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TOTAL FORCING SETS AND ZERO FORCING SETS IN TREES

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Abstract

A dynamic coloring of the vertices of a graph G starts with an initial subset S of colored vertices, with all remaining vertices being non-colored. At each discrete time interval, a colored vertex with exactly one non-colored neighbor forces this non-colored neighbor to be colored. The initial set S is called a forcing set of G if, by iteratively applying the forcing process, every vertex in G becomes colored. If the initial set S has the added property that it induces a subgraph of G without isolated vertices, then S is called a total forcing set in G. The minimum cardinality of a total forcing set in G is its total forcing number, denoted $F_t(G)$. We prove that if T is a tree of order $n \geq 3$ with maximum degree Δ and with n_1 leaves, then $n_1 \leq F_t(T) \leq \frac{1}{\Delta}((\Delta - 1)n + 1)$. In both lower and upper bounds, we characterize the infinite family of trees achieving equality. Further we show that $F_t(T) \geq F(T) + 1$, and we characterize the extremal trees for which equality holds.

Keywords: forcing set, forcing number, total forcing set, total forcing number.

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1. INTRODUCTION

A dynamic coloring of the vertices in a graph is a coloring of the vertex set which may change, or propagate, throughout the vertices during discrete time intervals. Of the dynamic colorings, the notion of *forcing sets* (*zero forcing sets*), and the associated graph invariant known as the *forcing number* (*zero forcing number*), are arguably the most prominent, see for example [1, 3, 7, 11, 14, 15, 12, 22]. In the study of minimum forcing sets in graphs, it is natural to consider the initial structure of such sets. In particular, if a forcing set induces an isolate-free subgraph, then the set in question is called a *total forcing set*.

More formally, let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *forcing process* is defined in [10] as follows: Let $S \subseteq V$ be a set of initially "colored" vertices, all other vertices are said to be "non-colored". A vertex contained in S is said to be S-colored, while a vertex not in S is said to be S-uncolored. At each time step, if a colored vertex has exactly one non-colored neighbor, then this colored vertex *forces* its non-colored neighbor to become colored. If v is such a colored vertex, we say that v is a *forcing vertex*. We say that S is a *zero forcing set*, if by iteratively applying the forcing process, all of V becomes colored. We call such a set S, an *S-forcing set*. In addition, if S is an S-forcing set in G and v is a S-colored vertex that forces a new vertex to be colored, then v is an *S-forcing vertex*. The *zero forcing number* of a graph G, denoted $F(G)$ (also denoted $Z(G)$ in the literature), is the cardinality of a smallest forcing set in G. For notational simplicity, throughout the paper we simply refer to a zero forcing set as a *forcing set*, and the zero forcing number as the *forcing number*.

One of the most fundamental *domination-type* graph parameters is the *total domination number* which in general behaves very differently from the ordinary *domination number* and is heavily studied in the literature (see [18]). The following observation is well-known (see, for example, [16]), where γ, γ_t and γ_c denote the domination, total domination and connected domination numbers, respectively.

Observation 1. *For every connected graph G on at least two vertices,*

$$\gamma(G) \le \gamma_t(G) \le \gamma_c(G).$$

Analogously, we study the concept of a total forcing set in a graph which is a fundamental *forcing-type* parameter. In doing so, we show that the total forcing number and the zero forcing number in general, behave very differently.

If S is a forcing set of a graph G that induces a connected subgraph, then S is a *connected forcing set*. The cardinality of a smallest connected forcing set in G is its *connected forcing number*, denoted $F_c(G)$. If S is a forcing set that induces a graph without isolated vertices, then S is a *total forcing set*, abbreviated as a

TF-*set* of G. The *total forcing number* of G, written $F_t(G)$, is the cardinality of a minimum TF-set in G. The concept of a total forcing set was first introduced and studied by Davila in [8], and is further studied, for example, by the authors in [10]. Since every connected forcing set is a TF-set, and every TF-set is a forcing set, the total forcing number is squeezed between the forcing number and the connected forcing number.

Observation 2. *For every connected graph G on at least two vertices,*

$$F(T) \leq F_t(G) \leq F_c(G).$$

Thus the total forcing number provides an upper bound on the well-studied forcing number and a lower bound on the connected forcing number. We highlight that the total forcing number is also intimately related to the *power domination number*, denoted $\gamma_P(G)$, which was introduced in [17] and is now very well studied in the literature; a rough estimate says that it occurs in more than 50 papers to date. The following theorem illustrates this fundamental relationship.

Theorem 3 [10]**.** *If G is a graph without isolated vertices, then $\gamma_P(G) \leq \frac{1}{2}F_t(G)$.*

In particular, whenever a TF-set is known, one may construct a power dominating set on half the vertices in the TF-set. Thus, total forcing in graphs is directly applicable to the power dominating set problem. The motivation behind the power dominating set problem is the placement of power monitoring units (PMU's) in electrical grids such that the number of PMU's (PMU's are expensive) used is minimized (precisely $\gamma_P(G)$). However, one can imagine a scenario where the maintenance and repair of said PMU's is time consuming, or that distance between PMU's incurs some additional cost. Thus, it may be beneficial to minimize the number of PMU's used along with minimizing distance from one PMU to another. This concept is related to total forcing (or *connected forcing*), since one can imagine each colored vertex in a TF-set as a PMU, and so, all vertices will be monitored, and every PMU is connected with another PMU.

In this paper, we study total forcing sets in trees. In particular, we study trees with smallest possible total forcing number, as well as trees with largest possible forcing number. The following results, first observed in [10], will prove helpful when proving our main results.

Observation 4 [10]**.** *If G is an isolate-free graph of order $n \geq 3$, then $F_t(G) \leq n-1$, with equality if and only if $G = K_n$ or $G = K_{1,n-1}$.*

Observation 5 [10]**.** *Every TF-set in an isolate-free graph contains every strong support vertex of the graph and all except possibly one leaf neighbor of each strong support vertex.*

2. Definitions and Notations

For notation and graph terminology, we will typically follow [18]. Throughout this paper, all graphs will be considered undirected, simple and finite. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n = |V(G|$ and size $m = |E(G)|$. If the graph G is clear from the context, we simply write V and E rather than $V(G)$ and $E(G)$, and we write $G = (V, E)$. A *non-trivial* graph is a graph with at least two vertices.

Two vertices v and w are adjacent, or neighbors, in G if $vw \in E$. The open neighborhood of a vertex $v \in V$, is the set of neighbors of v, denoted $N_G(v)$, whereas its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The open neighborhood of $S \subseteq V$ is the set of all neighbors of vertices in S, denoted $N_G(S)$, whereas the closed neighborhood of S is $N_G[S] = N_G(S) \cup S$. We denote the *degree* of a vertex v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. Thus, $d_G(v) = \big|N_G(v)\big|$. The minimum and maximum degree among the vertices of G is denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For a set of vertices $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we simply write $G - v$ rather than $G - S$.

A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. A *strong support vertex* is a vertex with at least two leaf neighbors. A *star* is a non-trivial tree with at most one vertex that is not a leaf. Thus, a star is the tree $K_{1,k}$ for some $k \geq 1$. For $r, s \geq 1$, a *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. We will denote a path on n vertices by P_n. We define a *pendant edge* of a graph to be an edge incident with a vertex of degree 1.

The *distance* between two vertices v and w in G is the length of a shortest (v, w)-path in G, and is denoted by $d_G(v, w)$. If no (v, w)-path exists in G, then we define $d_G(v, w) = \infty$. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. The *eccentricity* of a vertex v in G is the maximum distance of a vertex from v in G. A vertex of minimum eccentricity is called a *central vertex* of G. In particular, the central vertex of a star of order at least 3 is the vertex that is not a leaf, while a double star contains two central vertices, namely the two vertices that are not leaves.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v, while an *ancestor* of v is a vertex $u \neq v$ that belongs to the (r, v)-path in T. In particular, every child of v is a descendant of v while the parent of v is an ancestor of v. The *grandparent* of v is the ancestor of v at distance 2 from v. A *grandchild*

of v is the descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

We use the standard notation $[k] = \{1, \ldots, k\}$.

3. Main Results

Computing the forcing, total forcing and connected forcing number for a general graph is NP-hard [5, 7, 10, 22]. Hence it is of interest to determine computationally efficient lower and upper bounds on the forcing-type parameters. Recently, the authors established the following upper bound on the total forcing number of a graph with minimum degree at least two in terms of the order and maximum degree of the graph.

Theorem 6 [10]**.** *If G is a connected graph of order $n \geq 3$ with minimum degree at least 2 and with maximum degree Δ, then*

$$F_t(G) \leq \left(\frac{\Delta}{\Delta + 1}\right) n,$$

with equality if and only if $G \cong K_n$.

In this paper, we study the total forcing number of a tree. We have four immediate aims. First to prove that the upper bound established in Theorem 6 also holds for the class of trees. Secondly, to establish a much stronger upper bound on the total forcing number of a tree in terms of its order and maximum degree, and to characterize the extremal trees. Thirdly, to establish a lower bound on the total forcing number of a tree in terms of the number of leaves in the tree, and once again to characterize the extremal trees. Fourthly, to show that the total forcing number of a tree is strictly larger than its forcing number. More precisely, we shall prove the following four results, where $\mathcal{T}_\Delta$, $\mathcal{F}$, and $\mathcal{H}$ are families of trees we construct in Sections 4 and 5.

Theorem 7. *If T is a tree of order $n \geq 3$ with maximum degree Δ, then*

$$F_t(T) \leq \left(\frac{\Delta}{\Delta + 1}\right) n,$$

with equality if and only if $T \cong K_{1,\Delta}$.

Theorem 8. *If T is a tree of order $n \geq 3$ with maximum degree Δ, then*

$$F_t(T) \leq \frac{(\Delta - 1)n + 1}{\Delta},$$

with equality if and only if $T \in \mathcal{T}_\Delta$.

Theorem 9. *If T is a non-trivial tree with n_1 leaves, then $F_t(T) \geq n_1$, with equality if and only if $T \in \mathcal{F}$.*

Theorem 10. *If T is a non-trivial tree, then $F_t(T) \geq F(T) + 1$, with equality if and only if $T \in \mathcal{H}$.*

We remark that the proof techniques and strategies to obtain the bounds (and characterizations) in Theorem 7 and Theorem 8 are very different from those in [9, 10] used to obtain the bound in Theorem 6. Indeed, the bound in Theorem 6 relies heavily on the minimum degree at least two restriction on the graph G, while the bounds in Theorem 7 and Theorem 8 are heavily reliant on the structure of a tree. As observed earlier, the result of Theorem 8 for the class of trees is a much stronger result than the bound of Theorem 6 for general graphs. We remark further that the result of Theorem 8 and Theorem 9 establish very easily computable upper and lower bounds, respectively, on the total forcing number of a tree.

4. Upper Bounds on Total Forcing in Trees

In this section, we present a proof of Theorem 8, which establishes an upper bound on the total forcing number of a tree in terms of its order and maximum degree, and characterize the extremal trees. For this purpose, we define a family $\mathcal{T}_\Delta$ of trees as follows.

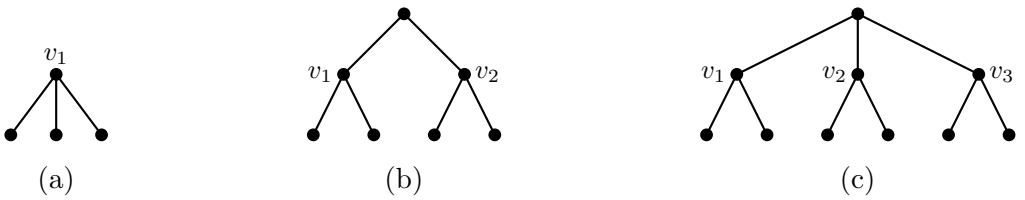


Figure 1. The three trees in the family $\mathcal{T}_3$.

The family $\mathcal{T}_\Delta$. Let $\mathcal{T}_\Delta$ be the family of all trees T with maximum degree Δ whose vertex set $V(T)$ can be partitioned into sets $\big(V_1, \ldots, V_k\big)$ such that the following holds, where $T_i = T\big[V_i\big]$ for $i \in [k]$.

- $T_1 \cong K_{1,\Delta}$, and if $k \geq 2$, then $T_i \cong K_{1,\Delta-1}$ for $i \in [k] \setminus \{1\}$.
- For $i \in [k]$, the central vertex v_i of the star T_i is a strong support vertex of degree Δ in the tree T.
- The set $\big\{v_1, \ldots, v_k\big\}$ is an independent set in T.

We call the trees $T_1, \ldots, T_k$ the *underlying subtrees* of the tree T. We note that $\mathcal{T}_2$ consists only of the path P_3; that is, $\mathcal{T}_2 = \{P_3\}$. The family $\mathcal{T}_3$ consists of the three trees shown in Figure 1(a), 1(b) and 1(c).

We first establish useful properties of trees in the family $\mathcal{T}_\Delta$.

Lemma 11. *If T is a tree of order n that belongs to the family $\mathcal{T}_\Delta$, then the following holds.*

(a) $F_t(T) = \frac{1}{\Delta}\big((\Delta-1)n+1\big)$.

(b) *The set consisting of all vertices of T, except for exactly one leaf neighbor in T of the central vertex of each underlying subtree of T is a minimum TF-set of T.*

Proof. Let $T \in \mathcal{T}_\Delta$ be a tree of order n with maximum degree Δ. We proceed by induction on the number, k, of underlying subtrees of the tree T. If $k = 1$, then $T \cong K_{1,\Delta}$ and by Observation 4, $F_t(T) = \Delta = \big((\Delta-1)n+1\big)/\Delta$ noting that here $n = \Delta + 1$. Further, property (b) is immediate in this case. This establishes the base case. Let $k \geq 2$ and assume that if T' is a tree of order n' in the family $\mathcal{T}_\Delta$ with k' underlying subtrees, where $k' < k$, then $F_t(T') = \frac{1}{\Delta}\big((\Delta-1)n'+1\big)$.

Let T be a tree of order n in the family $\mathcal{T}_\Delta$ with k underlying subtrees given by $T_1, T_2, \ldots, T_k$, where $T_1 \cong K_{1,\Delta}$ and $T_i \cong K_{1,\Delta-1}$ for $i \in [k] \setminus \{1\}$. Recall that $V_i = V(T_i)$ for $i \in [k]$. We note that $n = k\Delta + 1$. Let S consist of all vertices of T, except for exactly one leaf neighbor of the central vertex of each subtree of T. Further, let $S_i = S \cap V_i$ for $i \in [k]$. Thus, $|S_1| = \Delta$ and $|S_i| = \Delta - 1$ for $i \in [k] \setminus \{1\}$. The set S is a TF-set, and so

$$\begin{aligned} F_t(T) \leq |S| &= \sum_{i=1}^{k} |S_i| = \Delta + \sum_{i=2}^{k} (\Delta - 1) \\ &= k(\Delta-1) + 1 = \tfrac{1}{\Delta}\big((\Delta-1)n+1\big). \end{aligned}$$

If $F_t(T) = \frac{1}{\Delta}\big((\Delta-1)n+1\big)$, then we must have equality throughout the above inequality chain, implying that $F_t(T) = |S|$ and hence that S is a minimum TF-set of T. Therefore, if property (a) holds, then property (b) holds. Hence, it suffices for us to prove property (a).

Let T^* be the graph of order k whose vertices correspond to the k subtrees of T, and where we add an edge between two vertices of T^* if the corresponding subtrees of T are joined by an edge in T. Since T is a tree, so too is T^*. Since $k \geq 2$, at least one leaf of T^* corresponds to an underlying subtree T_i for some $i \geq 2$. Among the subtrees $T_2, \ldots, T_k$, there exists a subtree, T_k say, which is joined by an edge to only one other subtree, T_j say, where the central vertex v_k of T_k is adjacent to a leaf, x say, of T_j where $j \in [k-1]$. By definition of the family

$\mathcal{T}_\Delta$, the set of central vertices of the k underlying subtrees form an independent set in T, implying that $x \neq v_j$.

Let T' be the tree of order n' obtained from T by deleting the vertices in $V(T_k)$. We note that $T' \in \mathcal{T}_\Delta$ with underlying subtrees $T_1, \ldots, T_{k-1}$. Further, $n' = n - \Delta$. Applying the inductive hypothesis to the tree T', $F_t\big(T'\big) = \frac{1}{\Delta}\big((\Delta - 1)n' + 1\big)$. Let X be a minimum TF-set in T. By Observation 5, the set X contains the vertex v_k and all except possibly one leaf neighbor of v_k. If the set X contains all leaf neighbors of v_k, then by the minimality of the set X, we note that $x \notin X$. However, in this case, we can simply remove exactly one leaf neighbor of v_k from X and add the vertex x to the set X to produce a new minimum TF-set in T. Thus, we may choose the set X to contain all except exactly one leaf neighbor of v_k. Let X' be the restriction of X to $V\big(T'\big)$; that is, $X' = X \cap V\big(T'\big)$. Since X is a TF-set of T, the set X' is a TF-set of T'. Thus, $F_t\big(T'\big) \leq \big|X'\big| = |X| - (\Delta - 1) = F_t(T) - \Delta + 1$, implying that

$$\begin{aligned} F_t(T) &\geq F_t\big(T'\big) + \Delta - 1 = \tfrac{1}{\Delta}\big((\Delta - 1)n' + 1\big) + \Delta - 1 \\ &= \tfrac{1}{\Delta}\big((\Delta - 1)(n - \Delta) + 1\big) + \Delta - 1 = \tfrac{1}{\Delta}\big((\Delta - 1)n + 1\big). \end{aligned}$$

As observed earlier, $F_t(T) \leq \frac{1}{\Delta}\big((\Delta - 1)n + 1\big)$. Consequently, $F_t(T) = \frac{1}{\Delta}\big((\Delta - 1)n + 1\big)$. ∎

With the aid of Lemma 11, we next prove Theorem 8.

Proof of Theorem 8. We proceed by induction on the order $n \geq 3$ of a tree T with maximum degree Δ. If $n = 3$, then $T \cong P_3 \in \mathcal{T}_2$, $\Delta = 2$ and $F_t(T) = 2 = \big((\Delta - 1)n + 1\big)/\Delta$. This establishes the base case. Let $n \geq 4$ and assume that if T' is a tree of order n' and maximum degree Δ', where $3 \leq n' < n$ and $\Delta' \leq \Delta$, then $F_t\big(T'\big) \leq \big((\Delta' - 1)n' + 1\big)/\Delta'$, with equality if and only if $T' \in \mathcal{T}_{\Delta'}$. If $\Delta' < \Delta$, then, by monotonicity, we note that $\big((\Delta' - 1)n' + 1\big)/\Delta' < \big((\Delta - 1)n' + 1\big)/\Delta$. Let T be a tree of order n and maximum degree Δ. We note that $\Delta \geq 2$. Recall that $n \geq 4$.

If $\Delta = 2$, then $T \cong P_n$ and $F_t(T) = 2 < (n + 1)/2 = \big((\Delta - 1)n + 1\big)/\Delta$. If $\operatorname{diam}(T) = 2$ and $\Delta \geq 3$, then T is a star. In this case, $\Delta = n - 1$ and $T \cong K_{1,\Delta} \in \mathcal{T}_\Delta$. By Observation 4, $F_t(T) = \Delta = \big((\Delta - 1)n + 1\big)/\Delta$. Hence, we may assume that $\Delta \geq 3$ and $\operatorname{diam}(T) \geq 3$, for otherwise the desired result follows.

Suppose that $\operatorname{diam}(T) = 3$, and so $T \cong S(r, s)$ is a double star, where $1 \leq r \leq s$. Let u and v be the two vertices of T that are not leaves, where u has r leaf neighbors and v has s leaf neighbors. Since T has maximum degree Δ, we note that $s = \Delta - 1$, and so $n = r + s + 2 = r + \Delta + 1$. Let u' and v' be arbitrary leaf neighbors of u and v, respectively. The set $V(T) \setminus \big\{u', v\big\}$ is a TF-set of T, implying that $F_t(T) \leq r + s = r + \Delta - 1$. Moreover, since $\Delta \geq r + 1$,

$$\frac{(\Delta-1)n+1}{\Delta} > \frac{(\Delta-1)n}{\Delta} = \frac{1}{\Delta}\big((\Delta-1)(r+\Delta+1)\big)$$
$$= r+\Delta-1+\frac{1}{\Delta}(\Delta-r-1) \geq r+\Delta-1 \geq F_t(T).$$

Hence, we may assume that $\mathrm{diam}(T) \geq 4$, for otherwise the desired result holds. Let u and r be two vertices at maximum distance apart in T. Necessarily, u and r are leaves and $d(u,r) = \mathrm{diam}(T)$. We now root the tree T at the vertex r. Let v be the parent of u, w the parent of v, x be the parent of w, and y the parent of x. We note that if $\mathrm{diam}(T) = 4$, then $y = r$; otherwise, $y \neq r$.

Let $d_T(v) = \ell$, where we note that $\ell \leq \Delta$. Let T' be the tree obtained from T by deleting v and its children; that is, $T' = T - V\big(T_v\big)$ where recall that T_v denotes the maximal subtree of T at v induced by $D[v]$. Let T' have order n', and so $n' = n - d_T(v) = n - \ell$. Since $\mathrm{diam}(T) \geq 4$, we note that $n' \geq 3$. Applying the inductive hypothesis to the tree T', $F_t\big(T'\big) \leq \big((\Delta'-1)n'+1\big)/\Delta' \leq \big((\Delta-1)n'+1\big)/\Delta$. Further, if $F_t\big(T'\big) = \big((\Delta-1)n'+1\big)/\Delta$, then $\Delta' = \Delta$ and $T' \in \mathcal{T}_\Delta$. Let S' be a minimum TF-set in T', and so $|S'| = F_t\big(T'\big)$. We note that every child of v is a leaf. Let S be the set obtained from S' by adding to it v and all children of v different from u; that is, $S = S' \cup \big(D[v] \setminus \{u\}\big)$. The set S is a TF-set of T, implying that

$$F_t(T) \leq |S| = |S'| + \ell - 1 = F_t\big(T'\big) + \ell - 1$$
$$\leq \tfrac{1}{\Delta}\big((\Delta-1)n'+1\big) + \ell - 1 = \tfrac{1}{\Delta}\big((\Delta-1)(n-\ell)+1\big) + \ell - 1$$
$$= \tfrac{1}{\Delta}\big((\Delta-1)n+1\big) + \tfrac{1}{\Delta}(\ell-\Delta) \leq \tfrac{1}{\Delta}\big((\Delta-1)n+1\big),$$

which establishes the desired upper bound of the theorem. Suppose that

$$F_t(T) = \tfrac{1}{\Delta}\big((\Delta-1)n+1\big)$$

(and still $\Delta \geq 3$ and $\mathrm{diam}(T) \geq 4$). Then, we must have equality throughout the above inequality chain, implying that $F_t\big(T'\big) = \frac{1}{\Delta}\big((\Delta-1)n'+1\big)$ and $d_T(v) = \ell = \Delta$. By the inductive hypothesis, $T' \in \mathcal{T}_\Delta$. If the parent w of v in T is a central vertex of one of the underlying trees of T', then it would have degree $\Delta+1$ in T, a contradiction. Hence, w is a leaf in one of the underlying trees of $T' \in \mathcal{T}_\Delta$. Let $T_1, \ldots, T_k$ denote the underlying trees of T', and let v_i be the central vertex of the tree T_i for $i \in [k]$. Further, let w belong to the subtree T_j, where $j \in [k]$. As observed earlier, w is a leaf of the underlying tree T_j of T'.

Since $T' \in \mathcal{T}_\Delta$, we note that $T_1 \cong K_{1,\Delta}$, and if $k \geq 2$, then $T_i \cong K_{1,\Delta-1}$ for $i \in [k] \setminus \{1\}$. Further, for $i \in [k]$, the central vertex v_i of the star T_i is a strong support vertex of degree Δ in the tree T', and the set $\big\{v_1, \ldots, v_k\big\}$ is an independent set in T'.

Suppose that the central vertex v_j of T_j is not a strong support vertex in T. Since v_j is a strong support vertex in T', this implies that v_j has precisely two leaf neighbors in T', one of which is necessarily the vertex w. Thus, w is a leaf in T' but has degree 2 in T, with v and x as its neighbors. This in turn implies that for $i \in [k] \setminus \{j\}$, every leaf neighbor of the vertex v_i in T' is also a leaf neighbor of v_i in T, and so the vertex v_i is a strong support vertex of degree Δ in the tree T. For $i \in [k]$, let v'_i be a leaf neighbor of v_i in the tree T. We note that v'_j is the unique leaf neighbor of v_j in T. Let $L = \bigcup_{i=1}^{k} \{v'_i\}$.

We now consider the set $S = V(T) \setminus \big(L \cup \{u, x\}\big)$. We note that the vertex v and all its neighbors different from u belong to S. Thus, playing the vertex v in the first step of the forcing process starting with the initial set S of colored vertices, the vertex v forces its leaf neighbor u to be colored. In the second step of the forcing process, we play the vertex w which forces the vertex $x = v_j$ to be colored. At this stage of the forcing process, we note that all vertices of $V(T)$ are colored, except for the k leaves $v'_1, \ldots, v'_k$. Since $\{v_1, \ldots, v_k\}$ is an independent set in T, we now simply play each of the vertices $v_1, \ldots, v_k$ in turn in the forcing process, thereby forcing the leaves $v'_1, \ldots, v'_k$ to be colored. In this way, we color all of $V(T)$ starting with the initial set S. Further, since $T[S]$ contains no isolated vertex, the set S is therefore a TF-set. We note that S contains all, except for $k+1$, vertices of $V\big(T'\big)$. By Lemma 11(b), $F_t\big(T'\big) = n' - k$, implying that $\big|S \cap V\big(T'\big)\big| = n' - (k+1) < F_t\big(T'\big)$. Thus,

$$\begin{aligned} F_t(T) \le |S| &= \big|S \cap V\big(T'\big)\big| + \Delta - 1 < F_t\big(T'\big) + \Delta - 1 \\ &= \tfrac{1}{\Delta}\big((\Delta-1)n' + 1\big) + \Delta - 1 = \tfrac{1}{\Delta}\big((\Delta-1)(n-\Delta) + 1\big) + \Delta - 1 \\ &= \tfrac{1}{\Delta}\big((\Delta-1)n + 1\big), \end{aligned}$$

a contradiction. Hence, the central vertex v_j of T_j is a strong support vertex in T. We now let $T_{k+1} = T_v$, where as defined earlier T_v is the maximal subtree of T at v induced by $D[v]$, and we let $v_{k+1} = v$. We note that $T_{k+1} \cong K_{1,\Delta-1}$. Further, we note that the central vertex v_i of the star T_i is a strong support vertex of degree Δ in the tree T for all $i \in [k+1]$, and the set $\{v_1, v_2, \ldots, v_{k+1}\}$ is an independent set in T. Thus, $T \in \mathcal{T}_\Delta$. Conversely, if $T \in \mathcal{T}_\Delta$, then by Lemma 11, $F_t(T) = \frac{1}{\Delta}\big((\Delta-1)n + 1\big)$. This completes the proof of Theorem 8. ∎

For all graphs of order n with maximum degree Δ, we note that $n \ge \Delta + 1$, implying that

$$\frac{(\Delta-1)n+1}{\Delta} \le \left(\frac{\Delta}{\Delta+1}\right) n. \tag{1}$$

Further, equality holds in inequality (1) if and only if $n = \Delta + 1$. Thus, the upper bound of Theorem 7 follows as an immediate consequence of the upper bound of

Theorem 8. Moreover, if T is a tree for which equality holds in inequality (1), then $n = \Delta + 1$, implying that $T = K_{1,\Delta}$.

5. Lower Bounds on Total Forcing in Trees

In this section, we present a proof of Theorem 9, which establishes a lower bound on the total forcing number of a tree in terms of the number of leaves in the tree, and characterizes the extremal trees. For this purpose, we present a series of preliminary lemmas which will be used in our subsequent argument to establish the desired lower bound.

Lemma 12. *Let G be an isolate-free graph that contains an edge e incident with a vertex of degree at most 2. If G' is obtained from G by subdividing the edge e any number of times, then $F_t(G) = F_t(G')$.*

Proof. Let G be an isolate-free graph that contains an edge $e = uv$, where v has degree 1 or 2 in G. Let G' be obtained from G by subdividing the edge e any number of times. We may assume the edge e is subdivided at least once, for otherwise $G' = G$ and the result is immediate. Let $P\colon uu_1 \cdots u_k v$ denote the resulting (u, v)-path in G', where $k \geq 1$. If $d_G(v) = 2$, then let w denote its neighbor different from u in G.

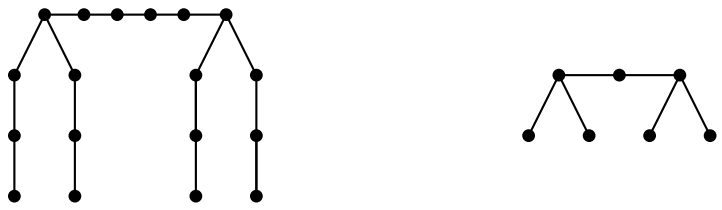
We first show that $F_t\big(G'\big) \leq F_t(G)$. Let $S \subseteq V(G)$ be a minimum TF-set of G, and so $|S| = F_t(G)$. Suppose that $v \in S$. Since S is a TF-set of G, the graph $G[S]$ contains no isolated vertex. In particular, S contains a neighbor of v. Suppose that both u and w belong to S. If v is the only neighbor of u in S, then the set $S \setminus \{u\}$ is a TF-set of G, contradicting the minimality of S. Hence, u has at least two neighbors in S. Analogously, w has at least two neighbors in S. Thus, the set $\big(S \setminus \{v\}\big) \cup \big\{u_1\big\}$ is a TF-set of G', and so $F_t\big(G'\big) \leq |S| = F_t(G)$, as desired. If $u \in S$ and $w \notin S$, then once again the set $\big(S \setminus \{v\}\big) \cup \big\{u_1\big\}$ is a TF-set of G', and $F_t(G') \leq F_t(G)$, as desired. If $u \notin S$ and $w \in S$, then the set S is a TF-set of G', and so $F_t\big(G'\big) \leq |S| = F_t(G)$, as desired. Hence we may assume that $v \notin S$.

Since S is a TF-set of G, there is a sequence $s\colon x_1, \ldots, x_t$ of played vertices in the forcing process that results in all $V(G)$ colored, where x_i denotes the forcing colored vertex played in the ith step of the process. In particular, $v = x_\ell$ for some integer ℓ where $\ell \in [t]$. Before the vertex v is colored, at least one neighbor of v is colored. If u is already colored before v, then starting with the same initial colored set S, the sequence obtained from s by replacing the vertex x_ℓ by the subsequence $u_1, \ldots, u_k, v$ results in a sequence of played vertices in the forcing process that results in all $V(G')$ colored. If u is colored after v, then w is necessarily colored before v. In this case, starting with the same initial colored

set S, the sequence obtained from s by replacing the vertex x_ℓ by the subsequence $v, u_k, u_{k-1}, \ldots, u_1$ results in a sequence of played vertices in the forcing process that results in all $V(G')$ colored. Thus, once again $F_t\big(G'\big) \le |S| = F_t(G)$, as desired.

We prove next that $F_t(G) \le F_t\big(G'\big)$. Let $S' \subseteq V(G)$ be a minimum TF-set of G', and so $|S| = F_t\big(G'\big)$. If $\big|S' \cap V(P)\big| \le 1$, then since the graph $G'[S]$ contains no isolated vertex, the set S' contains no vertex of P, except possibly for one of the ends of P, namely the vertex u or the vertex v. In both cases, the set S' is a TF-set of G, and so $F_t(G) \le \big|S'\big| = F_t\big(G'\big)$, as desired. If $\big|S' \cap V(P)\big| \ge 2$, then the set $\big(S' \setminus V(P)\big) \cup \{u, v\}$ is a TF-set of G, and so $F_t(G) \le |S'| - |S' \cap V(P)| + 2 \le |S'| = F_t(G')$, as desired. Thus, $F_t\big(G'\big) \le F_t(G)$ and $F_t(G) \le F_t\big(G'\big)$. Consequently, $F_t(G) = F_t\big(G'\big)$. ■

The contraction of an edge $e = xy$ in a graph G is the graph obtained from G by replacing the vertices x and y by a new vertex and joining this new vertex to all vertices that were adjacent to x or y in G. Given a non-trivial tree T, the *trimmed tree* of T, denoted $\mathrm{trim}(T)$, is the tree obtained from T by iteratively contracting edges with one of its incident vertices of degree exactly 2 and with the other incident vertex of degree at most 2 until no such edge remains. We note that if the original tree T is a path, then $\mathrm{trim}(T)$ is a path P_2, while if T is not a path, then every edge in $\mathrm{trim}(T)$ is incident with a vertex of degree at least 3. In particular, if T is not a path, then every support vertex in $\mathrm{trim}(T)$ has degree at least 3. As an illustration, the trimmed tree $\mathrm{trim}(T)$ of the tree T shown in Figure 2(a) is shown in Figure 2(b).



(a) T (b) $\mathrm{trim}(T)$

Figure 2. A tree T and its trimmed tree $\mathrm{trim}(T)$.

Since every non-trivial tree T can be reconstructed from its trimmed tree $\mathrm{trim}(T)$ by applying a sequence of subdivisions of edges incident with a vertex of degree at most 2, as an immediate consequence of Lemma 12 we note that $F_t(T) = F_t\big(\mathrm{trim}(T)\big)$. We remark that the number of leaves in T is equal to the number of leaves in $\mathrm{trim}(T)$. We state these properties of a trimmed tree formally as follows.

Lemma 13. *If T is a non-trivial tree, then the following hold.*
(a) $F_t(T) = F_t\big(\mathrm{trim}(T)\big)$.
(b) *The trees T and* $\mathrm{trim}(T)$ *have the same number of leaves.*

We proceed further by constructing a family $\mathcal{F}$ of trees with small total forcing number.

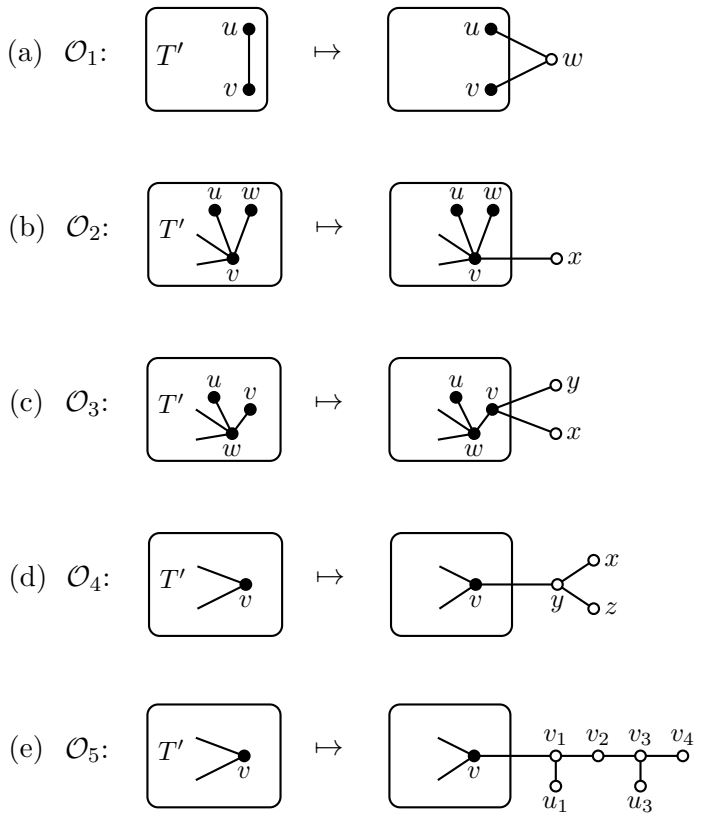


Figure 3. The operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}_5$.

The family $\mathcal{F}$. Let $\mathcal{F}$ be the family of trees that contains a path P_2 and is closed under the five operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_5$ below, which extend a tree T' to a new tree T. In Figure 3, the vertices of T' are colored black and the new vertices of T are colored white.

Operation $\mathcal{O}_1$: If uv is an edge of T' where at least one of u and v has degree at most 2 in T', then T is obtained from T' by subdividing the edge uv once. See Figure 3(a), where w denotes the new vertex (of degree 2 in T) obtained from subdividing the edge uv.

Operation $\mathcal{O}_2$: If v is a strong support vertex in T', then T is obtained from T'

by adding an additional pendant edge to the vertex v. See Figure 3(b), where u and w are leaf neighbors of v in T' and x is the new leaf added to T'.

Operation $\mathcal{O}_3$: If w is a strong support vertex in T' and v is a leaf neighbor of w, then T is obtained from T' by adding two pendant edges to v. See Figure 3(c), where u and v are leaf neighbors of w in T', and vx and vy are the pendant edges added to v.

Operation $\mathcal{O}_4$: If v is a vertex of degree at least 2 in T', then T is obtained from T' by adding a path P_3 and joining v to the central vertex of the path. See Figure 3(d) where xyz is the added path and vy the added edge.

Operation $\mathcal{O}_5$: If v is a vertex of degree at least 2 in T', then T is obtained from T' by adding a star $K_{1,3}$ with one edge subdivided twice and adding an edge joining the resulting support vertex of degree 2 to the vertex v. See Figure 3(e) where the added star has central vertex v_3 with leaf neighbors u_1, u_3 and v_4, and where the edge u_1v_3 of the star is subdivided twice resulting in the path $u_1v_1v_2v_3$ with v_1 a support vertex of degree 2 in the subdivided star.

In operations $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ and $\mathcal{O}_5$ illustrated in Figure 3, we call the vertex v the *link vertex* of T'. We are now in a position to establish the following lower bound on the total forcing number of a tree in terms of the number of leaves in the tree.

Lemma 14. *If T is a non-trivial tree with n_1 leaves, then $F_t(T) \geq n_1$. Further, if $F_t(T) = n_1$, then $T \in \mathcal{F}$.*

Proof. We proceed by induction on the order $n \geq 2$ of a tree T with n_1 leaves. If $n = 2$, then $T = P_2 \in \mathcal{F}$ and $F_t(T) = 2 = n_1$. This establishes the base case. Let $n \geq 3$ and assume that if T' is a non-trivial tree of order n' where $n' < n$ having n'_1 leaves, then $F_t\big(T'\big) \geq n'_1$ and that if $F_t\big(T'\big) = n'_1$, then $T' \in \mathcal{F}$. Let T be a tree of order n with n_1 leaves. We proceed further with the following series of claims.

Claim 15. *If $T \neq \mathrm{trim}(T)$, then $F_t(T) \geq n_1$ and if $F_t(T) = n_1$, then $T \in \mathcal{F}$.*

Proof. Suppose that the trimmed tree, $\mathrm{trim}(T)$, is different from T. This implies that T contains an edge e with one of its incident vertices of degree exactly 2 and with the other incident vertex of degree at most 2. Let T' be obtained from T by contracting the edge e. By Lemma 12, $F_t(T) = F_t\big(T'\big)$. Let T' have n'_1 leaves. We note that T' has order $n - 1$, and that $n_1 = n'_1$. Applying the inductive hypothesis to the tree T', $F_t(T) = F_t\big(T'\big) \geq n'_1 = n_1$. Further, if $F_t(T) = n_1$, then $F_t\big(T'\big) = n'_1$ and by the inductive hypothesis, $T' \in \mathcal{F}$. In this case, we can restore the tree T by applying operation $\mathcal{O}_1$ to the tree T', implying that $T \in \mathcal{F}$.

□

By Claim 15, we may assume that $T = \text{trim}(T)$, for otherwise the desired result follows. With this assumption, we note that every edge in T is incident with a vertex of degree at least 3. In particular, every support vertex in T has degree at least 3.

Claim 16. *If T contains a support vertex with at least three leaf neighbors, then $F_t(T) \geq n_1$ and if $F_t(T) = n_1$, then $T \in \mathcal{F}$.*

Proof. Suppose that T contains a support vertex v with at least three leaf neighbors. Let S be a minimum TF-set of F, and so $|S| = F_t(F)$. By Observation 5, the set S contains the vertex v and all except possibly one leaf neighbor of v. Let u and u' be two distinct leaf neighbors of v. If S contains every leaf neighbor of v, then by the minimality of S, there is a neighbor, w say, of v not in S. In this case, replacing the vertex u in S with the vertex w produces a new minimum TF-set of F. Hence, renaming the leaf neighbors of v if necessary, we may assume that $u \notin S$. We now consider the tree $T' = T - u'$. Let T' have n_1' leaves, and so $n_1' = n_1 - 1$. The set $S \setminus \{u'\}$ is necessarily a TF-set of T', implying that $F_t(T') \leq |S| - 1 = F_t(T) - 1$. Applying the inductive hypothesis to the tree T', we therefore have $n_1 - 1 = n_1' \leq F_t(T') \leq F_t(T) - 1$, implying that $F_t(T) \geq n_1$. Further, suppose that $F_t(T) = n_1$. In this case, $F_t(T') = n_1'$. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{F}$. We note that in the tree T' the vertex v is a strong support vertex. Hence, T can be obtained from the tree T' by applying operation $\mathcal{O}_2$ with v as the link vertex, and so $T \in \mathcal{F}$. □

By Claim 16, we may assume that every support vertex in T and has at most two leaf neighbors. Recall that every support vertex in T has degree at least 3. With these assumptions, we note that T is not a star, and so $\text{diam}(T) \geq 3$.

Claim 17. *If $\text{diam}(T) = 3$, then $F_t(T) = n_1$ and $T \in \mathcal{F}$.*

Proof. Suppose that $\text{diam}(T) = 3$, and so $T \cong S(r, s)$ is a double star, where $1 \leq r \leq s$. Since $T = \text{trim}(T)$, both vertices of T that are not leaves are strong support vertices, and so $r \geq 2$. By assumption, every support vertex in T has at most two leaf neighbors, and so $s \leq 2$. Thus, $r = s = 2$ and T is the double star $S(2, 2)$. By Observation 5, $F_t(T) \geq 4 = n_1$. If ℓ_1 and ℓ_2 are arbitrary leaves at distance 3 apart in T, then the set $V(T) \setminus \{\ell_1, \ell_2\}$ is a TF-set of T, and so $F_t(T) \leq |V(T)| - 2 = 4 = n_1$. Consequently, $F_t(T) = n_1$. Further, we note that the double star T can be constructed from a path P_2 by first applying operation $\mathcal{O}_1$ and then applying operation $\mathcal{O}_4$. Thus, $T \in \mathcal{F}$. □

By Claim 17, we may assume that $\text{diam}(T) \geq 4$, for otherwise the desired result holds. Let u and r be two vertices at maximum distance apart in T. Necessarily, u and r are leaves and $d(u, r) = \text{diam}(T)$. We now root the tree T at the vertex r. Let v be the parent of u, let w be the parent of v, let x

be the parent of w, and let y be the parent of x. Possibly, $y = r$. We note that every child of v is a leaf. Since $T = \text{trim}(T)$, we note that $d_T(v) \geq 3$. By assumption, every support vertex in T has at most two leaf neighbors. Thus, $d_T(v) \leq 3$. Consequently, $d_T(v) = 3$. Let u' be the child of v different from u. Thus, $N_T(v) = \{u, u', w\}$, where recall that w is the parent of v in T.

Let S be a minimum TF-set of F, and so $|S| = F_t(F)$. If S contains both children u and u' of v, then by the minimality of S, the vertex $w \notin S$. In this case, replacing u in S with the vertex w produces a new minimum TF-set of F. Hence, renaming the children of v if necessary, we may assume that $u \notin S$. By Observation 5, the set S therefore contains the vertex v and its child u'. Thus, $\{u', v\} \subseteq S$ and $u \notin S$.

Claim 18. *If $d_T(w) \geq 3$, then $F_t(T) \geq n_1$ and if $F_t(T) = n_1$, then $T \in \mathcal{F}$.*

Proof. Suppose that $d_T(w) \geq 3$. We now consider the tree $T' = T - \{u, u', v\}$. Let T' have n_1' leaves, and so $n_1' = n_1 - 2$. Let $S' = S \setminus \{u', v\}$. If S' is a TF-set of T', then, by the inductive hypothesis, $n_1 - 2 = n_1' \leq F_t(T') \leq |S'| = |S| - 2 = F_t(T) - 2$, implying that $F_t(T) \geq n_1$. Further, suppose that $F_t(T) = n_1$. In this case, $F_t(T') = n_1'$. Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{F}$. We note that in the tree T' the vertex w has degree at least 2, and therefore T can be obtained from the tree T' by applying operation $\mathcal{O}_4$ with w as the link vertex, and so $T \in \mathcal{F}$.

Hence, we may assume that the set S' is not a TF-set of T', implying that S contains w but no neighbor of w except for its child v. In particular, S contains no neighbor of w in T'. If a child v' of w different from v is not a leaf, then identical arguments as shown with the vertex v show that $d_T(v') = 3$ and that $v' \in S$. Thus, S contains a neighbor of w different from v, a contradiction. Hence, every child of w different from v is a leaf. If w has at least two leaf neighbors, then by Observation 5, the set S contains w and all except possibly one leaf neighbor of w, implying once again that S contains a neighbor of w different from v, a contradiction. Therefore, $d_T(w) = 3$ and the child, v' say, of w different from v is a leaf.

We now consider the tree $T'' = T - \{u, u'\}$. Let T'' have n_1'' leaves, and so $n_1'' = n_1 - 1$ noting that the vertex v is a leaf in T'' but not in T. Recall that both v and w belong to S. The set $S \setminus \{u'\}$ is a TF-set of T'', implying that $F_t(T'') \leq |S| - 1 = F_t(T) - 1$. Applying the inductive hypothesis to the tree T', we therefore have $n_1 - 1 = n_1'' \leq F_t(T'') \leq F_t(T) - 1$, implying that $F_t(T) \geq n_1$. Further, suppose that $F_t(T) = n_1$. In this case, $F_t(T'') = n_1''$. Applying the inductive hypothesis to the tree T'', we have $T'' \in \mathcal{F}$. We note that in the tree T'' the vertex w is a strong support vertex with v as one of its leaf neighbors. Hence, T can be obtained from the tree T'' by applying operation $\mathcal{O}_3$ with v as the link vertex, and so $T \in \mathcal{F}$. □

By Claim 18, we may assume that $d_T(w) = 2$, for otherwise the desired result holds. Since $T = \text{trim}(T)$, this implies that $d_T(x) \geq 3$.

Claim 19. *If $w \notin S$ or if $\{w, x\} \subset S$, then $F_t(T) \geq n_1 + 1$.*

Proof. Suppose that $w \notin S$ or $\{w, x\} \subset S$. In this case, we consider the tree $T' = T - \{u, u', v\}$. Let T' have n_1' leaves, and so $n_1' = n_1 - 1$, noting that the vertex w is a leaf in T' but not in T. Let $S' = S \setminus \{u', v\}$. Since $w \notin S$ or $\{w, x\} \subset S$, the set S' is a TF-set of T', and so, by the inductive hypothesis, $n_1 - 1 = n_1' \leq F_t(T') \leq |S'| = |S| - 2 = F_t(T) - 2$, implying that $F_t(T) \geq n_1 + 1$. □

By Claim 19, we may assume that $w \in S$ and $x \notin S$, for otherwise the desired result holds. Thus, by our earlier assumptions, $S \cap \{u, u', v, w, x\} = \{u', v, w\}$. Recall that $d_T(x) \geq 3$.

Claim 20. *If $d_T(v) \geq 4$ or if $d_T(x) = 3$ and the child of x different from w is not a leaf, then $F_t(T) \geq n_1$ and if $F_t(T) = n_1$, then $T \in \mathcal{F}$.*

Proof. Suppose that $d_T(v) \geq 4$ or $d_T(x) = 3$ and the child of x different from w is not a leaf. If x has at least two leaf neighbors, then by our earlier assumptions, x has exactly two leaf neighbors. In this case, by Observation 5 the set S contains the vertex x (and at least one leaf neighbor of x). This contradicts our assumption that $x \notin S$. Hence, at most one child of x is a leaf. By assumption, there is a child w_1 of x different from w of degree at least 2. We note that either w_1 has a grandchild or every child of w_1 is a leaf.

Suppose firstly that w_1 has a grandchild, say u_1. Let v_1 be the parent of u_1. Using analogous arguments as before with the vertices v and w, we may assume that $d_T(v_1) = 3$ and $d_T(w_1) = 2$, for otherwise the desired result follows. Let u_2 denote the child of v_1 different from u_1. Analogously as before, we may assume that $S \cap \{u_1, u_2, v_1, w_1\} = \{u_1, v_1, w_1\}$, for otherwise the desired result follows. We now consider the tree $T' = T - \{u, u', v, w\}$, and let T' have n_1' leaves. Thus, $n_1' = n_1 - 2$. The set $S \setminus \{u', v, w\}$ is a TF-set of T', and so, by the inductive hypothesis, $n_1 - 2 = n_1' \leq F_t(T') \leq |S| - 3 = F_t(T) - 3$, implying that $F_t(T) \geq n_1 + 1$. Hence, we may assume that w_1 has no grandchild, for otherwise $F_t(T) \geq n_1 + 1$. Thus, every child of w_1 is a leaf.

Since $T = \text{trim}(T)$, we note that $d_T(w_1) \geq 3$. By assumption, every support vertex in T has at most two leaf neighbors. Thus, $d_T(w_1) \leq 3$. Consequently, $d_T(w_1) = 3$. Let v_1 and v_2 denote the two children of w_1. By Observation 5, the set S contains the vertex w_1 and at least one of v_1 and v_2. If S contains both v_1 and v_2, then replacing v_1 in S with the vertex x produces a new minimum TF-set of F that contains the vertex x, contradicting our earlier assumptions. Hence, renaming v_1 and v_2 if necessary, we may assume that $S \cap \{v_1, v_2, w_1\} = \{v_1, w_1\}$.

We now consider the tree $T'' = T - \{v_1, v_2, w_1\}$. Let T'' have n_1'' leaves. Thus, $n_1'' = n_1 - 2$. The set $S \setminus \{v_1, w_1\}$ is a TF-set of T'', and so, by the inductive hypothesis, $n_1 - 2 = n_1'' \le F_t(T'') \le |S| - 2 = F_t(T) - 2$, implying that $F_t(T) \ge n_1$. Further, if $F_t(T) = n_1$, then $F_t(T'') = n_1''$ and by the inductive hypothesis, $T'' \in \mathcal{F}$. In this case, we can restore the tree T by applying operation $\mathcal{O}_4$ to the tree T'' with x as the link vertex, implying that $T \in \mathcal{F}$. □

By Claim 20, we may assume that $d_T(x) = 3$ and the child, w' say, of x different from w is a leaf, for otherwise the desired result holds. Recall that by our earlier assumptions, $x \notin S$, implying that $w' \notin S$.

Claim 21. *If $d_T(y) = 2$, then $F_t(T) \ge n_1 + 1$.*

Proof. Suppose that $d_T(y) = 2$. In this case, we consider the tree $T' = T - \{u, u', v, w, w', x\}$; that is, T' is the tree obtained from T by deleting x and all its descendants. Let T' have n_1' leaves. Thus, $n_1' = n_1 - 2$, noting that y is a leaf in T' but is not a leaf in T. The set $S \setminus \{u', v, w\}$ is necessarily a TF-set of T', and so, by the inductive hypothesis, $n_1 - 2 = n_1' \le F_t(T') \le |S| - 3 = F_t(T) - 3$, implying that $F_t(T) \ge n_1 + 1$. □

By Claim 21, we may assume that $d_T(y) \ge 3$, for otherwise the desired result holds. We now consider the tree $T' = T - \{u, u', v, w, w', x\}$; that is, T' is the tree obtained from T by deleting x and all its descendants. Let T' have n_1' leaves. Thus, $n_1' = n_1 - 3$. The set $S \setminus \{u', v, w\}$ is necessarily a TF-set of T', and so, by the inductive hypothesis, $n_1 - 3 = n_1' \le F_t(T') \le |S| - 3 = F_t(T) - 3$, implying that $F_t(T) \ge n_1$. Further, if $F_t(T) = n_1$, then $F_t(T') = n_1'$ and by the inductive hypothesis, $T' \in \mathcal{F}$. In this case, we can restore the tree T by applying operation $\mathcal{O}_5$ to the tree T' with y as the link vertex, implying that $T \in \mathcal{F}$. This completes the proof of Lemma 14. ■

We show next that every tree in the family $\mathcal{F}$ has total forcing number equal to the number of leaves in the tree.

Lemma 22. *If T is a tree in the family $\mathcal{F}$ with n_1 leaves, then $F_t(T) = n_1$.*

Proof. We proceed by induction on the order $n \ge 2$ of a tree T in the family $\mathcal{F}$ with n_1 leaves. If $n = 2$, then $T = P_2$ and $F_t(T) = 2 = n_1$. This establishes the base case. Let $n \ge 3$ and assume that if $T' \in \mathcal{F}$ is a tree of order n' where $n' < n$ and with n_1' leaves, then $F_t(T') = n_1'$. Let T be a tree of order n in the family $\mathcal{F}$ with n_1 leaves. By definition of the family $\mathcal{F}$, there is a sequence $T_0, T_1, \ldots, T_k$ of trees where $T_0 = P_2$, $T_k = T$ and for $i \in [k]$, the tree T_i can be obtained from the tree T_{i-1} by one of the five operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_5$. Let $T' = T_{k-1}$. Hence, $T' \in \mathcal{F}$ and the tree T' has order less than n. Let T' have n_1' leaves. By the inductive hypothesis, $F_t(T) = n_1'$. Let S' be a minimum TF-set of T'.

Suppose that T is obtained from T' by applying operation $\mathcal{O}_1$. In this case, $n_1 = n'_1$ and, by Lemma 12, $F_t(T) = F_t\big(T'\big)$, implying that $F_t(T) = n_1$.

Suppose that T is obtained from T' by applying operation $\mathcal{O}_2$. In this case, $n_1 = n'_1 + 1$. Adopting the notation of Figure 3(b), by Observation 5 the set S' contains the vertex v and all except possibly one leaf neighbor. Thus, the set $S' \cup \{x\}$ is a TF-set of T, and so $F_t(T) \le |S'| + 1 = F_t\big(T'\big) + 1 = n'_1 + 1 = n_1$.

Suppose that T is obtained from T' by applying operation $\mathcal{O}_3$. In this case, $n_1 = n'_1 + 1$. Adopting the notation of Figure 3(c), by Observation 5 we may choose S' so that $\{v, w\} \subseteq S$. Thus, the set $S' \cup \{x\}$ is a TF-set of T, and so $F_t(T) \le |S'| + 1 = F_t\big(T'\big) + 1 = n'_1 + 1 = n_1$.

Suppose that T is obtained from T' by applying operation $\mathcal{O}_4$. In this case, $n_1 = n'_1 + 2$. Adopting the notation of Figure 3(d), every minimum TF-set of T' can be extended to a TF-set of T by adding to it x and y, and so $F_t(T) \le F_t\big(T'\big) + 2 = n'_1 + 2 = n_1$.

Suppose that T is obtained from T' by applying operation $\mathcal{O}_5$. In this case, $n_1 = n'_1 + 3$. Adopting the notation of Figure 3(e), every minimum TF-set of T' can be extended to a TF-set of T by adding to it the vertices v_2, v_3 and v_4, and so $F_t(T) \le F_t\big(T'\big) + 3 = n'_1 + 3 = n_1$. In all the above cases, $F_t(T) \le n_1$. By Lemma 14, $F_t(T) \ge n_1$. Consequently, $F_t(T) = n_1$. ■

Theorem 9 follows as an immediate consequence of Lemmas 14 and 22. We present next a proof of Theorem 10. We first observe that a simple adaptation of the proof given for Lemma 13 yields an analogous result on forcing. We state this formally with the following lemma.

Lemma 23. *If T is a non-trivial tree, then $F(T) = F\big(\mathrm{trim}(T)\big)$.*

We proceed further by defining a family $\mathcal{H}$ of trees as follows.

The family $\mathcal{H}$. Let $n \ge 2$ be an integer and let $\mathcal{H}$ be the family of all trees T of order n such that $T \cong P_n$ or $\mathrm{trim}(T) \cong K_{1,n-1}$ for $n \ge 3$.

We are now in a position to prove Theorem 10.

***Proof of Theorem* 10.** Let T be a non-trivial tree with n_1 leaves. By Theorem 9, $F_t(T) \ge n_1$. In [2], it was shown that $F(T) \le n_1 - 1$ holds for all trees T. Consequently, $F_t(T) \ge F(T) + 1$, which establishes the desired inequality relating the total forcing number and forcing number of a non-trivial tree. Suppose next that $F_t(T) = F(T) + 1$. Let T' denote the trimmed tree of T; that is, $T' = \mathrm{trim}(T)$. By Lemma 13, $F_t(T) = F_t\big(T'\big)$, and by Lemma 23, $F(T) = F\big(T'\big)$. Therefore, by supposition, $F_t\big(T'\big) = F\big(T'\big) + 1$.

Suppose that T' has at least two strong support vertices, say v and w. Let S' be a minimum TF-set in T', and so $|S'| = F_t\big(T'\big)$. By Observation 5, the set S' contains both v and w, and all except possibly one leaf neighbor of each of v

and w in T'. Let v' and w' be an arbitrary leaf neighbor of v and w, respectively, that belongs to the set S'. We now consider the set $S = S' \setminus \{v, w\}$. We claim that S is a forcing set of T'. In the first step of the forcing process starting with the initial set S, we play the vertex v' which forces its (unique) neighbor v to be colored. In the second step of the forcing process, we play the vertex w' which forces its (unique) neighbor w to be colored. At this stage of the forcing process, the resulting set of colored vertices is precisely the set S', which is TF-set of T' and therefore also a forcing set of T'. We now follow a sequence of played vertices in the total forcing process determined by the TF-set S' of T'. In this way, all vertices of $V(T')$ are colored. Thus, the set S is a forcing set of T', implying that $F_t(T') - 1 = F(T') \le |S| = |S'| - 2 = F_t(T') - 2$, a contradiction. Therefore, T' has at most one strong support vertex. If T' has no strong support vertex, then T is a path, and so $T \in \mathcal{H}$. If T' has exactly one strong support vertex, then $\mathrm{trim}(T) \cong K_{1,n_1}$ where $n_1 \ge 3$, and so $T \in \mathcal{H}$. Hence, if $F_t(T) = F(T) + 1$, then $T \in \mathcal{H}$.

It remains for us to prove that if $T \in \mathcal{H}$, then $F_t(T) = F(T) + 1$. Let $T \in \mathcal{H}$ have order n with n_1 leaves. If $T \cong P_n$, then it is well-known (and simple to observe) that $F(T) = 1 = n_1 - 1$. If $T \ncong P_n$, then $\mathrm{trim}(T) \cong K_{1,n-1}$ for some $n \ge 4$ and $F(T) = n - 2 = n_1 - 1$. In both cases, $F(T) = n_1 - 1$. Every tree in the family $\mathcal{H}$ can be constructed from a path P_2 by applying a sequence of operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and therefore belongs to the family $\mathcal{F}$; that is, $\mathcal{H} \subseteq \mathcal{F}$. Hence since $T \in \mathcal{H}$, we note that $T \in \mathcal{F}$, implying by Lemma 22 that $F_t(T) = n_1$. Thus, $F_t(T) = F(T) + 1$. ■

By Theorem 10, all trees T in the family $\mathcal{H}$ achieve equality in the inequality $F_t(T) \ge F(T) + 1$. We close by proving that the gap in this inequality can be made arbitrarily large.

Proposition 24. *For every integer $k \ge 1$, there exists a tree T such that*

$$F_t(T) = F(T) + k.$$

Proof. Let $k \ge 1$ be an arbitrary integer, and let $T' \cong P_k$ be a path on k vertices. Let T be the graph obtained from T' by adding two pendant edges to each vertex of T'. Thus, T has order $3k$ and every vertex in $V(T')$ is a strong support vertex of T. For each vertex of $V(T')$, select one of its leaf-neighbors and let S denote the resulting set of k leaves. The set $V(T) \setminus S$ is a TF-set of T, and so $F_t(T) \le |V(T)| - |S| = 2k$. Conversely, by Observation 5, $F_t(T) \ge 2k$. Consequently, $F_t(T) = 2k$. Moreover, the set S is a forcing set of T, and so $F(T) \le k$. However, every forcing set of T must contain at least one leaf neighbor of every vertex of $V(T')$ in T, implying that $F(T) \ge k$. Consequently, $F(T) = k$. Therefore, $F_t(T) - F(T) = k$. ■

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