# POWER DOMINATION IN THE GENERALIZED PETERSEN GRAPHS ${ }^{1}$ 

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#### Abstract

The problem of monitoring an electric power system by placing as few measurement devices in the system can be formulated as a power dominating set problem in graph theory. The power domination number of a graph is the minimum cardinality of a power dominating set. Xu and Kang [On the power domination number of the generalized Petersen graphs, J. Comb. Optim. 22 (2011) 282-291] study the exact power domination number for the generalized Petersen graph $P(3 k, k)$, and propose the following problem: determine the power domination number for the generalized Petersen graph $P(4 k, k)$ or $P(c k, k)$. In this paper we give the power domination number for $P(4 k, k)$ and present a sharp upper bound on the power domination number for the generalized Petersen graph $P(c k, k)$.


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## 1. Introduction

Electric power companies need to continually monitor the state of their systems as in the case of voltage magnitude at loads and machine phase angle at generators. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. Due to the high cost of the PMUs, the number of PMUs used to monitor the network must be minized.

The power system monitoring problem can be formulated as a domination problem in graph theory by Haynes et al. in [7]. Let $G=(V(G), E(G))$ be a graph representing an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed, its incident edges and their ends. All these vertices and edges are said to be observed by the PMU. We can apply Ohm's law and Kirchoff's current law to deduce the other three observation rules.

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k \geq 2$ edges and if $k-1$ of these edges are observed, then all $k$ of these edges are observed.
We consider only graphs without loops or multiple edges. For a vertex $v$ of $G=(V(G), E(G))$, let $N(v)$ denote the open neighborhood of $v$, and for a subset $S \subseteq V(G)$, let $N(S)=\bigcup_{v \in S} N(v) \backslash S$. The closed neighborhood $N[S]$ of a subset $S$ is the set $N[S]=N(S) \cup S$. For any $X \subseteq V(G)$, the subgraph induced in $G$ by $X \subseteq V(G)$ is denoted by $G[X]$ and $E(X)$ denotes the edge set of $G[X]$. For other terminology and notation not given here, we refer to $[4,7,8]$.

A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex (node) in $V \backslash S$ has at least one neighbor in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The theory of dominating sets in graphs is well developed (see, for example, [8]). Considering the power system monitoring problem as a variation of the dominating set problem, a set $S$ is a power dominating set (PDS) if every vertex and every edge in $G$ is monitored by $S$ after applying the observation rules. The power domination number of $G$, denoted by $\gamma_{P}(G)$, is the minimum cardinality of a power dominating set of $G$. A power dominating set of $G$ with minimum cardinality is called a $\gamma_{P}(G)$-set.

Let $G$ be a connected graph and $S$ a subset of its vertices. In [3], Brueni and Heath first provided a new simplified definition of the observation rules that requires only 2 rules. In this paper, we shall use the following equivalent algorithm [4]. We denote by $M(S)$ the set of vertices in $G$ that is monitored by $S$.

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Algorithm 1: An alternative approach to the observation rules
    Input: A connected graph \(G(V, E)\) and \(S \subseteq V(G)\)
    Output: The set \(M(S)\) monitored by \(S\)
    1.Domination Step
        \(M(S) \leftarrow S \cup N(S) ;\)
    2.Propagation Step
    foreach \(w \in V(G) \backslash M(S)\) do
        if there exists \(v \in M(S)\) such that \(N(v) \cap(V(G) \backslash M(S))=\{w\}\) then
        \(M(S) \leftarrow M(S) \cup\{w\} ;\)
        end
    end
    Output \(M(S)\);
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Power domination in graphs was introduced and studied by Haynes et al. in [7]. It has received considerable attention from the algorithmic point of view. Haynes et al. [7] showed that the power dominating set problem is NP-complete even when restricted to bipartite graphs or chordal graphs and provided a linear algorithm to solve the PDS for trees. There are polynomial algorithms to solve this problem for graphs with bounded treewidth [6], block graphs [15], block-cactus graphs [9], interval graphs [10], grids [12], honeycomb meshes [13] and circular-arc graphs [11]. On the other hand, upper bounds on the power domination number are given for connected graphs with at least three vertices, for connected clawfree cubic graphs [19], for hypercubes [3], and for generalized Petersen graphs [1]. Dorbec et al. [4] determined the power domination number for product graphs.

As a generalization of the well-known Petersen graph, the generalized Petersen graph has attracted much attention. The generalized Petersen graph $P(n, k)(k \geq 1)$ is the graph with vertex set $U \cup V$, where $U=\left\{u_{i} \mid 1 \leq i \leq n\right\}$ and $V=\left\{v_{i} \mid 1 \leq i \leq n\right\}$, and edge set $E=\left\{u_{i} u_{i+1}, u_{i} v_{i}, v_{i} v_{i+k} \mid 1 \leq i \leq n\right\}$, where the subscripts are to be read as integers modulo $n$. The graph $P(5,2)$ is the Petersen graph. Domination and its variations have been extensively investigated in the class of generalized Petersen graphs in $[2,5,14,17,18]$.

Xu and Kang [16] gave a sharp upper bound on power domination number for generalized Petersen graph $P(n, k)$ and determined the exact power domination number for $P(3 k, k)$. They posed the following open problem: find the exact power domination number for the generalized Petersen graph $P(4 k, k)$ (or even for $P(c k, k)$, where $c \geq 4$ is a constant integer). In this paper we give an upper bound on the power domination number for $P(c k, k)$ and determine the exact power domination number for $P(4 k, k)$.

The remainder of this paper is organized as follows. In Section 2, we prove $\gamma_{P}(P(c k, k)) \leq\left\lceil\frac{2 k+2}{3}\right\rceil$ for integer $k \geq 2$ and $c \geq 4$ by providing an explicit con-
struction for the upper bounds. In Section 3, we first show that $\gamma_{P}(P(4 k, k)) \leq$ $\left\lceil\frac{2 k}{3}\right\rceil$ for $k \geq 4$ and $\gamma_{P}(P(4(k-3), k-3)) \leq \gamma_{P}(P(4 k, k))-2$ for $k \geq 7$. Then using the above results we prove that $\gamma_{P}(P(4 k, k)) \geq\left\lceil\frac{2 k}{3}\right\rceil$ for $k \geq 4$ and determine the exact power domination number for $P(4 k, k)$. In Section 4, we conclude this paper and propose a conjecture.

## 2. An Upper Bound on $\gamma(P(c k, k))$

For the generalized Petersen graph $P(c k, k)$, when $k=1$, it is easily seen that $\gamma_{P}(P(2,1))=\gamma_{P}(P(3,1))=1$ and $\gamma_{P}(P(n, 1))=2$ for $n \geq 4$. When $k \geq 2$, Xu and Kang [16] showed that $\gamma_{P}(P(n, k)) \leq \min \left\{\left\lceil\frac{n}{3}\right\rceil, k\right\}$ for $n \geq 4$.

We now restrict our attention to the generalized Petersen graph $P(c k, k)$ for $k \geq 2$. We give a sharp upper bound on $\gamma_{P}(P(c k, k))$ for $k \geq 2, c \geq 4$. The bound improves the Xu and Kang's bound for $P(n, k)$.

Theorem 1. For $k \geq 2$ and $c \geq 4, \gamma_{P}(P(c k, k)) \leq\left\lceil\frac{2 k+2}{3}\right\rceil$, and this bound is sharp.

Proof. To obtain the upper bound, we directly construct the power dominating set for $P(c k, k)$ with $|S|=\left\lceil\frac{2 k+2}{3}\right\rceil$.

Let $2 k=3 a+b$ with $b \in\{0,1,-1\}$ and set $S=\left\{u_{1}, u_{4}, \ldots, u_{3 a+1}\right\}$. We have $|S|=a+1=\left\lceil\frac{2 k+2}{3}\right\rceil$. Next, we show that $S$ is a PDS for $P(c k, k)$. In Figure 1, the following procedure is shown when $2 k=3 a$.
(1) After the domination step, the set $S \cup N(S)=\left\{u_{c k}, u_{1}, u_{2}, \ldots, u_{3 a+2}\right\} \cup$ $\left\{v_{1}, v_{4}, \ldots, v_{3 a+1}\right\}$ (i.e., vertices with bold circles in Figure 1) is monitored.
(2) For $2 \leq i \leq 3 a, i \not \equiv 1(\bmod 3)$, the vertex $u_{i}$ monitors $v_{i}$ (i.e., vertices with gray circles in Figure 1) by the propagation step. And as a result $\left\{u_{1}, u_{2}, \ldots, u_{3 a+2}\right\} \cup\left\{v_{1}, v_{2}, \ldots, v_{3 a+1}\right\} \supseteq\left\{u_{1}, u_{2}, \ldots, u_{2 k+1}\right\} \cup\left\{v_{1}, v_{2}, \ldots, v_{2 k}\right\}$ is monitored.
(3) For $t=1,2, \ldots, c-2$,

- first, for $i=1, \ldots, k$, vertex $v_{t k+i}$ monitors $v_{(t+1) k+i}$ by the propagation (i.e., vertices contained in the dotted rectangle in Figure 1 are monitored when $t=1$ ), and then
- for $i=1, \ldots, k$, vertex $u_{(t+1) k+i}$ monitors $u_{(t+1) k+i+1}$ by the propagation (i.e., vertices contained in the solid ellipse in Figure 1 are monitored when $t=1$ ).

So $S$ is a PDS for $P(c k, k)$ and $\gamma_{P}(P(c k, k)) \leq|S|=\left\lceil\frac{2 k+2}{3}\right\rceil$. By Theorem 8 (we will prove it in Section 3), we have $\gamma_{P}(P(4 k, k))=\left\lceil\frac{2 k}{3}\right\rceil$. Since $\left\lceil\frac{2 k+2}{3}\right\rceil=\left\lceil\frac{2 k}{3}\right\rceil$ when $2 k \equiv 1(\bmod 3)$, the bound is sharp.


Figure 1. Vertices monitored by $S$ for $P(c k, k)$ when $2 k=3 a$ and $t=1$.

## 3. The Power Domination Number for $P(4 k, k)$

In this section we determine the power domination number for the generalized Petersen graph $P(4 k, k)$. The main result is given in Theorem 8.

For $G=P(4 k, k)$, we denote $F_{i}=\left\{u_{i}, v_{i}, u_{i+k}, v_{i+k}, u_{i+2 k}, v_{i+2 k}, u_{i+3 k}, v_{i+3 k}\right\}$ and $F_{i}^{u}=\left\{u_{i}, u_{i+k}, u_{i+2 k}, u_{i+3 k}\right\}, F_{i}^{v}=\left\{v_{i}, v_{i+k}, v_{i+2 k}, v_{i+3 k}\right\}$, where $1 \leq i \leq k$. In the following statements, the subscripts for $u_{i}$ and $v_{i}$ are used modulo $4 k$, and the subscripts for $F_{i}, F_{i}^{u}$ and $F_{i}^{v}$ are used modulo $k$.

Observation 2. For $G=P(4 k, k)$, if all vertices of $\left\{u_{i}, v_{i}, u_{i+k}, v_{i+k}\right\}(1 \leq i \leq$ $k$ ) are monitored by a set $D$, then $G\left[F_{i}\right]$ is monitored by $D$.

Proof. We note that $N\left(v_{i+k}\right)=\left\{v_{i}, u_{i+k}, v_{i+2 k}\right\}$ and $N\left(v_{i}\right)=\left\{u_{i}, v_{i+k}, v_{i+3 k}\right\}$. The vertices of $N\left(v_{i+k}\right)$ other than $v_{i+2 k}$ and the vertices of $N\left(v_{i}\right)$ other than $v_{i+3 k}$ are monitored by $D$ during the domination step, so $v_{i+2 k}$ and $v_{i+3 k}$ are monitored
by the propagation step. Similarly, the vertices of $N\left(v_{i+2 k}\right)$ other than $u_{i+2 k}$ and the vertices of $N\left(v_{i+3 k}\right)$ other than $u_{i+3 k}$ are monitored by domination step, so $u_{i+2 k}$ and $u_{i+3 k}$ can be monitored by the propagation. Hence, $G\left[F_{i}\right]$ is completely monitored by $D$.

We begin to study the power domination problem on $P(4 k, k)$. First, we give an upper bound on $\gamma_{P}(P(4 k, k))$.

Lemma 3. For $k \geq 4, \gamma_{P}(P(4 k, k)) \leq\left\lceil\frac{2 k}{3}\right\rceil$.
Proof. By Theorem 1, $\gamma_{P}(P(4 k, k)) \leq\left\lceil\frac{2 k}{3}\right\rceil$ for $2 k \equiv 1(\bmod 3)$ since $\left\lceil\frac{2 k+2}{3}\right\rceil=$ $\left\lceil\frac{2 k}{3}\right\rceil$ when $2 k \equiv 1(\bmod 3)$. Now, we show that the result also holds for $2 k \equiv 0$ or $2(\bmod 3)$.

Let $2 k=3 a+b$ with $b \in\{2,3\}$ and set $S=\left\{u_{1}, u_{4}, \ldots, u_{3 a+1}\right\}$. We have $|S|=a+1=\left\lceil\frac{2 k}{3}\right\rceil$. Similar to the proof of Theorem 1, $\left\{u_{1}, u_{2}, \ldots, u_{3 a+2}\right\} \cup$ $\left\{v_{1}, v_{2}, \ldots, v_{3 a+1}\right\}$ is monitored by $S$. So by the propagation, $\bigcup_{i=1}^{3 a+1-k} F_{i}$ is monitored and as a result $u_{2 k}, v_{2 k}$ are monitored by $S$. Hence $S$ is a PDS for $P(4 k, k)$.

Therefore, $\gamma_{P}(P(4 k, k)) \leq|S| \leq\left\lceil\frac{2 k}{3}\right\rceil$.
Next, we prove that $\left\lceil\frac{2 k}{3}\right\rceil$ is also a lower bound on $\gamma_{P}(P(4 k, k))$ for $k \geq 4$. To prove the result, we will show that $\gamma_{P}(P(4(k-3), k-3)) \leq \gamma_{P}(P(4 k, k))-2$ for $k \geq 7$ (Lemma 5). The following property is required.

Lemma 4. For $k \geq 7$, there is a $\gamma_{P}$-set $S$ of $P(4 k, k)$ such that

$$
\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 3
$$

Proof. We denote $M_{j}=\left\{u_{i+j k}, v_{i+j k} \mid i=1,2,3,4,5\right\}$ for $j=0,1,2,3$, and the subscripts are used modulo 4 in the following proof. For a $\gamma_{P}$-set $S$ for $P(4 k, k)$, we claim that $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 2$. Otherwise, if there is a unique vertex $w \in M_{j} \cap S$, then $u_{(j+2) k+3}$ cannot be monitored by $S$, a contradiction. So $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 2$.

If $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 3$, the result is proven. Suppose now that $\left|S^{\prime} \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|$ $=2$ for a $\gamma_{P}$-set $S^{\prime}$ of $P(4 k, k)$. We will prove that $\left|S^{\prime} \cap F_{3}^{u}\right|=2,\left|S^{\prime} \cap\left(F_{6} \cup F_{7}\right)\right| \geq$ 1 and $\left|S^{\prime} \cap\left(\bigcup_{i=3}^{7} F_{i}\right)\right| \geq 3$. By the symmetry of $P(4 k, k)$, the result holds.

Let $S^{\prime} \cap\left(\bigcup_{i=1}^{5} F_{i}\right)=\left\{w_{1}, w_{2}\right\}$. Then we claim that $w_{1}$ and $w_{2}$ are in $M_{j}$ and $M_{j+1}$, respectively, for some $j=0,1,2,3$. Otherwise, if $w_{1}, w_{2} \in M_{0} \cap S^{\prime}$, then there is at most one vertex in $N\left(v_{k+3}\right)$ (or $N\left(v_{3 k+3}\right)$ ) monitored by $\left\{w_{1}, w_{2}\right\}$, and $v_{2 k+3}$ cannot be monitored by $\left\{w_{1}, w_{2}\right\}$, a contradiction. So $\left|M_{0} \cap S^{\prime}\right| \leq 1$. Similarly, $\left|M_{j} \cap S^{\prime}\right| \leq 1$ for each $j=0,1,2,3$. If $w_{1}$ and $w_{2}$ are in $M_{j}$ and $M_{j+2}$ respectively for $j=0$ or 1 , then there is at least one vertex in $M_{j+1} \cup M_{j+3}$ which cannot be monitored by $S^{\prime}$, a contradiction. Hence $w_{1}$ and $w_{2}$ are in $M_{j}$ and $M_{j+1}$, respectively, for some $j=0,1,2,3$.

Next, we show that $\left|S^{\prime} \cap F_{3}\right|=2$. Otherwise, suppose $\mid S^{\prime} \cap\left[\left(\bigcup_{i=1}^{2} F_{i}\right) \cup\right.$ $\left.\left(\bigcup_{i=4}^{5} F_{i}\right)\right] \mid=2$, then $w_{1}$ and $w_{2}$ are in $M_{j} \cap\left[\left(\bigcup_{i=1}^{2} F_{i}\right) \cup\left(\bigcup_{i=4}^{5} F_{i}\right)\right]$ and $M_{j+1} \cap$ $\left[\left(\bigcup_{i=1}^{2} F_{i}\right) \cup\left(\bigcup_{i=4}^{5} F_{i}\right)\right]$, respectively, for $j=0,1,2,3$. Without loss of generality, let $w_{1} \in M_{j} \cap\left(F_{1} \cup F_{2}\right)$. If $w_{2} \in M_{j+1} \cap\left(F_{4} \cup F_{5}\right)$, then $v_{j k+3}$ and $v_{(j+1) k+3}$ cannot be monitored by $S^{\prime}$, a contradiction. So $w_{2} \in M_{j+1} \cap\left(F_{1} \cup F_{2}\right)$ or $w_{2} \in$ $M_{j-1} \cap\left(F_{1} \cup F_{2}\right)$ for $j=1,2,3,4$. Similarly, if $w_{1} \in M_{j} \cap\left(F_{4} \cup F_{5}\right)$, then $w_{2} \in$ $M_{j+1} \cap\left(F_{4} \cup F_{5}\right)$ or $w_{2} \in M_{j-1} \cap\left(F_{4} \cup F_{5}\right)$ for $j=1,2,3,4$. Without loss of generality, let $w_{1} \in M_{0} \cap\left(F_{1} \cup F_{2}\right)$ and $w_{2} \in M_{1} \cap\left(F_{1} \cup F_{2}\right)$. Then $v_{4}$ and $v_{k+4}$ cannot be monitored by $S^{\prime}$, a contradiction. So $\left|S^{\prime} \cap\left[\left(\bigcup_{i=1}^{2} F_{i}\right) \cup\left(\bigcup_{i=4}^{5} F_{i}\right)\right]\right| \leq 1$ and $\left|S^{\prime} \cap F_{3}\right| \geq 1$.

Without loss of generality, let $w_{1} \in\left\{u_{1}, v_{1}, u_{2}, v_{2}\right\}$ and $w_{2} \in\left\{u_{k+3}, v_{k+3}\right\}$. If $w_{1} \in\left\{u_{1}, u_{2}\right\}$ and $w_{2}=v_{k+3}$, then there is at most one vertex $u_{1} \in N\left(v_{1}\right)$ (or $u_{2} \in N\left(v_{2}\right)$ ) monitored by $\left\{w_{1}, w_{2}\right\}$, and $v_{k+1}$ and $v_{k+2}$ cannot be monitored by $S^{\prime}$, a contradiction. Similarly, if $w_{1}=v_{1}$ and $w_{2}=v_{k+3}$, then $u_{3}$ cannot be monitored by $S^{\prime}$, a contradiction; if $w_{1}=v_{2}$ and $w_{2}=v_{k+3}$, then $u_{k+2}$ cannot be monitored by $S^{\prime}$, a contradiction; if $w_{2}=u_{k+3}$, then $v_{3}$ cannot be monitored by $S^{\prime}$, a contradiction. So $\left|S^{\prime} \cap F_{3}\right|=2$. Without loss of generality, suppose $w_{1} \in S^{\prime} \cap\left\{u_{3}, v_{3}\right\}, w_{2} \in S^{\prime} \cap\left\{u_{k+3}, v_{k+3}\right\}$.

Now, we show that $\left|S^{\prime} \cap F_{3}^{u}\right|=2$ and there are at least two vertices in $F_{5}^{u}$ that are monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$. If $w_{1}=v_{3}$, then $u_{2}$ cannot be monitored by $S^{\prime}$ since there is at most one vertex $v_{3} \in N\left(u_{3}\right)$ (or $u_{4 k} \in N\left(u_{1}\right)$ or $v_{k+2} \in N\left(v_{2}\right)$ ) monitored by $S^{\prime}$, a contradiction. So $S^{\prime} \cap F_{3}^{v}=\emptyset$ and $\left|S^{\prime} \cap F_{3}^{u}\right|=2$. Without loss of generality, let $u_{3}, u_{k+3} \in S^{\prime} \cap F_{3}^{u}$. If $u_{5}$ cannot be monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$, then $v_{4}$ and $v_{5}$ cannot be monitored by $S^{\prime} \cap F_{3}^{u}$, a contradiction. Then $u_{1}, u_{5}, u_{k+1}, u_{k+5}$ are monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$. Hence $\left|S^{\prime} \cap F_{3}^{u}\right|=2$ and there are at least two vertices in $F_{5}^{u}$ monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$.

If $u_{3}, u_{k+3} \in S^{\prime}$, then $u_{5}, u_{k+5}$ must be monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$ and $u_{1}, u_{k+1} \in F_{1}^{u}$ must be monitored by $S^{\prime} \backslash \bigcup_{i=1}^{5} F_{i}$. If $S^{\prime} \cap\left(F_{6} \cup F_{7}\right)=\emptyset$, then $G\left[F_{6}^{u} \cup F_{7}^{u}\right]$ cannot be monitored by $S^{\prime}$, a contradiction. So $\left|S^{\prime} \cap\left(F_{6} \cup F_{7}\right)\right| \geq 1$ and $\left|S^{\prime} \cap\left(\bigcup_{i=3}^{7} F_{i}\right)\right| \geq 3$.

The result holds.
In [16], Xu and Kang gave a procedure which constructs a smaller generalized Petersen graph from $P(3 k, k)$. Now we construct a smaller generalized Petersen graph $P(4(k-3), k-3)$ from $P(4 k, k)$ as follows.

Let $G=P(4 k, k)$ with vertex $U \cup V$ and edge set $E$. The graph $G^{\prime}=$ $\left(U^{\prime} \cup V^{\prime}, E^{\prime}\right)$ is defined by $U^{\prime}=U \backslash\left\{u_{q k+i} \mid q \in\{0,1,2,3\}, i \in\{2,3,4\}\right\}, V^{\prime}=$ $V \backslash\left\{v_{q k+i} \mid q \in\{0,1,2,3\}, i \in\{2,3,4\}\right\}$ and $E^{\prime}=E\left(U^{\prime} \cup V^{\prime}\right) \cup\left\{u_{q k+1} u_{q k+5} \mid q=\right.$ $0,1,2,3\}$. Obviously, $G^{\prime}$ is isomorphic to $P(4(k-3), k-3)$.

Figure 2 gives an illustration of the construction for $P(4(k-3), k-3)$ from $P(4 k, k)$.


Figure 2. The illustration of the construction for $P(4(k-3), k-3)$ from $P(4 k, k)$.
Using the above lemma and procedure, we now prove the following results.
Lemma 5. For $k \geq 7$, we have $\gamma_{P}(P(4(k-3), k-3)) \leq \gamma_{P}(P(4 k, k))-2$.
Proof. Applying the above procedure to process $P(4 k, k)$, we get a graph $G^{\prime}$ which is isomorphic to $P(4(k-3), k-3)$. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 3$ since such a set exists by Lemma 4. If $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 5$, then set $D=\left\{u_{k+5}, v_{1}, v_{3 k+5}\right\}$. We show that $D$ can monitor $G^{\prime}\left[F_{1} \cup F_{5}\right]$. In $G^{\prime}$, notice that $\left\{v_{k+5}, u_{k+1}, u_{1}, v_{k+1}, v_{3 k+1}, v_{5}, v_{2 k+5}, u_{3 k+5}\right\}$ can be monitored by $D$ during the domination step. The vertices of $N\left(v_{5}\right)$ other than $u_{5}$ are monitored, so $u_{5}$ is monitored by the propagation. By Observation 2, $G^{\prime}\left[F_{1} \cup F_{5}\right]$ can be monitored by $D$ since $\left\{u_{i}, v_{i} \mid i=1,5, k+1, k+5\right\}$ is monitored by $D$. The vertices in $V\left(G^{\prime}\right) \backslash F_{1} \cup F_{5}$ are monitored by $S \backslash S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ together with the vertices in $F_{1}^{u} \cup F_{5}^{u} \cup F_{6}^{u} \cup F_{k}^{u}$ that are monitored by $D$. Therefore, $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right| \leq|S|-2$. The result is proven.

We consider the following two cases: $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$ and $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=$ 4. We use $M(S)$ to denote the set of vertices monitored by $S$, and set $A=$ $M\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cap\left(F_{1}^{u} \cup F_{5}^{u}\right)$ and $B=M\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cap\left(F_{k}^{u} \cup F_{6}^{u}\right)$. Let $C=$ $\bigcup_{i=2}^{4} F_{i}, C^{u}=\bigcup_{i=2}^{4} F_{i}^{u}, C^{v}=\bigcup_{i=2}^{4} F_{i}^{v}$. Next, we show that there exists a set $D \subseteq F_{1} \cup F_{5}$ such that $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right| \leq|S|-2$.

Now, we will present it as a sequence of claims as follows.


Figure 3. Vertices in dotted rectangle are in $C$. If $v_{3 k+5} \in S$ and $u_{5}$ is monitored by $S \cap C$, then $u_{3 k+5}, v_{5}, v_{2 k+5}$ are monitored by $v_{3 k+5}$ during the domination step, and $v_{k+5}, u_{k+5}$ are monitored by the propagation.

Claim 1. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=4$. Then at least one of the following two statements is true.
(1) $A \neq \emptyset$, or
(2) $B$ contains two vertices $u_{i}$ and $u_{j}$ with $\left|u_{i}-u_{j}\right| \notin\{2 k, 2 k+6,2 k-6\}$.

Proof. Suppose to the contrary that neither (1) nor (2) holds, that is, one of the following statements holds.
(a) $A=\emptyset$ and $|B| \leq 1$, or
(b) $A=\emptyset$ and $|B| \geq 2$ but $|i-j| \in\{2 k, 2 k+6,2 k-6\}$ for each pair of vertices $u_{i}, u_{j} \in B$.

If (a) holds (i.e., $A=\emptyset$ and $|B| \leq 1$ ), then we claim that $|S \cap C| \leq 2$. In fact, if $|B|=0$, then $F_{1} \cup F_{5}$ must be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ and there is at least one vertex in $S \cap F_{1}$ and $S \cap F_{5}$, respectively. Then $|S \cap C| \leq 2$. If $|B|=1$, without loss of generality, let $u_{4 k} \in B$. Then $F_{1} \backslash\left\{u_{1}, v_{1}\right\}$ must be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ and there is at least one vertex in $S \cap F_{1}$. Similarly, $F_{5}$ must
be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ and there is at least one vertex in $S \cap F_{5}$. So $|S \cap C| \leq 2$ and there is at least one vertex in $S \cap F_{1}$ and $S \cap F_{5}$, respectively.

Suppose that there is a vertex in $F_{5}^{v} \cap S$, without loss of generality, let $v_{3 k+5} \in S \cap F_{5}^{v}$. To monitor $v_{k+5}$, there is a vertex in $\left(F_{5} \backslash\left\{u_{3 k+5}, v_{3 k+5}\right\}\right) \cap S$, or there is a vertex in $\left\{u_{5}, u_{2 k+5}\right\}$ monitored by $S \cap C$. If there is a vertex in $\left(F_{5} \backslash\left\{u_{3 k+5}, v_{3 k+5}\right\}\right) \cap S$, then $|S \cap C| \leq 1$ and $C \backslash\left\{u_{2}\right\}$ must be monitored by $S \cap C$ since $F_{6} \cap B=\emptyset$. However, no vertex in $S \cap C$ can completely monitor $G\left[C \backslash\left\{u_{2}\right\}\right]$, a contradiction. So $\left|F_{5} \cap S\right|=1,|S \cap C| \geq 2$ and there is a vertex in $\left\{u_{5}, u_{2 k+5}\right\}$ monitored by $S \cap C$ (Figure 3 illustrates the case of $u_{5}$ monitored by $S \cap C$ ).

Suppose that there is a vertex in $F_{5}^{u} \cap S$, without loss of generality let $u_{3 k+5} \in$ $S \cap F_{5}^{u}$. To monitor $v_{k+5}$, there must be a vertex in $\left(F_{5} \backslash\left\{u_{3 k+5}\right\}\right) \cap S$, since $u_{k+6} \notin$ $B$. Then $|S \cap C| \leq 1$. Similar to the above proof, we obtain the contradiction.

By the above proof, if $A=\emptyset$ and $|B| \leq 1$, then $|S \cap C|=2,\left|S \cap F_{1}\right|=1$, $\left|S \cap F_{5}\right|=1$ and there is at least a vertex in $F_{5}^{u}$ monitored by $S \cap C$.

Now, we show that there is also at least one vertex in $F_{1}^{u}$ monitored by $S \cap C$. If $|B|=0$, then we obtain the conclusion similar to the above proof. Suppose $|B|=1$ and, without loss of generality, let $B=\left\{u_{4 k}\right\}$. Since $\left|S \cap F_{1}\right|=1$, we have $u_{1} \notin S$. Otherwise, if $u_{1} \in S$, then there is a vertex in $S \cap\left(F_{1} \backslash\left\{u_{1}\right\}\right)$ to monitor $u_{2 k+1}$, contradicting the assumption of $\left|S \cap F_{1}\right|=1$. If $u_{2 k+1} \in S$, then $v_{k+1}$ and $v_{3 k+1}$ cannot be monitored by $S$ since $u_{k}, u_{3 k} \notin B$ and $\left|F_{1} \cap S\right|=1$, a contradiction. So $u_{1}, u_{2 k+1} \notin S$. If $u_{k+1} \in S$ (or $u_{3 k+1} \in S$ ), then $u_{1}$ must be monitored by $S \cap C$ to monitor $v_{3 k+1}$ (or $v_{k+1}$ ) since $u_{4 k} \in B$ and $\left|S \cap F_{1}\right|=1$. If $v_{1} \in S$, then there is a vertex in $\left\{u_{k+1}, u_{3 k+1}\right\}$ monitored by $S \cap C$ to monitor $v_{2 k+1}$. If there is a vertex in $\left\{v_{k+1}, v_{2 k+1}, v_{3 k+1}\right\} \cap S$, then there is a vertex in $F_{1}^{u}$ monitored by $S \cap C$.

Hence, there are at least two vertices $w_{1} \in F_{1}^{u}$ and $w_{2} \in F_{5}^{u}$ monitored by $S \cap C$. However, no two vertices in $S \cap C$ can completely monitor the induced subgraph $G\left[C \cup\left\{w_{1}, w_{2}\right\}\right]$, contradicting the assumption that $S$ is a PDS for $P(4 k, k)$.

If (b) holds (i.e., $A=\emptyset$ and $|B| \geq 2$ but $|i-j| \in\{2 k, 2 k+6,2 k-6\}$ for each pair of vertices $u_{i}, u_{j} \in B$ ), then let $u_{4 k} \in B$. Then $u_{k}, u_{3 k} \notin B$ and $\left\{v_{k+1}, v_{3 k+1}\right\} \subseteq F_{1}$ cannot be monitored by $S \backslash F_{1}$. So $\left|S \cap F_{1}\right| \geq 1$. Similarly, $\left|S \cap F_{5}\right| \geq 1$ and hence $|S \cap C| \leq 2$. Similar to the above proof, there are at least two vertices $w_{1} \in F_{1}^{u}$ and $w_{2} \in F_{5}^{u}$ monitored by $S \cap C$ and we obtain the contradiction. Therefore, (1) or (2) holds.

Claim 2. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=4$. If $A \neq \emptyset$, then there exists a set $D \subseteq F_{1} \cup F_{5}$ with $|D|=2$ such that $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$.

Proof. If $u_{i} \in A \cap F_{5}^{u}$, then set $D=\left\{v_{i+k}, v_{i+3 k-4}\right\}$; if $u_{i} \in A \cap F_{1}^{u}$, then set
$D=\left\{v_{i+k}, v_{i+3 k+4}\right\}$. In each case, $D$ together with $A$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ in $G^{\prime}$. So $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right|=|S|-2$.

Claim 3. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=4$. If $B$ contains two vertices $u_{i}$ and $u_{j}$ with $|i-j| \notin\{2 k, 2 k+6,2 k-6\}$, then there exists a set $D \subseteq F_{1} \cup F_{5}$ with $|D|=2$ such that $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$.

Proof. If $u_{i}, u_{j} \in B$, then it is one of the following cases:
(1) $u_{i}, u_{j} \in F_{6}^{u}$,
(2) $u_{i}, u_{j} \in F_{k}^{u}$,
(3) $u_{i} \in F_{6}^{u}, u_{j} \in F_{k}^{u}$ or $u_{i} \in F_{k}^{u}, u_{j} \in F_{6}^{u}$.

If $u_{i}, u_{j} \in B \cap F_{6}^{u}$ and $|i-j| \neq 2 k$, then let $u_{6}, u_{k+6} \in B$ and set $D=$ $\left\{u_{1}, u_{k+1}\right\}$. In $G^{\prime}$, the vertices of $N\left(u_{5}\right)$ other than $v_{5}$ and the vertices of $N\left(u_{k+5}\right)$ other than $v_{k+5}$ are monitored by $D$ by the domination step, so $v_{5}$ and $v_{k+5}$ are monitored by the propagation. So $G^{\prime}\left[F_{1} \cup F_{5}\right]$ is monitored by $D$ together with $B$ by Observation 2 (see Figure 4). So by the symmetry of $P(4 k, k)$, if $u_{i}, u_{j} \in B \cap F_{6}^{u}$ and $|i-j| \neq 2 k$, then we set $D=\left\{u_{i-5}, u_{j-5}\right\}$ and $D$ together with $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$. Similarly, if $u_{i}, u_{j} \in B \cap F_{k}^{u}$ and $|i-j| \neq 2 k$, then set $D=\left\{u_{i+5}, u_{j+5}\right\}$ and $D$ together with $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$.

Next we discuss the third case. Now we denote $N_{l}=\left\{u_{l k}, u_{l k+6}\right\}$ for $l=$ $1,2,3,4$. In the following proof, the subscripts for $N_{i}$ are used modulo 4. If $u_{i}, u_{j} \in B \cap N_{l}$ for some $l=1,2,3,4$, then let $u_{i} \in F_{k}^{u} \cap B \cap N_{l}, u_{j} \in F_{6}^{u} \cap B \cap N_{l}$ and set $D=\left\{u_{i-k+1}, v_{j-1}\right\}$ (if $u_{6}, u_{4 k} \in B$, then $D=\left\{u_{3 k+1}, v_{5}\right\}$ together with $B$ monitors $\left.G^{\prime}\left[F_{1} \cup F_{5}\right]\right)$. If $u_{i} \in F_{6}^{u} \cap B \cap N_{l}$ and $u_{j} \in F_{k}^{u} \cap B \cap N_{l+1}$ for some $l=1,2,3,4$, then set $D=\left\{v_{i-1}, v_{j+1}\right\}$ (if $u_{6}, u_{k} \in B$, then $D=\left\{v_{5}, v_{k+1}\right\}$ together with $B$ monitors $\left.G^{\prime}\left[F_{1} \cup F_{5}\right]\right)$. If $u_{i} \in F_{6}^{u} \cap B \cap N_{l}, u_{j} \in F_{k}^{u} \cap B \cap N_{l+3}$ for some $l=1,2,3,4$, then set $D=\left\{v_{i-1}, v_{j+1}\right\}$ (if $u_{6}, u_{3 k} \in B$, then $D=\left\{v_{5}, v_{3 k+1}\right\}$ together with $B$ monitors $\left.G^{\prime}\left[F_{1} \cup F_{5}\right]\right)$.

In each case, $D$ together with $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ in $G^{\prime}$. So $S^{\prime}=$ $\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right|=|S|-2$.

Claim 4. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$. Then $A \neq \emptyset$.
Proof. Suppose to the contrary that $A=\emptyset$, then each vertex in $F_{1}^{u} \cup F_{5}^{u}$ is monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$. To monitor $F_{1} \cup F_{2}$, it follows that $\left|S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)\right| \geq$ 2. Similarly, $\left|S \cap\left(\bigcup_{i=3}^{5} F_{i}\right)\right| \geq 2$ holds to monitor $F_{4} \cup F_{5}$. Since $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=$ 3, we have $\left|S \cap F_{3}\right|=1,\left|S \cap\left(F_{1} \cup F_{2}\right)\right|=1$ and $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$.

If $S \cap F_{3}=\left\{u_{t k+3}\right\}$ for some $t=0,1,2,3$, then since $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$, there is one vertex in $\left\{v_{t k+4}, v_{t k+5}, v_{(t+1) k+4}, v_{(t+3) k+4}\right\} \cap S$ to monitor $F_{4} \cup F_{5}$. Similarly, there is one vertex in $\left\{v_{t k+1}, v_{t k+2}, v_{(t+1) k+2}, v_{(t+3) k+2}\right\} \cap S$ to monitor $F_{1} \cup F_{2}$. However, $u_{(t+1) k+3}$ and $v_{(t+1) k+3}$ cannot be monitored by $S$, a contradiction.


Figure 4. If $u_{6}, u_{k+6} \in B$, then set $D=\left\{u_{1}, u_{k+1}\right\}$. In $G^{\prime}, u_{4 k}, u_{5}, v_{1}, u_{k}, u_{k+5}, v_{k+1}$ are monitored by $D$ by the domination step, and $v_{5}, v_{k+5}$ are monitored by $D$ together with $B$ by the propagation.

If $S \cap F_{3}=\left\{v_{t k+3}\right\}$ for some $t=0,1,2,3$, then since $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$, it follows that $v_{t k+4} \in S$ to monitor $F_{4} \cup F_{5}$. Similarly, $v_{t k+2} \in S$ holds to monitor $F_{1} \cup F_{2}$. However, $\left\{u_{(t+1) k+2}, u_{(t+1) k+3}, u_{(t+1) k+4}\right\}$ cannot be monitored, a contradiction.

Hence, $A \neq \emptyset$.
Claim 5. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)$ with $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$. Then there exists a set $D \subseteq F_{1} \cup F_{5}$ with $|D|=1$ such that $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$.

Proof. By Claim 4, $A \neq \emptyset$. We consider the following cases.
Case 1. $|A|=1$. We claim that the following properties hold.
(1) If $A=\left\{u_{i}\right\} \subseteq F_{1}$, then $u_{i+5} \in B$.
(2) If $A=\left\{u_{i}\right\} \subseteq F_{1}$, then $u_{i+k-1}, u_{i+k+5} \in B$ or $u_{i-k-1}, u_{i-k+5} \in B$.
(3) If $A=\left\{u_{i}\right\} \subseteq F_{5}$, then $u_{i-5} \in B$.
(4) If $A=\left\{u_{i}\right\} \subseteq F_{5}$, then $u_{i+k+1}, u_{i+k-5} \in B$ or $u_{i-k+1}, u_{i-k-5} \in B$.

Without loss of generality, let $u_{1} \in F_{1}^{u} \cap A$. Then $u_{4 k} \in B$. If $u_{6} \notin B$, then $\left|S \cap\left(F_{4} \cup F_{5}\right)\right| \geq 2$ holds to monitor $F_{5}$ since $|A|=1$. To monitor $F_{1} \cup F_{2}$, there are at least two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$. So we have $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 4$, contradicting to the assumption of $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$. Hence $u_{6} \in B$ and (1) holds. Similarly, (3) also holds.

Now, we show that if $u_{1} \in F_{1}^{u} \cap A$, then $u_{k}, u_{k+6} \in B$ or $u_{3 k}, u_{3 k+6} \in$ $B$. Otherwise, suppose that there is at least one vertex in both $\left\{u_{k}, u_{k+6}\right\}$ and $\left\{u_{3 k}, u_{3 k+6}\right\}$ which is not in $B$. Without loss of generality, let $u_{k}, u_{3 k+6} \notin B$. Then $u_{k}, u_{3 k+6}$ are monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$. However, no three vertices in $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ can completely monitor $\left(\bigcup_{i=1}^{5} F_{i} \backslash\left\{u_{1}\right\}\right) \cup\left\{u_{k}, u_{3 k+6}\right\}$. In fact, there are two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$ to monitor $F_{1} \cup F_{2}$, and there are two vertices in $S \cap\left(\bigcup_{i=3}^{5} F_{i}\right)$ to monitor $F_{4} \cup F_{5}$. Since $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$, we have $\left|S \cap F_{3}\right|=1,\left|S \cap\left(F_{1} \cup F_{2}\right)\right|=1$ and $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$.

If $S \cap F_{3}=\left\{u_{3 k+3}\right\}$, then $u_{3 k+5}$ and $v_{3 k+5}$ are monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ since $u_{3 k+6} \notin B$. Because $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$ and $\left|S \cap F_{3}\right|=1$, we have $v_{3 k+5} \in S$. But $v_{k+5}$ cannot be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$, a contradiction. So $u_{3 k+3} \notin S$. Similarly, we have $u_{k+3}, u_{2 k+3} \notin S$.

If $S \cap F_{3}=\left\{u_{3}\right\}$, then there is one vertex in $\left\{v_{4}, v_{5}, v_{k+4}, v_{3 k+4}\right\} \cap S$ to monitor $u_{5}$. If $v_{4} \in S$ or $v_{5} \in S$, then $v_{2 k+5}$ cannot be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ since $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$, a contradiction. So $v_{4}, v_{5} \notin S$. If $v_{k+4} \in S$ or $v_{3 k+4} \in S$, then $v_{3 k+5}$ cannot be monitored by $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ since $u_{3 k+6} \notin B$, a contradiction. Hence $u_{3} \notin S \cap F_{3}$.

If $S \cap F_{3}=\left\{v_{3}\right\}$, then $\left|\left\{u_{k+2}, u_{k+4}, u_{3 k+2}, u_{3 k+4}\right\} \cap S\right| \geq 1$, or there is one vertex in $S \cap\left\{u_{2 k+4}, u_{2 k+5}\right\}$ and $S \cap\left\{u_{2 k+1}, u_{2 k+2}\right\}$, respectively, to monitor $v_{2 k+3}$. In all cases, $S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)$ cannot completely monitor $\bigcup_{i=1}^{3} F_{i}$ since $\left|S \cap F_{3}\right|=$ $1,\left|S \cap\left(F_{1} \cup F_{2}\right)\right|=1$ and $\left|S \cap\left(F_{4} \cup F_{5}\right)\right|=1$, a contradiction. So $v_{3} \notin S \cap F_{3}$. Similarly, it follows that $v_{k+3}, v_{2 k+3}, v_{3 k+3} \notin S \cap F_{3}$.

So $u_{k}, u_{k+6} \in B$ or $u_{3 k}, u_{3 k+6} \in B$ and (2) holds. Similarly, (4) also holds.
Without loss of generality, suppose that $A=\left\{u_{1}\right\}$ and $u_{6}, u_{k}, u_{k+6} \in B$. Let $D=\left\{v_{k+1}\right\}$. In $G^{\prime}$, we notice that $v_{1}, v_{k+1}, v_{2 k+1}$ and $u_{k+1}$ are monitored by the domination step. The vertices of $N\left(u_{k+1}\right)$ other than $u_{k+5}$ and the vertices of $N\left(u_{1}\right)$ other than $u_{5}$ are monitored since $u_{1} \in A$ and $u_{k} \in B$. So $u_{5}$ and $u_{k+5}$ are monitored by the propagation. Similarly, $v_{5}$ and $v_{k+5}$ are monitored by the propagation since $u_{6}, u_{k+6} \in B$. Hence, by Observation $2, G^{\prime}\left[F_{1} \cup F_{5}\right]$ is monitored by $\left\{v_{k+1}\right\}$ together with $A$ and $B$. By the symmetry of $G^{\prime}$, we can find a set $D \subseteq F_{1} \cup F_{5}$ with $|D|=1$ which together with $A$ and $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ for other cases. So $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right|=|S|-2$.

Case 2. $|A|=2$. We claim that $\left|F_{1}^{u} \cap A\right|=1,\left|F_{5}^{u} \cap A\right|=1$, and if $u_{i} \in$ $F_{1}^{u} \cap A, u_{j} \in F_{5}^{u} \cap A$, then $|i-j| \notin\{2 k+4,2 k-4\}$. Moreover, the following properties hold.
(1) If $u_{i} \in F_{1}^{u} \cap A, u_{j} \in F_{5}^{u} \cap A$ and $j-i=4$, then there is at least one vertex in $B \cap\left\{u_{i+k-1}, u_{i-k-1}, u_{j+k+1}, u_{j-k+1}\right\}$.
(2) If $u_{i} \in F_{1}^{u} \cap A, u_{j} \in F_{5}^{u} \cap A$ and $|i-j| \in\{k-4, k+4,3 k-4,3 k+4\}$, then there is at least one vertex in $B \cap\left\{u_{i+5}, u_{j-5}\right\}$.
Suppose $\left|F_{1}^{u} \cap A\right|=2$ and $F_{5}^{u} \cap A=\emptyset$. We claim that there must be two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$ to monitor $\bigcup_{i=1}^{3} F_{i}$. Otherwise, suppose that there is a vertex in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$. Without loss of generality, let $u_{1}, u_{k+1} \in F_{1}^{u} \cap A$. If there is a vertex $w \in\left(\bigcup_{i=1}^{3} F_{i}^{u}\right) \cap S$, then $\bigcup_{i=1}^{3} F_{i}^{v}$ cannot completely be monitored by $S$; if there is a vertex $w \in\left(\bigcup_{i=1}^{3} F_{i}^{v}\right) \cap S$, then $F_{2}$ cannot completely be monitored by $S$. So $\left|S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)\right| \geq 2$. Similarly, there must be two vertices in $S \cap\left(F_{4} \cup F_{5}\right)$ to monitor $F_{5}$. Then $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 4$, a contradiction. So $\left|F_{1}^{u} \cap A\right|=1$ and $\left|F_{5}^{u} \cap A\right|=1$.

Suppose $u_{1} \in F_{1}^{u} \cap A, u_{2 k+5} \in F_{5}^{u} \cap A$. Then there are two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$ to monitor $F_{1} \cup F_{2}$, and there are two vertices in $S \cap\left(\bigcup_{i=3}^{5} F_{i}\right)$ to monitor $F_{4} \cup F_{5}$. Since $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right|=3$, we have $\left|S \cap F_{3}\right|=1$. Similar to the proof of Claim 3, no such three vertices monitor $\bigcup_{i=1}^{5} F_{i} \backslash\left\{u_{1}, u_{2 k+5}\right\}$. Similarly, we obtain contradictions for $u_{i} \in F_{1}^{u} \cap A$ and $u_{j} \in F_{5}^{u} \cap A$ with $|i-j| \in\{2 k+4,2 k-4\}$. So if $u_{i} \in F_{1}^{u} \cap A, u_{j} \in F_{5}^{u} \cap A$, then $|i-j| \notin\{2 k+4,2 k-4\}$.

Next, we show that (1) and (2) hold.
Without loss of generality, let $u_{1} \in F_{1}^{u} \cap A, u_{5} \in F_{5}^{u} \cap A$. If $B \cap\left\{u_{k}, u_{3 k}, u_{k+6}\right.$, $\left.u_{3 k+6}\right\}=\emptyset$, then there is one vertex in $S \cap F_{1}$ to monitor $F_{1}$ and there is one vertex in $S \cap F_{5}$ to monitor $F_{5}$. However, there are two vertices in $S \cap\left(\bigcup_{i=2}^{4} F_{i}\right)$ to monitor $F_{3}$ and $\left|S \cap\left(\bigcup_{i=1}^{5} F_{i}\right)\right| \geq 4$, a contradiction. So $\left|B \cap\left\{u_{k}, u_{3 k}, u_{k+6}, u_{3 k+6}\right\}\right| \geq$ 1. Hence (1) holds.

Now, we prove that (2) holds. Without loss of generality, let $u_{1} \in F_{1}^{u} \cap A$ and $u_{k+5} \in F_{5}^{u} \cap A$. If $u_{6}, u_{k} \notin B$, then there are two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$ to monitor $\bigcup_{i=1}^{3} F_{i}$ and there are two vertices in $S \cap\left(\bigcup_{i=3}^{5} F_{i}\right)$ to monitor $\bigcup_{i=3}^{5} F_{i}$. So $\left|S \cap F_{3}\right|=1$. But no such three vertices monitor $\bigcup_{i=1}^{5} F_{i} \backslash\left\{u_{1}, u_{k+5}\right\}$, a contradiction. So $\left|B \cap\left\{u_{6}, u_{k}\right\}\right| \geq 1$.

Hence, (1) and (2) hold.
If $A=\left\{u_{1}, u_{5}\right\}$, then $u_{4 k}, u_{6} \in B$. Without loss of generality, let $u_{3 k} \in$ $B$ and set $D=\left\{u_{3 k+5}\right\}$. In $G^{\prime}$, the vertices of $N\left(u_{1}\right)$ other than $v_{1}$ and the vertices of $N\left(u_{5}\right)$ other than $v_{5}$ are monitored, so $v_{1}$ and $v_{5}$ are monitored by the propagation. Similarly, $v_{3 k+1}$ is monitored by the propagation since $u_{3 k+1}, v_{3 k+5}$ and the vertices of $N\left(u_{3 k+1}\right)$ other than $v_{3 k+1}$ are monitored by the domination step. So $G^{\prime}\left[F_{1} \cup F_{5}\right]$ is monitored by $D$ together with $A$ and $B$ by Observation 2 .

If $A=\left\{u_{1}, u_{k+5}\right\}$, then $u_{4 k}, u_{k+6} \in B$. Without loss of generality, let $u_{6} \in B$ and set $D=\left\{v_{k+1}\right\}$. Similar to the above proof, $D$ together with $A$ and $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ in $G^{\prime}$. By the symmetry of $G^{\prime}$, we can find a set $D \subseteq F_{1} \cup F_{5}$
with $|D|=1$ which together with $A$ and $B$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ for other cases. So $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right|=|S|-2$.

Case 3. $|A| \geq 3$. We have the following statements.
(1) $\left|F_{1}^{u} \cap A\right| \geq 1$ and $\left|F_{5}^{u} \cap A\right| \geq 1$.
(2) If $\left|F_{1}^{u} \cap A\right| \geq 2$ or $\left|F_{5}^{u} \cap A\right| \geq 2$, then there are two vertices $u_{i}, u_{j} \in F_{1}^{u} \cap A$ (or $F_{5}^{u} \cap A$ ) such that $|i-j| \neq 2 k$.

Similar to the proof of Case 2, we can prove that (1) holds. Suppose (2) does not hold. Without loss of generality, let $u_{1}, u_{2 k+1} \in F_{1}^{u} \cap A$. Then $u_{k+1}, u_{3 k+1} \notin$ $F_{1}^{u} \cap A$. There are two vertices in $S \cap\left(\bigcup_{i=1}^{3} F_{i}\right)$ to monitor $F_{1} \cup F_{2}$. Similarly, there are two vertices in $S \cap\left(\bigcup_{i=3}^{5} F_{i}\right)$ to monitor $F_{4} \cup F_{5}$. So $\left|F_{3} \cap S\right|=1$. However, no such three vertices monitor $\bigcup_{i=1}^{5} F_{i} \backslash\left\{u_{1}, u_{2 k+1}\right\}$.

If $u_{1}, u_{5}, u_{k+5} \in A$, then set $D=\left\{v_{2 k+1}\right\}$. If $u_{1}, u_{k+1}, u_{2 k+5} \in A$, then set $D=\left\{v_{3 k+5}\right\}$. If $u_{1}, u_{k+1}, u_{3 k+5} \in A$, then set $D=\left\{v_{5}\right\}$. In each case, $D$ together with $A$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ in $G^{\prime}$ by the propagation step and Observation 2. By the symmetry of $G^{\prime}$, we can find a set $D \subseteq F_{1} \cup F_{5}$ with $|D|=1$ which together with $A$ monitors $G^{\prime}\left[F_{1} \cup F_{5}\right]$ for other cases. So $S^{\prime}=\left(S \backslash \bigcup_{i=1}^{5} F_{i}\right) \cup D$ is a PDS for $G^{\prime}$ and $\left|S^{\prime}\right|=|S|-2$.

In each case, we obtain a PDS $S^{\prime}$ for $G^{\prime}$ with $|S|-2$ vertices. Therefore, the assertion follows.

This completes the proof of Lemma 5.
We are now ready to give the exact power domination number for $P(4 k, k)$. The following observation is useful.

Observation 6. For $k \geq 2, \gamma_{P}(P(4 k, k)) \geq \gamma_{P}(P(4(k-1), k-1))$.
Proof. Let $S$ be a $\gamma_{P}$-set for $P(4 k, k)=(U, V)$. By Lemma 3, $\gamma_{P}(P(4 k, k)) \leq$ $\left\lceil\frac{2 k}{3}\right\rceil$ holds. Then there is one set $F_{i}$ for some $i \in\{1,2, \ldots, k\}$ such that $S \cap F_{i}=\emptyset$. Let $U^{\prime}=U \backslash\left\{u_{q k+i} \mid q \in\{0,1,2,3\}\right\}, V^{\prime}=V \backslash\left\{v_{q k+i} \mid q \in\{0,1,2,3\}\right\}$ and $E^{\prime}=$ $E\left(U^{\prime} \cup V^{\prime}\right) \cup\left\{u_{q k+i-1} u_{q k+i+1} \mid q=0,1,2,3\right\}$. Then $G^{\prime}=\left(U^{\prime} \cup V^{\prime}, E^{\prime}\right)$ is isomorphic to $P(4(k-1), k-1)$. Obviously, $S$ is also a PDS for $G^{\prime}=P(4(k-1), k-1)$. So $\gamma_{P}(P(4(k-1), k-1)) \leq|S|=\gamma_{P}(P(4 k, k))$.
Lemma 7. For the generalized Petersen graph $P(4 k, k)$ with $1 \leq k \leq 6$, we have

$$
\gamma_{P}(P(4 k, k))= \begin{cases}2 & \text { for } k=1 \text { or } 2, \\ 3 & \text { for } k=3 \text { or } 4, \\ 4 & \text { for } k=5 \text { or } 6 .\end{cases}
$$

Proof. We notice that no vertex in $G=P(4,1)$ or $P(8,2)$ can completely monitor it and there are two vertices (see Figure 5(a) and 5(b)) which form a PDS

(a) $\gamma_{P}(P(4,1))=2$
(c) $\gamma_{P}(P(12,3))=3$

(b) $\gamma_{P}(P(8,2))=2$

(e) $\gamma_{P}(P(20,5))=4$

(d) $\gamma_{P}(P(16,4))=3$


(f) $\gamma_{P}(P(24,6))=4$

Figure 5. Vertices with bold circles form a $\gamma_{P}$-set for $P(4 k, k)$ with $1 \leq k \leq 6$.
for $G$. So $\gamma_{P}(P(4,1))=\gamma_{P}(P(8,2))=2$. By Observation 6, $\gamma_{P}(P(12,3)) \geq$ $\gamma_{P}(P(8,2))=2$. Note that no two vertices in $P(12,3)$ can completely monitor it and we can find three vertices (see Figure $5(\mathrm{c})$ ) which form a PDS for $P(12,3)$. So $\gamma_{P}(P(12,3))=3$. Since we can find three vertices (see Figure $5(\mathrm{~d}))$ in $P(16,4)$ which form a PDS and $\gamma_{P}(P(16,4)) \geq \gamma_{P}(P(12,3))=3$, $\gamma_{P}(P(16,4))=3$. Similarly, no three vertices in $P(20,5)$ can completely monitor it and $\gamma_{P}(P(20,5))=\gamma_{P}(P(24,6))=4$ (see Figure $5(\mathrm{e})$ and $\left.5(\mathrm{f})\right)$.

Theorem 8. For the generalized Petersen graph $P(4 k, k)$,

$$
\gamma_{P}(P(4 k, k))= \begin{cases}2 & \text { for } k=1 \\ 3 & \text { for } k=3 \\ \left\lceil\frac{2 k}{3}\right\rceil & \text { for } k=2 \text { or } k \geq 4\end{cases}
$$

Proof. We show that $\gamma_{P}(P(4 k, k)) \geq\left\lceil\frac{2 k}{3}\right\rceil$ for all $k \geq 4$ holds by induction on $k$ as follows. By Lemma $7, \gamma_{P}(P(4 k, k))=\left\lceil\frac{2 k}{3}\right\rceil$ for $k \in\{4,5,6\}$. Suppose the
result holds for $k \geq 7$. Then $\gamma_{P}(P(4(k+3), k+3)) \geq\left\lceil\frac{2 k}{3}\right\rceil$ by Lemma 5 together with $\left\lceil\frac{2(k+3)}{3}\right\rceil=\left\lceil\frac{2 k}{3}\right\rceil+2$.

Combining with Lemma 3 and Lemma 7, we obtain the desired result.

## 4. Closing Remarks

In this paper, we give an upper bound on the power domination number for $P(c k, k)$ with $c \geq 4$ and determine the exact power domination number for $P(4 k, k)$. Finally, we conjecture that the following result holds.

Conjecture 9. For $k \geq 2$ and $c \geq 5, \gamma_{P}(P(c k, k))=\left\lceil\frac{2 k}{3}\right\rceil$ if $2 k \equiv 1(\bmod 3)$, and $\gamma_{P}(P(c k, k)) \in\left\{\left\lceil\frac{2 k}{3}\right\rceil,\left\lceil\frac{2 k}{3}\right\rceil+1\right\}$ if $2 k \equiv 0$ or $2 k \equiv 2(\bmod 3)$.

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