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POWER DOMINATION IN THE GENERALIZED PETERSEN GRAPHS[1]

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Abstract

The problem of monitoring an electric power system by placing as few measurement devices in the system can be formulated as a power dominating set problem in graph theory. The power domination number of a graph is the minimum cardinality of a power dominating set. Xu and Kang [*On the power domination number of the generalized Petersen graphs*, J. Comb. Optim. 22 (2011) 282–291] study the exact power domination number for the generalized Petersen graph $P(3k, k)$, and propose the following problem: determine the power domination number for the generalized Petersen graph $P(4k, k)$ or $P(ck, k)$. In this paper we give the power domination number for $P(4k, k)$ and present a sharp upper bound on the power domination number for the generalized Petersen graph $P(ck, k)$.

Keywords: power domination, domination, generalized Petersen graph, electric power system.

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1. Introduction

Electric power companies need to continually monitor the state of their systems as in the case of voltage magnitude at loads and machine phase angle at generators. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. Due to the high cost of the PMUs, the number of PMUs used to monitor the network must be minized.

The power system monitoring problem can be formulated as a domination problem in graph theory by Haynes *et al.* in [7]. Let $G = (V(G), E(G))$ be a graph representing an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed, its incident edges and their ends. All these vertices and edges are said to be *observed* by the PMU. We can apply Ohm's law and Kirchoff's current law to deduce the other three observation rules.

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k \geq 2$ edges and if $k-1$ of these edges are observed, then all k of these edges are observed.

We consider only graphs without loops or multiple edges. For a vertex v of $G = (V(G), E(G))$, let $N(v)$ denote the open neighborhood of v, and for a subset $S \subseteq V(G)$, let $N(S) = \bigcup_{v\in S} N(v) \setminus S$. The closed neighborhood $N[S]$ of a subset S is the set $N[S] = N(S) \cup S$. For any $X \subseteq V(G)$, the subgraph induced in G by $X \subseteq V(G)$ is denoted by $G[X]$ and $E(X)$ denotes the edge set of $G[X]$. For other terminology and notation not given here, we refer to [4, 7, 8].

A *dominating set* of a graph G is a set S of vertices of G such that every vertex (node) in $V \setminus S$ has at least one neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The theory of dominating sets in graphs is well developed (see, for example, [8]). Considering the power system monitoring problem as a variation of the dominating set problem, a set S is a *power dominating set* (PDS) if every vertex and every edge in G is monitored by S after applying the observation rules. The *power domination number* of G, denoted by $\gamma_P(G)$, is the minimum cardinality of a power dominating set of G. A power dominating set of G with minimum cardinality is called a $\gamma_P(G)$-set.

Let G be a connected graph and S a subset of its vertices. In [3], Brueni and Heath first provided a new simplified definition of the observation rules that requires only 2 rules. In this paper, we shall use the following equivalent algorithm [4]. We denote by $M(S)$ the set of vertices in G that is *monitored* by S.

Algorithm 1: An alternative approach to the observation rules

Input: A connected graph $G(V, E)$ and $S \subseteq V(G)$
Output: The set $M(S)$ monitored by S

1.*Domination Step*
  $M(S) \leftarrow S \cup N(S)$;

2.*Propagation Step*
foreach $w \in V(G) \setminus M(S)$ **do**
  if *there exists* $v \in M(S)$ *such that* $N(v) \cap (V(G) \setminus M(S)) = \{w\}$ **then**
    $M(S) \leftarrow M(S) \cup \{w\}$;
  end
end

Output $M(S)$;

Power domination in graphs was introduced and studied by Haynes *et al.* in [7]. It has received considerable attention from the algorithmic point of view. Haynes *et al.* [7] showed that the power dominating set problem is NP-complete even when restricted to bipartite graphs or chordal graphs and provided a linear algorithm to solve the PDS for trees. There are polynomial algorithms to solve this problem for graphs with bounded treewidth [6], block graphs [15], block-cactus graphs [9], interval graphs [10], grids [12], honeycomb meshes [13] and circular-arc graphs [11]. On the other hand, upper bounds on the power domination number are given for connected graphs with at least three vertices, for connected claw-free cubic graphs [19], for hypercubes [3], and for generalized Petersen graphs [1]. Dorbec *et al.* [4] determined the power domination number for product graphs.

As a generalization of the well-known Petersen graph, the generalized Petersen graph has attracted much attention. The *generalized Petersen graph* $P(n, k)$ $(k \geq 1)$ is the graph with vertex set $U \cup V$, where $U = \{u_i \mid 1 \leq i \leq n\}$ and $V = \{v_i \mid 1 \leq i \leq n\}$, and edge set $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 1 \leq i \leq n\}$, where the subscripts are to be read as integers modulo n. The graph $P(5, 2)$ is the Petersen graph. Domination and its variations have been extensively investigated in the class of generalized Petersen graphs in [2, 5, 14, 17, 18].

Xu and Kang [16] gave a sharp upper bound on power domination number for generalized Petersen graph $P(n, k)$ and determined the exact power domination number for $P(3k, k)$. They posed the following open problem: find the exact power domination number for the generalized Petersen graph $P(4k, k)$ (or even for $P(ck, k)$, where $c \geq 4$ is a constant integer). In this paper we give an upper bound on the power domination number for $P(ck, k)$ and determine the exact power domination number for $P(4k, k)$.

The remainder of this paper is organized as follows. In Section 2, we prove $\gamma_P(P(ck, k)) \leq \lceil \frac{2k+2}{3} \rceil$ for integer $k \geq 2$ and $c \geq 4$ by providing an explicit construction for the upper bounds. In Section 3, we first show that $\gamma_P(P(4k,k)) \leq \lceil \frac{2k}{3} \rceil$ for $k \geq 4$ and $\gamma_P(P(4(k-3),k-3)) \leq \gamma_P(P(4k,k)) - 2$ for $k \geq 7$. Then using the above results we prove that $\gamma_P(P(4k,k)) \geq \lceil \frac{2k}{3} \rceil$ for $k \geq 4$ and determine the exact power domination number for $P(4k,k)$. In Section 4, we conclude this paper and propose a conjecture.

2. An Upper Bound on $\gamma(P(ck,k))$

For the generalized Petersen graph $P(ck,k)$, when $k=1$, it is easily seen that $\gamma_P(P(2,1)) = \gamma_P(P(3,1)) = 1$ and $\gamma_P(P(n,1)) = 2$ for $n \geq 4$. When $k \geq 2$, Xu and Kang [16] showed that $\gamma_P(P(n,k)) \leq \min\left\{\lceil \frac{n}{3} \rceil, k\right\}$ for $n \geq 4$.

We now restrict our attention to the generalized Petersen graph $P(ck,k)$ for $k \geq 2$. We give a sharp upper bound on $\gamma_P(P(ck,k))$ for $k \geq 2$, $c \geq 4$. The bound improves the Xu and Kang's bound for $P(n,k)$.

Theorem 1. *For $k \geq 2$ and $c \geq 4$, $\gamma_P(P(ck,k)) \leq \lceil \frac{2k+2}{3} \rceil$, and this bound is sharp.*

Proof. To obtain the upper bound, we directly construct the power dominating set for $P(ck,k)$ with $|S| = \lceil \frac{2k+2}{3} \rceil$.

Let $2k = 3a+b$ with $b \in \{0,1,-1\}$ and set $S = \{u_1, u_4, \ldots, u_{3a+1}\}$. We have $|S| = a+1 = \lceil \frac{2k+2}{3} \rceil$. Next, we show that S is a PDS for $P(ck,k)$. In Figure 1, the following procedure is shown when $2k = 3a$.

(1) After the domination step, the set $S \cup N(S) = \{u_{ck}, u_1, u_2, \ldots, u_{3a+2}\} \cup \{v_1, v_4, \ldots, v_{3a+1}\}$ (i.e., vertices with bold circles in Figure 1) is monitored.

(2) For $2 \leq i \leq 3a$, $i \not\equiv 1 \pmod 3$, the vertex u_i monitors v_i (i.e., vertices with gray circles in Figure 1) by the propagation step. And as a result $\{u_1, u_2, \ldots, u_{3a+2}\} \cup \{v_1, v_2, \ldots, v_{3a+1}\} \supseteq \{u_1, u_2, \ldots, u_{2k+1}\} \cup \{v_1, v_2, \ldots, v_{2k}\}$ is monitored.

(3) For $t = 1, 2, \ldots, c-2$,

- first, for $i = 1, \ldots, k$, vertex v_{tk+i} monitors $v_{(t+1)k+i}$ by the propagation (i.e., vertices contained in the dotted rectangle in Figure 1 are monitored when $t=1$), and then
- for $i = 1, \ldots, k$, vertex $u_{(t+1)k+i}$ monitors $u_{(t+1)k+i+1}$ by the propagation (i.e., vertices contained in the solid ellipse in Figure 1 are monitored when $t=1$).

So S is a PDS for $P(ck,k)$ and $\gamma_P(P(ck,k)) \leq |S| = \lceil \frac{2k+2}{3} \rceil$. By Theorem 8 (we will prove it in Section 3), we have $\gamma_P(P(4k,k)) = \lceil \frac{2k}{3} \rceil$. Since $\lceil \frac{2k+2}{3} \rceil = \lceil \frac{2k}{3} \rceil$ when $2k \equiv 1 \pmod 3$, the bound is sharp. ∎

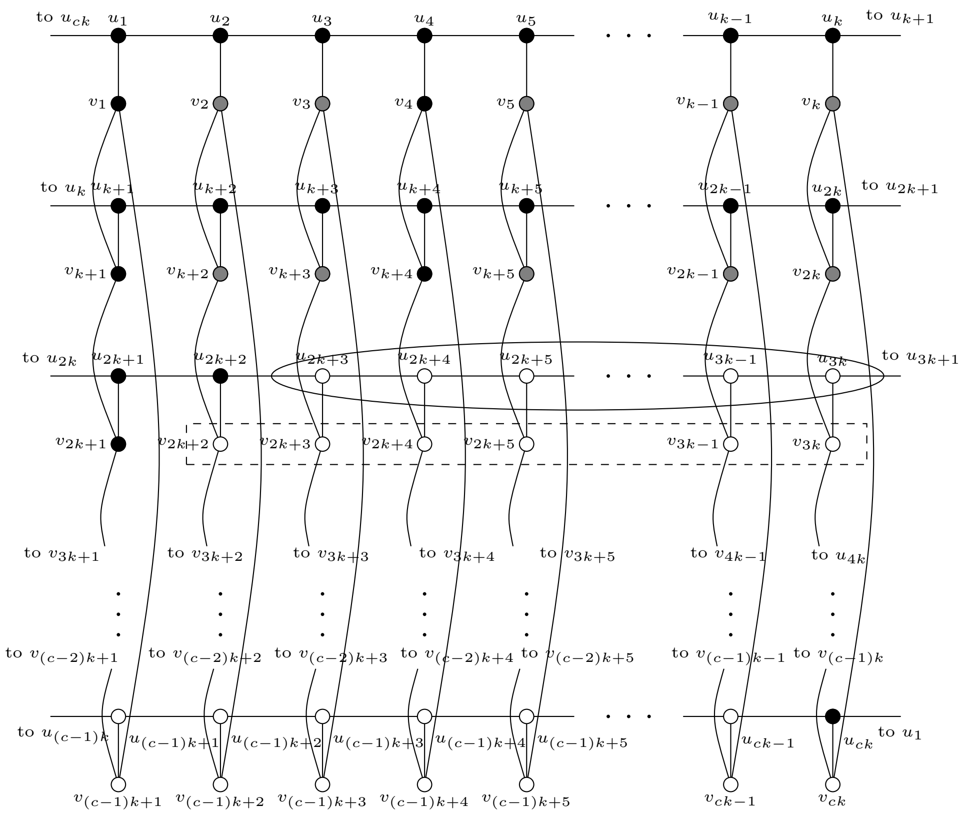


Figure 1. Vertices monitored by S for $P(ck, k)$ when $2k = 3a$ and $t = 1$.

3. THE POWER DOMINATION NUMBER FOR $P(4k, k)$

In this section we determine the power domination number for the generalized Petersen graph $P(4k, k)$. The main result is given in Theorem 8.

For $G = P(4k, k)$, we denote $F_i = \{u_i, v_i, u_{i+k}, v_{i+k}, u_{i+2k}, v_{i+2k}, u_{i+3k}, v_{i+3k}\}$ and $F_i^u = \{u_i, u_{i+k}, u_{i+2k}, u_{i+3k}\}$, $F_i^v = \{v_i, v_{i+k}, v_{i+2k}, v_{i+3k}\}$, where $1 \leq i \leq k$. In the following statements, the subscripts for u_i and v_i are used modulo $4k$, and the subscripts for F_i, F_i^u and F_i^v are used modulo k.

Observation 2. *For $G = P(4k, k)$, if all vertices of $\{u_i, v_i, u_{i+k}, v_{i+k}\}$ $(1 \leq i \leq k)$ are monitored by a set D, then $G[F_i]$ is monitored by D.*

Proof. We note that $N(v_{i+k}) = \{v_i, u_{i+k}, v_{i+2k}\}$ and $N(v_i) = \{u_i, v_{i+k}, v_{i+3k}\}$. The vertices of $N(v_{i+k})$ other than v_{i+2k} and the vertices of $N(v_i)$ other than v_{i+3k} are monitored by D during the domination step, so v_{i+2k} and v_{i+3k} are monitored by the propagation step. Similarly, the vertices of $N(v_{i+2k})$ other than u_{i+2k} and the vertices of $N(v_{i+3k})$ other than u_{i+3k} are monitored by domination step, so u_{i+2k} and u_{i+3k} can be monitored by the propagation. Hence, $G[F_i]$ is completely monitored by D. ■

We begin to study the power domination problem on $P(4k,k)$. First, we give an upper bound on $\gamma_P(P(4k,k))$.

Lemma 3. *For $k \geq 4$, $\gamma_P(P(4k,k)) \leq \lceil \frac{2k}{3} \rceil$.*

Proof. By Theorem 1, $\gamma_P(P(4k,k)) \leq \lceil \frac{2k}{3} \rceil$ for $2k \equiv 1 \pmod 3$ since $\lceil \frac{2k+2}{3} \rceil = \lceil \frac{2k}{3} \rceil$ when $2k \equiv 1 \pmod 3$. Now, we show that the result also holds for $2k \equiv 0$ or $2 \pmod 3$.

Let $2k = 3a+b$ with $b \in \{2,3\}$ and set $S = \{u_1, u_4, \ldots, u_{3a+1}\}$. We have $|S| = a+1 = \lceil \frac{2k}{3} \rceil$. Similar to the proof of Theorem 1, $\{u_1, u_2, \ldots, u_{3a+2}\} \cup \{v_1, v_2, \ldots, v_{3a+1}\}$ is monitored by S. So by the propagation, $\bigcup_{i=1}^{3a+1-k} F_i$ is monitored and as a result u_{2k}, v_{2k} are monitored by S. Hence S is a PDS for $P(4k,k)$.

Therefore, $\gamma_P(P(4k,k)) \leq |S| \leq \lceil \frac{2k}{3} \rceil$. ■

Next, we prove that $\lceil \frac{2k}{3} \rceil$ is also a lower bound on $\gamma_P(P(4k,k))$ for $k \geq 4$. To prove the result, we will show that $\gamma_P(P(4(k-3),k-3)) \leq \gamma_P(P(4k,k)) - 2$ for $k \geq 7$ (Lemma 5). The following property is required.

Lemma 4. *For $k \geq 7$, there is a γ_P-set S of $P(4k,k)$ such that*

$$\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 3.$$

Proof. We denote $M_j = \{u_{i+jk}, v_{i+jk} \mid i = 1,2,3,4,5\}$ for $j = 0,1,2,3$, and the subscripts are used modulo 4 in the following proof. For a γ_P-set S for $P(4k,k)$, we claim that $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 2$. Otherwise, if there is a unique vertex $w \in M_j \cap S$, then $u_{(j+2)k+3}$ cannot be monitored by S, a contradiction. So $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 2$.

If $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 3$, the result is proven. Suppose now that $\left|S' \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 2$ for a γ_P-set S' of $P(4k,k)$. We will prove that $|S' \cap F_3^u| = 2$, $\left|S' \cap \left(F_6 \cup F_7\right)\right| \geq 1$ and $\left|S' \cap \left(\bigcup_{i=3}^{7} F_i\right)\right| \geq 3$. By the symmetry of $P(4k,k)$, the result holds.

Let $S' \cap \left(\bigcup_{i=1}^{5} F_i\right) = \{w_1, w_2\}$. Then we claim that w_1 and w_2 are in M_j and M_{j+1}, respectively, for some $j = 0,1,2,3$. Otherwise, if $w_1, w_2 \in M_0 \cap S'$, then there is at most one vertex in $N(v_{k+3})$ (or $N(v_{3k+3})$) monitored by $\{w_1, w_2\}$, and v_{2k+3} cannot be monitored by $\{w_1, w_2\}$, a contradiction. So $|M_0 \cap S'| \leq 1$. Similarly, $|M_j \cap S'| \leq 1$ for each $j = 0,1,2,3$. If w_1 and w_2 are in M_j and M_{j+2} respectively for $j = 0$ or 1, then there is at least one vertex in $M_{j+1} \cup M_{j+3}$ which cannot be monitored by S', a contradiction. Hence w_1 and w_2 are in M_j and M_{j+1}, respectively, for some $j = 0,1,2,3$.

Next, we show that $|S' \cap F_3| = 2$. Otherwise, suppose $\big|S' \cap \big[\big(\bigcup_{i=1}^{2} F_i\big) \cup \big(\bigcup_{i=4}^{5} F_i\big)\big]\big| = 2$, then w_1 and w_2 are in $M_j \cap \big[\big(\bigcup_{i=1}^{2} F_i\big) \cup \big(\bigcup_{i=4}^{5} F_i\big)\big]$ and $M_{j+1} \cap \big[\big(\bigcup_{i=1}^{2} F_i\big) \cup \big(\bigcup_{i=4}^{5} F_i\big)\big]$, respectively, for $j = 0, 1, 2, 3$. Without loss of generality, let $w_1 \in M_j \cap (F_1 \cup F_2)$. If $w_2 \in M_{j+1} \cap (F_4 \cup F_5)$, then v_{jk+3} and $v_{(j+1)k+3}$ cannot be monitored by S', a contradiction. So $w_2 \in M_{j+1} \cap (F_1 \cup F_2)$ or $w_2 \in M_{j-1} \cap (F_1 \cup F_2)$ for $j = 1, 2, 3, 4$. Similarly, if $w_1 \in M_j \cap (F_4 \cup F_5)$, then $w_2 \in M_{j+1} \cap (F_4 \cup F_5)$ or $w_2 \in M_{j-1} \cap (F_4 \cup F_5)$ for $j = 1, 2, 3, 4$. Without loss of generality, let $w_1 \in M_0 \cap (F_1 \cup F_2)$ and $w_2 \in M_1 \cap (F_1 \cup F_2)$. Then v_4 and v_{k+4} cannot be monitored by S', a contradiction. So $\big|S' \cap \big[\big(\bigcup_{i=1}^{2} F_i\big) \cup \big(\bigcup_{i=4}^{5} F_i\big)\big]\big| \leq 1$ and $|S' \cap F_3| \geq 1$.

Without loss of generality, let $w_1 \in \{u_1, v_1, u_2, v_2\}$ and $w_2 \in \{u_{k+3}, v_{k+3}\}$. If $w_1 \in \{u_1, u_2\}$ and $w_2 = v_{k+3}$, then there is at most one vertex $u_1 \in N(v_1)$ (or $u_2 \in N(v_2)$) monitored by $\{w_1, w_2\}$, and v_{k+1} and v_{k+2} cannot be monitored by S', a contradiction. Similarly, if $w_1 = v_1$ and $w_2 = v_{k+3}$, then u_3 cannot be monitored by S', a contradiction; if $w_1 = v_2$ and $w_2 = v_{k+3}$, then u_{k+2} cannot be monitored by S', a contradiction; if $w_2 = u_{k+3}$, then v_3 cannot be monitored by S', a contradiction. So $|S' \cap F_3| = 2$. Without loss of generality, suppose $w_1 \in S' \cap \{u_3, v_3\}$, $w_2 \in S' \cap \{u_{k+3}, v_{k+3}\}$.

Now, we show that $|S' \cap F_3^u| = 2$ and there are at least two vertices in F_5^u that are monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$. If $w_1 = v_3$, then u_2 cannot be monitored by S' since there is at most one vertex $v_3 \in N(u_3)$ (or $u_{4k} \in N(u_1)$ or $v_{k+2} \in N(v_2)$) monitored by S', a contradiction. So $S' \cap F_3^v = \emptyset$ and $|S' \cap F_3^u| = 2$. Without loss of generality, let $u_3, u_{k+3} \in S' \cap F_3^u$. If u_5 cannot be monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$, then v_4 and v_5 cannot be monitored by $S' \cap F_3^u$, a contradiction. Then $u_1, u_5, u_{k+1}, u_{k+5}$ are monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$. Hence $|S' \cap F_3^u| = 2$ and there are at least two vertices in F_5^u monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$.

If $u_3, u_{k+3} \in S'$, then u_5, u_{k+5} must be monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$ and $u_1, u_{k+1} \in F_1^u$ must be monitored by $S' \setminus \bigcup_{i=1}^{5} F_i$. If $S' \cap (F_6 \cup F_7) = \emptyset$, then $G\big[F_6^u \cup F_7^u\big]$ cannot be monitored by S', a contradiction. So $\big|S' \cap (F_6 \cup F_7)\big| \geq 1$ and $\big|S' \cap \big(\bigcup_{i=3}^{7} F_i\big)\big| \geq 3$.

The result holds. ■

In [16], Xu and Kang gave a procedure which constructs a smaller generalized Petersen graph from $P(3k, k)$. Now we construct a smaller generalized Petersen graph $P(4(k-3), k-3)$ from $P(4k, k)$ as follows.

Let $G = P(4k, k)$ with vertex $U \cup V$ and edge set E. The graph $G' = (U' \cup V', E')$ is defined by $U' = U \setminus \{u_{qk+i} \mid q \in \{0, 1, 2, 3\},\ i \in \{2, 3, 4\}\}$, $V' = V \setminus \{v_{qk+i} \mid q \in \{0, 1, 2, 3\},\ i \in \{2, 3, 4\}\}$ and $E' = E(U' \cup V') \cup \{u_{qk+1}u_{qk+5} \mid q = 0, 1, 2, 3\}$. Obviously, G' is isomorphic to $P(4(k-3), k-3)$.

Figure 2 gives an illustration of the construction for $P(4(k-3), k-3)$ from $P(4k, k)$.

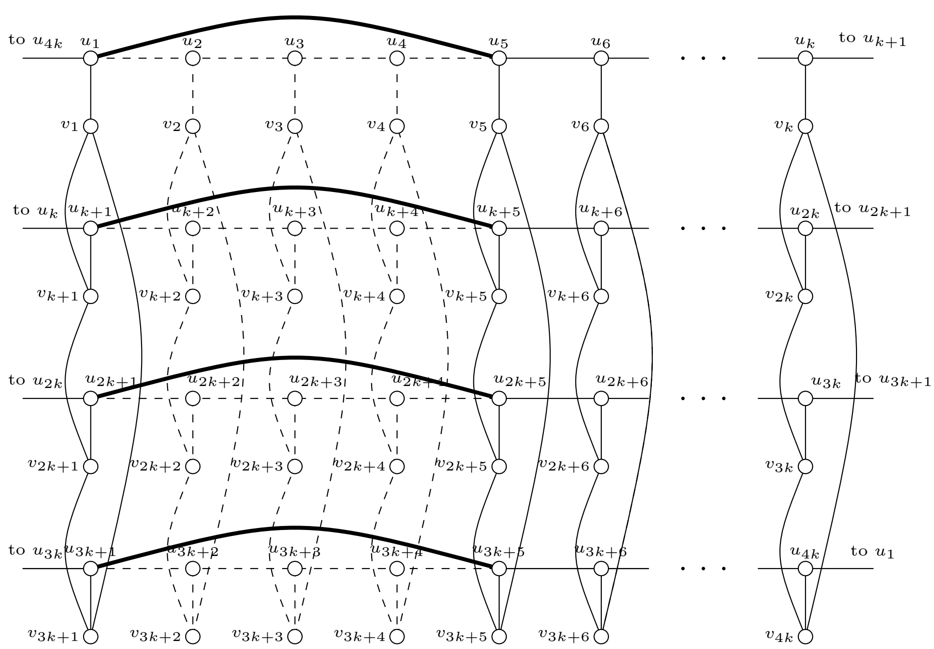


Figure 2. The illustration of the construction for $P(4(k-3), k-3)$ from $P(4k, k)$.

Using the above lemma and procedure, we now prove the following results.

Lemma 5. *For $k \geq 7$, we have $\gamma_P(P(4(k-3), k-3)) \leq \gamma_P(P(4k, k)) - 2$.*

Proof. Applying the above procedure to process $P(4k, k)$, we get a graph G' which is isomorphic to $P(4(k-3), k-3)$. Let S be a γ_P-set for $P(4k, k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 3$ since such a set exists by Lemma 4. If $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 5$, then set $D = \{u_{k+5}, v_1, v_{3k+5}\}$. We show that D can monitor $G'[F_1 \cup F_5]$. In G', notice that $\{v_{k+5}, u_{k+1}, u_1, v_{k+1}, v_{3k+1}, v_5, v_{2k+5}, u_{3k+5}\}$ can be monitored by D during the domination step. The vertices of $N(v_5)$ other than u_5 are monitored, so u_5 is monitored by the propagation. By Observation 2, $G'[F_1 \cup F_5]$ can be monitored by D since $\{u_i, v_i \mid i = 1, 5, k+1, k+5\}$ is monitored by D. The vertices in $V(G') \setminus F_1 \cup F_5$ are monitored by $S \setminus S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ together with the vertices in $F_1^u \cup F_5^u \cup F_6^u \cup F_k^u$ that are monitored by D. Therefore, $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G' and $|S'| \leq |S| - 2$. The result is proven.

We consider the following two cases: $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$ and $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 4$. We use $M(S)$ to denote the set of vertices monitored by S, and set $A = M\left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cap \left(F_1^u \cup F_5^u\right)$ and $B = M\left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cap \left(F_k^u \cup F_6^u\right)$. Let $C = \bigcup_{i=2}^{4} F_i, C^u = \bigcup_{i=2}^{4} F_i^u, C^v = \bigcup_{i=2}^{4} F_i^v$. Next, we show that there exists a set $D \subseteq F_1 \cup F_5$ such that $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G' and $|S'| \leq |S| - 2$.

Now, we will present it as a sequence of claims as follows.

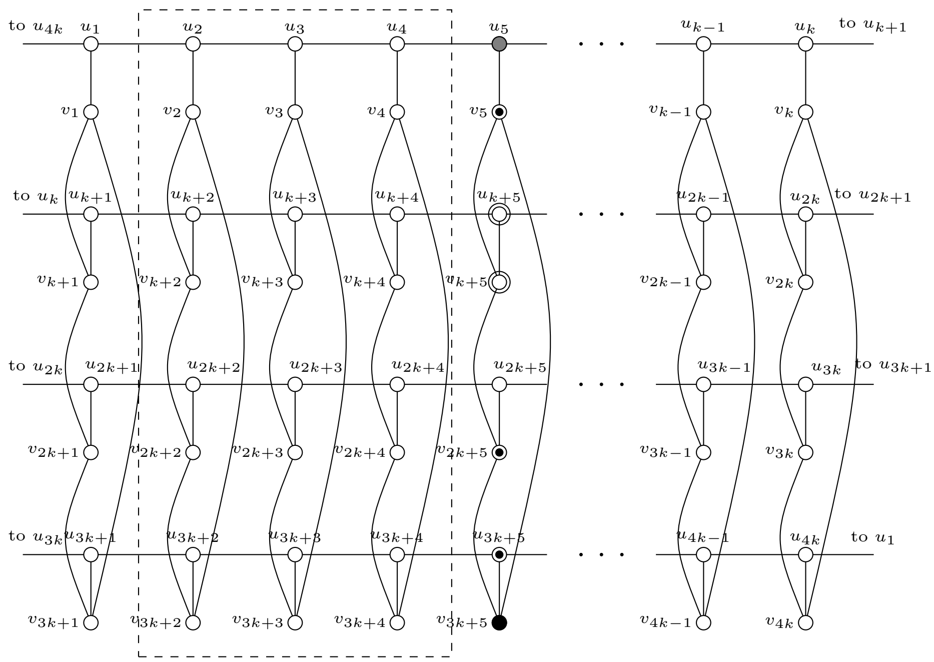


Figure 3. Vertices in dotted rectangle are in C. If $v_{3k+5} \in S$ and u_5 is monitored by $S \cap C$, then u_{3k+5}, v_5, v_{2k+5} are monitored by v_{3k+5} during the domination step, and v_{k+5}, u_{k+5} are monitored by the propagation.

Claim 1. *Let S be a γ_P-set for $P(4k,k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 4$. Then at least one of the following two statements is true.*

(1) *$A \neq \emptyset$, or*

(2) *B contains two vertices u_i and u_j with $|u_i - u_j| \notin \{2k, 2k+6, 2k-6\}$.*

Proof. Suppose to the contrary that neither (1) nor (2) holds, that is, one of the following statements holds.

(a) $A = \emptyset$ and $|B| \leq 1$, or

(b) $A = \emptyset$ and $|B| \geq 2$ but $|i-j| \in \{2k, 2k+6, 2k-6\}$ for each pair of vertices $u_i, u_j \in B$.

If (a) holds (i.e., $A = \emptyset$ and $|B| \leq 1$), then we claim that $|S \cap C| \leq 2$. In fact, if $|B| = 0$, then $F_1 \cup F_5$ must be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ and there is at least one vertex in $S \cap F_1$ and $S \cap F_5$, respectively. Then $|S \cap C| \leq 2$. If $|B| = 1$, without loss of generality, let $u_{4k} \in B$. Then $F_1 \setminus \{u_1, v_1\}$ must be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ and there is at least one vertex in $S \cap F_1$. Similarly, F_5 must be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ and there is at least one vertex in $S \cap F_5$. So $|S \cap C| \le 2$ and there is at least one vertex in $S \cap F_1$ and $S \cap F_5$, respectively.

Suppose that there is a vertex in $F_5^v \cap S$, without loss of generality, let $v_{3k+5} \in S \cap F_5^v$. To monitor v_{k+5}, there is a vertex in $\left(F_5 \setminus \{u_{3k+5}, v_{3k+5}\}\right) \cap S$, or there is a vertex in $\{u_5, u_{2k+5}\}$ monitored by $S \cap C$. If there is a vertex in $\left(F_5 \setminus \{u_{3k+5}, v_{3k+5}\}\right) \cap S$, then $|S \cap C| \le 1$ and $C \setminus \{u_2\}$ must be monitored by $S \cap C$ since $F_6 \cap B = \emptyset$. However, no vertex in $S \cap C$ can completely monitor $G[C \setminus \{u_2\}]$, a contradiction. So $|F_5 \cap S| = 1$, $|S \cap C| \ge 2$ and there is a vertex in $\{u_5, u_{2k+5}\}$ monitored by $S \cap C$ (Figure 3 illustrates the case of u_5 monitored by $S \cap C$).

Suppose that there is a vertex in $F_5^u \cap S$, without loss of generality let $u_{3k+5} \in S \cap F_5^u$. To monitor v_{k+5}, there must be a vertex in $\left(F_5 \setminus \{u_{3k+5}\}\right) \cap S$, since $u_{k+6} \notin B$. Then $|S \cap C| \le 1$. Similar to the above proof, we obtain the contradiction.

By the above proof, if $A = \emptyset$ and $|B| \le 1$, then $|S \cap C| = 2, |S \cap F_1| = 1$, $|S \cap F_5| = 1$ and there is at least a vertex in F_5^u monitored by $S \cap C$.

Now, we show that there is also at least one vertex in F_1^u monitored by $S \cap C$. If $|B| = 0$, then we obtain the conclusion similar to the above proof. Suppose $|B| = 1$ and, without loss of generality, let $B = \{u_{4k}\}$. Since $|S \cap F_1| = 1$, we have $u_1 \notin S$. Otherwise, if $u_1 \in S$, then there is a vertex in $S \cap (F_1 \setminus \{u_1\})$ to monitor u_{2k+1}, contradicting the assumption of $|S \cap F_1| = 1$. If $u_{2k+1} \in S$, then v_{k+1} and v_{3k+1} cannot be monitored by S since $u_k, u_{3k} \notin B$ and $|F_1 \cap S| = 1$, a contradiction. So $u_1, u_{2k+1} \notin S$. If $u_{k+1} \in S$ (or $u_{3k+1} \in S$), then u_1 must be monitored by $S \cap C$ to monitor v_{3k+1} (or v_{k+1}) since $u_{4k} \in B$ and $|S \cap F_1| = 1$. If $v_1 \in S$, then there is a vertex in $\{u_{k+1}, u_{3k+1}\}$ monitored by $S \cap C$ to monitor v_{2k+1}. If there is a vertex in $\{v_{k+1}, v_{2k+1}, v_{3k+1}\} \cap S$, then there is a vertex in F_1^u monitored by $S \cap C$.

Hence, there are at least two vertices $w_1 \in F_1^u$ and $w_2 \in F_5^u$ monitored by $S \cap C$. However, no two vertices in $S \cap C$ can completely monitor the induced subgraph $G[C \cup \{w_1, w_2\}]$, contradicting the assumption that S is a PDS for $P(4k, k)$.

If (b) holds (i.e., $A = \emptyset$ and $|B| \ge 2$ but $|i - j| \in \{2k, 2k+6, 2k-6\}$ for each pair of vertices $u_i, u_j \in B$), then let $u_{4k} \in B$. Then $u_k, u_{3k} \notin B$ and $\{v_{k+1}, v_{3k+1}\} \subseteq F_1$ cannot be monitored by $S \setminus F_1$. So $|S \cap F_1| \ge 1$. Similarly, $|S \cap F_5| \ge 1$ and hence $|S \cap C| \le 2$. Similar to the above proof, there are at least two vertices $w_1 \in F_1^u$ and $w_2 \in F_5^u$ monitored by $S \cap C$ and we obtain the contradiction. Therefore, (1) or (2) holds. □

Claim 2. *Let S be a γ_P-set for $P(4k, k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 4$. If $A \ne \emptyset$, then there exists a set $D \subseteq F_1 \cup F_5$ with $|D| = 2$ such that $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G'.*

Proof. If $u_i \in A \cap F_5^u$, then set $D = \{v_{i+k}, v_{i+3k-4}\}$; if $u_i \in A \cap F_1^u$, then set

$D = \{v_{i+k}, v_{i+3k+4}\}$. In each case, D together with A monitors $G'[F_1 \cup F_5]$ in G'. So $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G' and $|S'| = |S| - 2$. □

Claim 3. *Let S be a γ_P-set for $P(4k,k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 4$. If B contains two vertices u_i and u_j with $|i-j| \notin \{2k, 2k+6, 2k-6\}$, then there exists a set $D \subseteq F_1 \cup F_5$ with $|D| = 2$ such that $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G'.*

Proof. If $u_i, u_j \in B$, then it is one of the following cases:

(1) $u_i, u_j \in F_6^u$,

(2) $u_i, u_j \in F_k^u$,

(3) $u_i \in F_6^u$, $u_j \in F_k^u$ or $u_i \in F_k^u$, $u_j \in F_6^u$.

If $u_i, u_j \in B \cap F_6^u$ and $|i-j| \neq 2k$, then let $u_6, u_{k+6} \in B$ and set $D = \{u_1, u_{k+1}\}$. In G', the vertices of $N(u_5)$ other than v_5 and the vertices of $N(u_{k+5})$ other than v_{k+5} are monitored by D by the domination step, so v_5 and v_{k+5} are monitored by the propagation. So $G'[F_1 \cup F_5]$ is monitored by D together with B by Observation 2 (see Figure 4). So by the symmetry of $P(4k,k)$, if $u_i, u_j \in B \cap F_6^u$ and $|i-j| \neq 2k$, then we set $D = \{u_{i-5}, u_{j-5}\}$ and D together with B monitors $G'[F_1 \cup F_5]$. Similarly, if $u_i, u_j \in B \cap F_k^u$ and $|i-j| \neq 2k$, then set $D = \{u_{i+5}, u_{j+5}\}$ and D together with B monitors $G'[F_1 \cup F_5]$.

Next we discuss the third case. Now we denote $N_l = \{u_{lk}, u_{lk+6}\}$ for $l = 1,2,3,4$. In the following proof, the subscripts for N_i are used modulo 4. If $u_i, u_j \in B \cap N_l$ for some $l = 1,2,3,4$, then let $u_i \in F_k^u \cap B \cap N_l$, $u_j \in F_6^u \cap B \cap N_l$ and set $D = \{u_{i-k+1}, v_{j-1}\}$ (if $u_6, u_{4k} \in B$, then $D = \{u_{3k+1}, v_5\}$ together with B monitors $G'[F_1 \cup F_5]$). If $u_i \in F_6^u \cap B \cap N_l$ and $u_j \in F_k^u \cap B \cap N_{l+1}$ for some $l = 1,2,3,4$, then set $D = \{v_{i-1}, v_{j+1}\}$ (if $u_6, u_k \in B$, then $D = \{v_5, v_{k+1}\}$ together with B monitors $G'[F_1 \cup F_5]$). If $u_i \in F_6^u \cap B \cap N_l$, $u_j \in F_k^u \cap B \cap N_{l+3}$ for some $l = 1,2,3,4$, then set $D = \{v_{i-1}, v_{j+1}\}$ (if $u_6, u_{3k} \in B$, then $D = \{v_5, v_{3k+1}\}$ together with B monitors $G'[F_1 \cup F_5]$).

In each case, D together with B monitors $G'[F_1 \cup F_5]$ in G'. So $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G' and $|S'| = |S| - 2$. □

Claim 4. *Let S be a γ_P-set for $P(4k,k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$. Then $A \neq \emptyset$.*

Proof. Suppose to the contrary that $A = \emptyset$, then each vertex in $F_1^u \cup F_5^u$ is monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$. To monitor $F_1 \cup F_2$, it follows that $\left|S \cap \left(\bigcup_{i=1}^{3} F_i\right)\right| \geq 2$. Similarly, $\left|S \cap \left(\bigcup_{i=3}^{5} F_i\right)\right| \geq 2$ holds to monitor $F_4 \cup F_5$. Since $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$, we have $|S \cap F_3| = 1$, $|S \cap (F_1 \cup F_2)| = 1$ and $|S \cap (F_4 \cup F_5)| = 1$.

If $S \cap F_3 = \{u_{tk+3}\}$ for some $t = 0,1,2,3$, then since $|S \cap (F_4 \cup F_5)| = 1$, there is one vertex in $\{v_{tk+4}, v_{tk+5}, v_{(t+1)k+4}, v_{(t+3)k+4}\} \cap S$ to monitor $F_4 \cup F_5$. Similarly, there is one vertex in $\{v_{tk+1}, v_{tk+2}, v_{(t+1)k+2}, v_{(t+3)k+2}\} \cap S$ to monitor $F_1 \cup F_2$. However, $u_{(t+1)k+3}$ and $v_{(t+1)k+3}$ cannot be monitored by S, a contradiction.

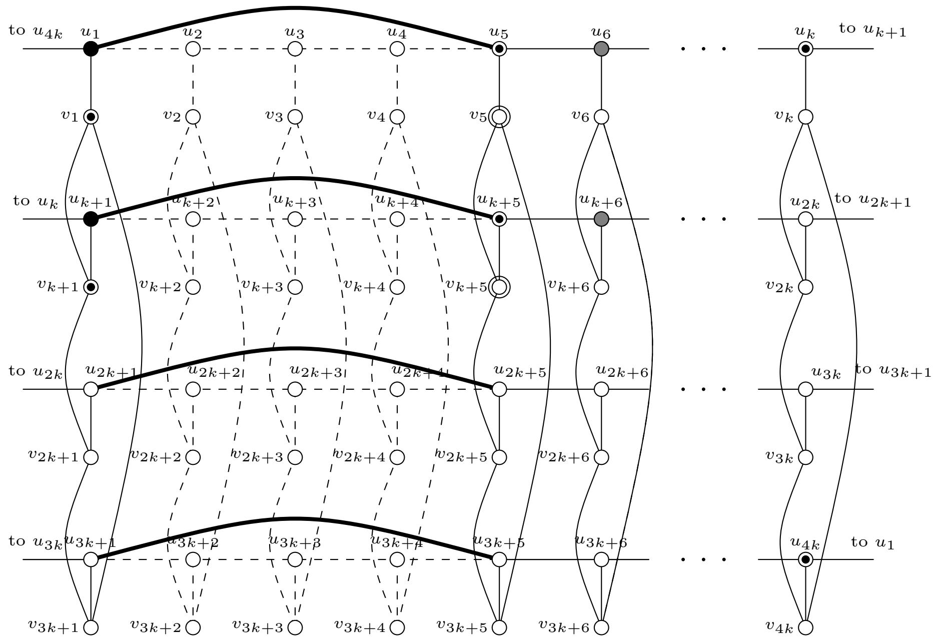


Figure 4. If $u_6, u_{k+6} \in B$, then set $D = \{u_1, u_{k+1}\}$. In G', $u_{4k}, u_5, v_1, u_k, u_{k+5}, v_{k+1}$ are monitored by D by the domination step, and v_5, v_{k+5} are monitored by D together with B by the propagation.

If $S \cap F_3 = \{v_{tk+3}\}$ for some $t = 0, 1, 2, 3$, then since $|S \cap (F_4 \cup F_5)| = 1$, it follows that $v_{tk+4} \in S$ to monitor $F_4 \cup F_5$. Similarly, $v_{tk+2} \in S$ holds to monitor $F_1 \cup F_2$. However, $\{u_{(t+1)k+2}, u_{(t+1)k+3}, u_{(t+1)k+4}\}$ cannot be monitored, a contradiction.

Hence, $A \neq \emptyset$. $\square$

Claim 5. *Let S be a γ_P-set for $P(4k, k)$ with $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$. Then there exists a set $D \subseteq F_1 \cup F_5$ with $|D| = 1$ such that $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G'.*

Proof. By Claim 4, $A \neq \emptyset$. We consider the following cases.

Case 1. $|A| = 1$. We claim that the following properties hold.

(1) If $A = \{u_i\} \subseteq F_1$, then $u_{i+5} \in B$.
(2) If $A = \{u_i\} \subseteq F_1$, then $u_{i+k-1}, u_{i+k+5} \in B$ or $u_{i-k-1}, u_{i-k+5} \in B$.
(3) If $A = \{u_i\} \subseteq F_5$, then $u_{i-5} \in B$.
(4) If $A = \{u_i\} \subseteq F_5$, then $u_{i+k+1}, u_{i+k-5} \in B$ or $u_{i-k+1}, u_{i-k-5} \in B$.

Without loss of generality, let $u_1 \in F_1^u \cap A$. Then $u_{4k} \in B$. If $u_6 \notin B$, then $|S \cap (F_4 \cup F_5)| \geq 2$ holds to monitor F_5 since $|A| = 1$. To monitor $F_1 \cup F_2$, there are at least two vertices in $S \cap \left(\bigcup_{i=1}^{3} F_i\right)$. So we have $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| \geq 4$, contradicting to the assumption of $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$. Hence $u_6 \in B$ and (1) holds. Similarly, (3) also holds.

Now, we show that if $u_1 \in F_1^u \cap A$, then $u_k, u_{k+6} \in B$ or $u_{3k}, u_{3k+6} \in B$. Otherwise, suppose that there is at least one vertex in both $\{u_k, u_{k+6}\}$ and $\{u_{3k}, u_{3k+6}\}$ which is not in B. Without loss of generality, let $u_k, u_{3k+6} \notin B$. Then u_k, u_{3k+6} are monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$. However, no three vertices in $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ can completely monitor $\left(\bigcup_{i=1}^{5} F_i \setminus \{u_1\}\right) \cup \{u_k, u_{3k+6}\}$. In fact, there are two vertices in $S \cap \left(\bigcup_{i=1}^{3} F_i\right)$ to monitor $F_1 \cup F_2$, and there are two vertices in $S \cap \left(\bigcup_{i=3}^{5} F_i\right)$ to monitor $F_4 \cup F_5$. Since $\left|S \cap \left(\bigcup_{i=1}^{5} F_i\right)\right| = 3$, we have $|S \cap F_3| = 1, |S \cap (F_1 \cup F_2)| = 1$ and $|S \cap (F_4 \cup F_5)| = 1$.

If $S \cap F_3 = \{u_{3k+3}\}$, then u_{3k+5} and v_{3k+5} are monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ since $u_{3k+6} \notin B$. Because $|S \cap (F_4 \cup F_5)| = 1$ and $|S \cap F_3| = 1$, we have $v_{3k+5} \in S$. But v_{k+5} cannot be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$, a contradiction. So $u_{3k+3} \notin S$. Similarly, we have $u_{k+3}, u_{2k+3} \notin S$.

If $S \cap F_3 = \{u_3\}$, then there is one vertex in $\{v_4, v_5, v_{k+4}, v_{3k+4}\} \cap S$ to monitor u_5. If $v_4 \in S$ or $v_5 \in S$, then v_{2k+5} cannot be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ since $|S \cap (F_4 \cup F_5)| = 1$, a contradiction. So $v_4, v_5 \notin S$. If $v_{k+4} \in S$ or $v_{3k+4} \in S$, then v_{3k+5} cannot be monitored by $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ since $u_{3k+6} \notin B$, a contradiction. Hence $u_3 \notin S \cap F_3$.

If $S \cap F_3 = \{v_3\}$, then $|\{u_{k+2}, u_{k+4}, u_{3k+2}, u_{3k+4}\} \cap S| \geq 1$, or there is one vertex in $S \cap \{u_{2k+4}, u_{2k+5}\}$ and $S \cap \{u_{2k+1}, u_{2k+2}\}$, respectively, to monitor v_{2k+3}. In all cases, $S \cap \left(\bigcup_{i=1}^{5} F_i\right)$ cannot completely monitor $\bigcup_{i=1}^{3} F_i$ since $|S \cap F_3| = 1, |S \cap (F_1 \cup F_2)| = 1$ and $|S \cap (F_4 \cup F_5)| = 1$, a contradiction. So $v_3 \notin S \cap F_3$. Similarly, it follows that $v_{k+3}, v_{2k+3}, v_{3k+3} \notin S \cap F_3$.

So $u_k, u_{k+6} \in B$ or $u_{3k}, u_{3k+6} \in B$ and (2) holds. Similarly, (4) also holds.

Without loss of generality, suppose that $A = \{u_1\}$ and $u_6, u_k, u_{k+6} \in B$. Let $D = \{v_{k+1}\}$. In G', we notice that v_1, v_{k+1}, v_{2k+1} and u_{k+1} are monitored by the domination step. The vertices of $N(u_{k+1})$ other than u_{k+5} and the vertices of $N(u_1)$ other than u_5 are monitored since $u_1 \in A$ and $u_k \in B$. So u_5 and u_{k+5} are monitored by the propagation. Similarly, v_5 and v_{k+5} are monitored by the propagation since $u_6, u_{k+6} \in B$. Hence, by Observation 2, $G'[F_1 \cup F_5]$ is monitored by $\{v_{k+1}\}$ together with A and B. By the symmetry of G', we can find a set $D \subseteq F_1 \cup F_5$ with $|D| = 1$ which together with A and B monitors $G'[F_1 \cup F_5]$ for other cases. So $S' = (S \setminus \bigcup_{i=1}^{5} F_i) \cup D$ is a PDS for G' and $|S'| = |S| - 2$.

Case 2. $|A| = 2$. We claim that $|F_1^u \cap A| = 1$, $|F_5^u \cap A| = 1$, and if $u_i \in F_1^u \cap A, u_j \in F_5^u \cap A$, then $|i - j| \notin \{2k + 4, 2k - 4\}$. Moreover, the following properties hold.

(1) If $u_i \in F_1^u \cap A, u_j \in F_5^u \cap A$ and $j - i = 4$, then there is at least one vertex in $B \cap \{u_{i+k-1}, u_{i-k-1}, u_{j+k+1}, u_{j-k+1}\}$.

(2) If $u_i \in F_1^u \cap A, u_j \in F_5^u \cap A$ and $|i - j| \in \{k-4, k+4, 3k-4, 3k+4\}$, then there is at least one vertex in $B \cap \{u_{i+5}, u_{j-5}\}$.

Suppose $|F_1^u \cap A| = 2$ and $F_5^u \cap A = \emptyset$. We claim that there must be two vertices in $S \cap \left(\bigcup_{i=1}^3 F_i\right)$ to monitor $\bigcup_{i=1}^3 F_i$. Otherwise, suppose that there is a vertex in $S \cap \left(\bigcup_{i=1}^3 F_i\right)$. Without loss of generality, let $u_1, u_{k+1} \in F_1^u \cap A$. If there is a vertex $w \in \left(\bigcup_{i=1}^3 F_i^u\right) \cap S$, then $\bigcup_{i=1}^3 F_i^v$ cannot completely be monitored by S; if there is a vertex $w \in \left(\bigcup_{i=1}^3 F_i^v\right) \cap S$, then F_2 cannot completely be monitored by S. So $\left|S \cap \left(\bigcup_{i=1}^3 F_i\right)\right| \geq 2$. Similarly, there must be two vertices in $S \cap (F_4 \cup F_5)$ to monitor F_5. Then $\left|S \cap \left(\bigcup_{i=1}^5 F_i\right)\right| \geq 4$, a contradiction. So $|F_1^u \cap A| = 1$ and $|F_5^u \cap A| = 1$.

Suppose $u_1 \in F_1^u \cap A, u_{2k+5} \in F_5^u \cap A$. Then there are two vertices in $S \cap \left(\bigcup_{i=1}^3 F_i\right)$ to monitor $F_1 \cup F_2$, and there are two vertices in $S \cap \left(\bigcup_{i=3}^5 F_i\right)$ to monitor $F_4 \cup F_5$. Since $\left|S \cap \left(\bigcup_{i=1}^5 F_i\right)\right| = 3$, we have $|S \cap F_3| = 1$. Similar to the proof of Claim 3, no such three vertices monitor $\bigcup_{i=1}^5 F_i \setminus \{u_1, u_{2k+5}\}$. Similarly, we obtain contradictions for $u_i \in F_1^u \cap A$ and $u_j \in F_5^u \cap A$ with $|i-j| \in \{2k+4, 2k-4\}$. So if $u_i \in F_1^u \cap A$, $u_j \in F_5^u \cap A$, then $|i-j| \notin \{2k+4, 2k-4\}$.

Next, we show that (1) and (2) hold.

Without loss of generality, let $u_1 \in F_1^u \cap A, u_5 \in F_5^u \cap A$. If $B \cap \{u_k, u_{3k}, u_{k+6}, u_{3k+6}\} = \emptyset$, then there is one vertex in $S \cap F_1$ to monitor F_1 and there is one vertex in $S \cap F_5$ to monitor F_5. However, there are two vertices in $S \cap \left(\bigcup_{i=2}^4 F_i\right)$ to monitor F_3 and $\left|S \cap \left(\bigcup_{i=1}^5 F_i\right)\right| \geq 4$, a contradiction. So $|B \cap \{u_k, u_{3k}, u_{k+6}, u_{3k+6}\}| \geq 1$. Hence (1) holds.

Now, we prove that (2) holds. Without loss of generality, let $u_1 \in F_1^u \cap A$ and $u_{k+5} \in F_5^u \cap A$. If $u_6, u_k \notin B$, then there are two vertices in $S \cap \left(\bigcup_{i=1}^3 F_i\right)$ to monitor $\bigcup_{i=1}^3 F_i$ and there are two vertices in $S \cap \left(\bigcup_{i=3}^5 F_i\right)$ to monitor $\bigcup_{i=3}^5 F_i$. So $|S \cap F_3| = 1$. But no such three vertices monitor $\bigcup_{i=1}^5 F_i \setminus \{u_1, u_{k+5}\}$, a contradiction. So $|B \cap \{u_6, u_k\}| \geq 1$.

Hence, (1) and (2) hold.

If $A = \{u_1, u_5\}$, then $u_{4k}, u_6 \in B$. Without loss of generality, let $u_{3k} \in B$ and set $D = \{u_{3k+5}\}$. In G', the vertices of $N(u_1)$ other than v_1 and the vertices of $N(u_5)$ other than v_5 are monitored, so v_1 and v_5 are monitored by the propagation. Similarly, v_{3k+1} is monitored by the propagation since u_{3k+1}, v_{3k+5} and the vertices of $N(u_{3k+1})$ other than v_{3k+1} are monitored by the domination step. So $G'[F_1 \cup F_5]$ is monitored by D together with A and B by Observation 2.

If $A = \{u_1, u_{k+5}\}$, then $u_{4k}, u_{k+6} \in B$. Without loss of generality, let $u_6 \in B$ and set $D = \{v_{k+1}\}$. Similar to the above proof, D together with A and B monitors $G'[F_1 \cup F_5]$ in G'. By the symmetry of G', we can find a set $D \subseteq F_1 \cup F_5$

with $|D| = 1$ which together with A and B monitors $G'[F_1 \cup F_5]$ for other cases. So $S' = \left(S \setminus \bigcup_{i=1}^{5} F_i\right) \cup D$ is a PDS for G' and $|S'| = |S| - 2$.

Case 3. $|A| \geq 3$. We have the following statements.

(1) $|F_1^u \cap A| \geq 1$ and $|F_5^u \cap A| \geq 1$.

(2) If $|F_1^u \cap A| \geq 2$ or $|F_5^u \cap A| \geq 2$, then there are two vertices $u_i, u_j \in F_1^u \cap A$ (or $F_5^u \cap A$) such that $|i - j| \neq 2k$.

Similar to the proof of Case 2, we can prove that (1) holds. Suppose (2) does not hold. Without loss of generality, let $u_1, u_{2k+1} \in F_1^u \cap A$. Then $u_{k+1}, u_{3k+1} \notin F_1^u \cap A$. There are two vertices in $S \cap \left(\bigcup_{i=1}^{3} F_i\right)$ to monitor $F_1 \cup F_2$. Similarly, there are two vertices in $S \cap \left(\bigcup_{i=3}^{5} F_i\right)$ to monitor $F_4 \cup F_5$. So $|F_3 \cap S| = 1$. However, no such three vertices monitor $\bigcup_{i=1}^{5} F_i \setminus \{u_1, u_{2k+1}\}$.

If $u_1, u_5, u_{k+5} \in A$, then set $D = \{v_{2k+1}\}$. If $u_1, u_{k+1}, u_{2k+5} \in A$, then set $D = \{v_{3k+5}\}$. If $u_1, u_{k+1}, u_{3k+5} \in A$, then set $D = \{v_5\}$. In each case, D together with A monitors $G'[F_1 \cup F_5]$ in G' by the propagation step and Observation 2. By the symmetry of G', we can find a set $D \subseteq F_1 \cup F_5$ with $|D| = 1$ which together with A monitors $G'[F_1 \cup F_5]$ for other cases. So $S' = (S \setminus \bigcup_{i=1}^{5} F_i) \cup D$ is a PDS for G' and $|S'| = |S| - 2$.

In each case, we obtain a PDS S' for G' with $|S| - 2$ vertices. Therefore, the assertion follows. □

This completes the proof of Lemma 5. ■

We are now ready to give the exact power domination number for $P(4k, k)$. The following observation is useful.

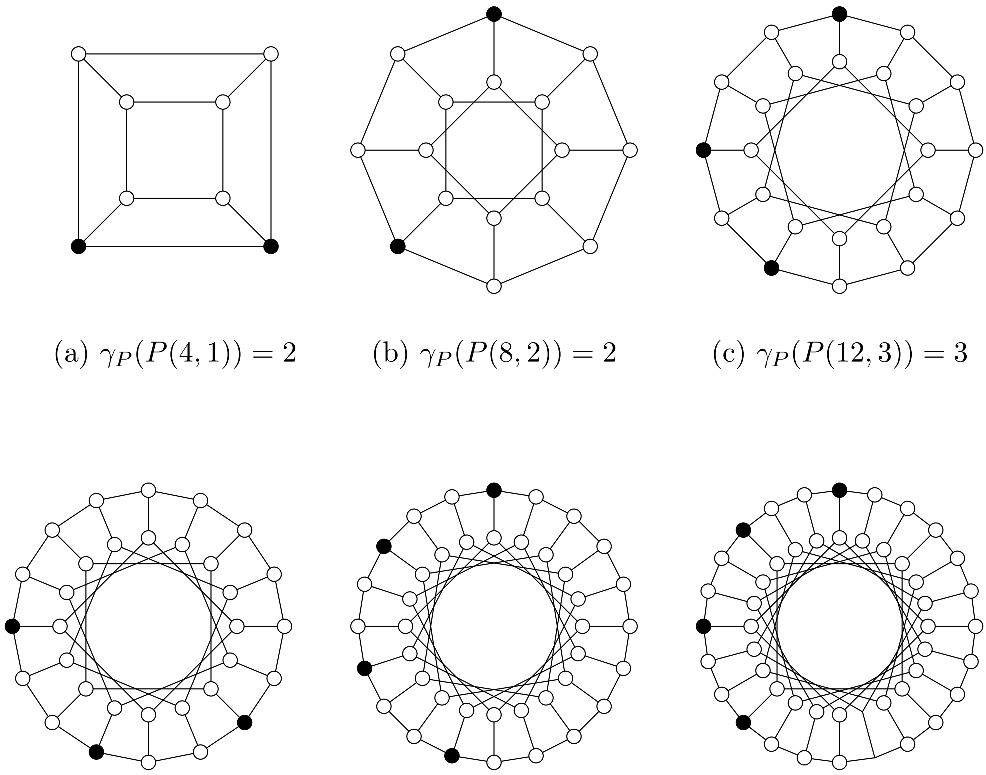
Observation 6. *For $k \geq 2$, $\gamma_P(P(4k, k)) \geq \gamma_P(P(4(k-1), k-1))$.*

Proof. Let S be a γ_P-set for $P(4k, k) = (U, V)$. By Lemma 3, $\gamma_P(P(4k, k)) \leq \left\lceil \frac{2k}{3} \right\rceil$ holds. Then there is one set F_i for some $i \in \{1, 2, \ldots, k\}$ such that $S \cap F_i = \emptyset$. Let $U' = U \setminus \{u_{qk+i} \,|\, q \in \{0, 1, 2, 3\}\}$, $V' = V \setminus \{v_{qk+i} \,|\, q \in \{0, 1, 2, 3\}\}$ and $E' = E(U' \cup V') \cup \{u_{qk+i-1} u_{qk+i+1} \,|\, q = 0, 1, 2, 3\}$. Then $G' = (U' \cup V', E')$ is isomorphic to $P(4(k-1), k-1)$. Obviously, S is also a PDS for $G' = P(4(k-1), k-1)$. So $\gamma_P(P(4(k-1), k-1)) \leq |S| = \gamma_P(P(4k, k))$. ■

Lemma 7. *For the generalized Petersen graph $P(4k, k)$ with $1 \leq k \leq 6$, we have*

$$\gamma_P(P(4k, k)) = \begin{cases} 2 & \text{for } k = 1 \text{ or } 2, \\ 3 & \text{for } k = 3 \text{ or } 4, \\ 4 & \text{for } k = 5 \text{ or } 6. \end{cases}$$

Proof. We notice that no vertex in $G = P(4, 1)$ or $P(8, 2)$ can completely monitor it and there are two vertices (see Figure 5(a) and 5(b)) which form a PDS



(a) $\gamma_P(P(4,1)) = 2$ (b) $\gamma_P(P(8,2)) = 2$ (c) $\gamma_P(P(12,3)) = 3$

(d) $\gamma_P(P(16,4)) = 3$ (e) $\gamma_P(P(20,5)) = 4$ (f) $\gamma_P(P(24,6)) = 4$

Figure 5. Vertices with bold circles form a γ_P-set for $P(4k,k)$ with $1 \leq k \leq 6$.

for G. So $\gamma_P(P(4,1)) = \gamma_P(P(8,2)) = 2$. By Observation 6, $\gamma_P(P(12,3)) \geq \gamma_P(P(8,2)) = 2$. Note that no two vertices in $P(12,3)$ can completely monitor it and we can find three vertices (see Figure 5(c)) which form a PDS for $P(12,3)$. So $\gamma_P(P(12,3)) = 3$. Since we can find three vertices (see Figure 5(d)) in $P(16,4)$ which form a PDS and $\gamma_P(P(16,4)) \geq \gamma_P(P(12,3)) = 3$, $\gamma_P(P(16,4)) = 3$. Similarly, no three vertices in $P(20,5)$ can completely monitor it and $\gamma_P(P(20,5)) = \gamma_P(P(24,6)) = 4$ (see Figure 5(e) and 5(f)). ■

Theorem 8. *For the generalized Petersen graph $P(4k,k)$,*

$$\gamma_P(P(4k,k)) = \begin{cases} 2 & \textit{for } k = 1, \\ 3 & \textit{for } k = 3, \\ \lceil \frac{2k}{3} \rceil & \textit{for } k = 2 \textit{ or } k \geq 4. \end{cases}$$

Proof. We show that $\gamma_P(P(4k,k)) \geq \lceil \frac{2k}{3} \rceil$ for all $k \geq 4$ holds by induction on k as follows. By Lemma 7, $\gamma_P(P(4k,k)) = \lceil \frac{2k}{3} \rceil$ for $k \in \{4,5,6\}$. Suppose the result holds for $k \geq 7$. Then $\gamma_P(P(4(k+3), k+3)) \geq \left\lceil \frac{2k}{3} \right\rceil$ by Lemma 5 together with $\left\lceil \frac{2(k+3)}{3} \right\rceil = \left\lceil \frac{2k}{3} \right\rceil + 2$.

Combining with Lemma 3 and Lemma 7, we obtain the desired result. ■

4. Closing Remarks

In this paper, we give an upper bound on the power domination number for $P(ck, k)$ with $c \geq 4$ and determine the exact power domination number for $P(4k, k)$. Finally, we conjecture that the following result holds.

Conjecture 9. *For $k \geq 2$ and $c \geq 5$, $\gamma_P(P(ck, k)) = \left\lceil \frac{2k}{3} \right\rceil$ if $2k \equiv 1 \pmod 3$, and $\gamma_P(P(ck, k)) \in \left\{ \left\lceil \frac{2k}{3} \right\rceil, \left\lceil \frac{2k}{3} \right\rceil + 1 \right\}$ if $2k \equiv 0$ or $2k \equiv 2 \pmod 3$.*

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