, \pi_1(e) = V_1 \,\wedge\, \pi_2(e) = V_2\}$.

We obtain
$\mathcal{A} = \mathcal{C} = \emptyset$ if and only if $A_1 = \emptyset$, $\mathcal{E}^l_1 = \emptyset$ and $\mathcal{E}^l_2 = \{\pi_2(e) \,|\, e \in \mathcal{E}_\vee\}$;
$\mathcal{B} = \mathcal{C} = \emptyset$ if and only if $A_2 = \emptyset$, $\mathcal{E}_2 = \emptyset$ and $\mathcal{E}^l_1 = \{\pi_1(e) \,|\, e \in \mathcal{E}_\vee\}$;
$\mathcal{C} \neq \emptyset$ if and only if $A_1 \neq \emptyset \neq A_2$.

It remains to investigate the case $\mathcal{C} \neq \emptyset$. Here we see that, to determine $\mathcal{E}^l_1$ and $\mathcal{E}^l_2$, it suffices to make use of the hyperedges in $\mathcal{C}$:

$$\mathcal{E}_1^l = \left\{ \{i \in V_1 \mid \pi_2^i(e) = V_2\} \mid e \in \mathcal{C} \right\} \text{ and } \mathcal{E}_2^l = \left\{ \{j \in V_2 \mid \pi_1^j(e) = V_1\} \mid e \in \mathcal{C} \right\}.$$

(Note that in case $\mathcal{A} \neq \emptyset$ we have $\mathcal{E}_1^l = \{\pi_1(e) \mid e \in \mathcal{A}\}$ and, analogously, if $\mathcal{B} \neq \emptyset$ it follows $\mathcal{E}_2^l = \{\pi_2(e) \mid e \in \mathcal{B}\}$.) ■

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