THE SLATER AND SUB-k-DOMINATION NUMBER OF A GRAPH WITH APPLICATIONS TO DOMINATION AND k-DOMINATION

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Abstract

In this paper we introduce and study a new graph invariant derived from the degree sequence of a graph G, called the *sub-k-domination number* and denoted $\mathrm{sub}_k(G)$. This invariant serves as a generalization of the *Slater number*; in particular, we show that $\mathrm{sub}_k(G)$ is a computationally efficient sharp lower bound on the k-domination number of G, and improves on several known lower bounds. We also characterize the sub-k-domination numbers of several families of graphs, provide structural results on sub-k-domination, and explore properties of graphs which are $\mathrm{sub}_k(G)$-critical with respect to addition and deletion of vertices and edges.

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1. Introduction

Domination is one of the most well-studied and widely applied concepts in graph theory. A set $S \subseteq V(G)$ is *dominating* for a graph G if every vertex of G is either in S, or is adjacent to a vertex in S. A related parameter of interest is the *domination number*, denoted $\gamma(G)$, which is the cardinality of the smallest dominating set of G. Much of the literature on domination is surveyed in the two monographs of Haynes, Hedetniemi and Slater [14, 15]. For more recent results on domination, see [4–6, 11, 19, 24] and the references therein.

In 1984, Fink and Jacobson [9] generalized domination by introducing the notion of k-domination and its associated graph invariant, the k-domination number. Given a positive integer k, $S \subseteq V(G)$ is a *k-dominating set* for a graph G if every vertex not in S is adjacent to at least k vertices in S. The minimum cardinality of a k-dominating set of G is the *k-domination number* of G, denoted $\gamma_k(G)$. When $k = 1$, the 1-domination number is precisely the domination number; that is, $\gamma_1(G) = \gamma(G)$. Like domination, k-domination has also been extensively studied; for results on k-domination related to this paper, we refer the reader to [2, 3, 8, 12, 18, 21, 22].

Computing the k-domination number is NP-hard [17], and as such, many researchers have sought computationally efficient upper and lower bounds for this parameter. In general, the degree sequence of a graph can be a useful tool for bounding NP-hard graph invariants, see for example [1, 7, 10, 20]. In relation to this paper, we highlight that the degree sequence derived invariant known as the *Slater number* serves as a lower bound on the domination number from below [23]. Recently, Caro and Pepper [1] introduced the *degree sequence index strategy*, or DSI-strategy, which provides a unified framework for using the degree sequence of a graph to bound NP-hard invariants. In this paper we generalize the Slater number, and in doing so, introduce a new degree sequence invariant called the sub-k-domination number, which serves as a sharp lower bound on the k-domination number; our investigation contributes to the known literature on both degree sequence invariants and domination.

Throughout this paper all graphs are simple and finite. Let $G = (V(G), E(G))$ be a graph. Two vertices v and w in G are *adjacent*, or *neighbors*, if there exists an edge $vw \in E$. A vertex is an *isolate* if it has no neighbors. The *complement* of G is the graph $\overline{G}$ with the same vertex set, in which two vertices are adjacent if and only if they are not adjacent in G. A set $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent; the cardinality of the largest independent set in G is denoted $\alpha(G)$. For any edge $e \in E(G)$, $G - e$ denotes the graph G with the edge e removed. For any vertex $v \in V(G)$, $G - v$ denotes the graph G with the vertex v and all edges incident to v removed; for any edge $e \in E(\overline{G})$, $G + e$ denotes the graph G with the edge e added.

The *degree* of a vertex v, denoted $d(v)$, is the number of vertices adjacent to v. We will use the notation $n(G) = |V(G)|$ to denote the order of G, $\Delta(G)$ to denote the maximum degree of G, and $\delta(G)$ to denote the minimum degree of G; when there is no scope for confusion, the dependence on G will be omitted. We will also use d_i to denote the ith element in the *degree sequence* of G, denoted $D(G) = \{\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta\}$, which lists the vertex degrees in non-increasing order. We may abbreviate $D(G)$ by only writing distinct degrees, with the number of vertices realizing each degree in superscript. For example, the star $K_{n-1,1}$ may have its degree sequence written as $D(K_{n-1,1}) = \{n-1, 1^{n-1}\}$, and the complete graph K_n may have degree sequence written as $D(K_n) = \{(n-1)^n\}$. If a graph is said to be r-regular, every vertex degree is r. More specifically, a cubic graph will be a graph where every vertex degree will be three. For other graph terminology and notation, we will generally follow [16].

This paper is organized as follows. In the next section, we introduce the sub-k-domination number of a graph and show that it is a lower bound on the k-domination number. In Section 3, we characterize the sub-k-domination numbers of several families of graphs and provide other structural results on sub-k-domination. In Section 4, we compare the sub-k-domination number to other known lower bounds on the k-domination number. In Section 5, we explore the properties of $\mathrm{sub}_k(G)$-critical graphs. We conclude with some final remarks and open questions in Section 6.

2. Sub-k-Domination

In this section we introduce the sub-k-domination number of a graph and prove that it is a lower bound on the k-domination number. We first recall a definition and result due to Slater [23], which is a special case of our result. For consistency in terminology, we will refer to Slater's definition as the *sub-domination number* of a graph; this invariant was originally denoted $sl(G)$, and for our purposes will be denoted $\mathrm{sub}(G)$.

Definition 1. If G is a graph with order n and degree sequence $D(G) = \{\Delta(G) = d_1, \ldots, d_n = \delta(G)\}$, the *sub-domination number*, denoted $\mathrm{sub}(G)$, and originally introduced as the *Slater number* $sl(G)$ in [23], is defined as the smallest integer t such that $t + \sum_{i=1}^{t} d_i \geq n$.

In [23], Slater showed that $sl(G) = \mathrm{sub}(G)$ serves as a lower bound on the domination number. We recall the statement of this result with the following theorem.

Theorem 1 (Slater [23]). *If G is a graph, then*

$$\gamma(G) \geq \mathrm{sub}(G),$$

and this bound is sharp.

As a strengthening of domination, we observe that for any $k \geq 1$, the k-domination number is monotonically increasing with respect to k; that is, $\gamma_k(G) \leq \gamma_{k+1}(G)$. Keeping monotonicity in mind, it is natural that a parameter generalizing $\mathrm{sub}(G)$ will need to increase with respect to increasing k. This idea motivates the following definition.

Definition 2. Let $k \geq 1$ be an integer. If G is a graph with order n and degree sequence $D(G) = \{\Delta(G) = d_1, \ldots, d_n = \delta(G)\}$, the *sub-$k$-domination number*, denoted $\mathrm{sub}_k(G)$, is defined as the smallest integer t such that $t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n$.

Since the vertex degrees of G are integers between 0 and $n-1$, the sorted degree sequence of G can be obtained in $O(n)$ time by counting sort (assuming vertex degrees can be accessed in $O(1)$ time). By maintaining the sum of the first t elements in $D(G)$ and incrementing t, $\mathrm{sub}_k(G)$ can be computed in linear time; we state this formally below.

Observation 1. *If G is a graph and $k \geq 1$ is an integer, then $\mathrm{sub}_k(G)$ can be computed in $O(n)$ time.*

Taking $k = 1$ in Definition 2, we observe $\mathrm{sub}_1(G) = \mathrm{sub}(G)$, and hence $\mathrm{sub}_1(G) \leq \gamma_1(G)$ by Theorem 1. More generally, we will now show that the k-domination number of a graph is bounded below by its sub-k-domination number.

Theorem 2. *If G is a graph and $k \geq 1$ is an integer, then*

$$\gamma_k(G) \geq \mathrm{sub}_k(G),$$

and this bound is sharp.

Proof. Let $S = \{v_1, \ldots, v_t\}$ be a minimum k-dominating set of G. By definition, each of the $n - t$ vertices in $V(G) \backslash S$ is adjacent to at least k vertices in S. Thus, the sum of the degrees of the vertices in S, i.e., $\sum_{i=1}^{t} d(v_i)$, is at least $k(n - t)$. Dividing by k and rearranging, we obtain

$$t + \frac{1}{k} \sum_{i=1}^{t} d(v_i) \geq n.$$

Since the degree sequence of G is non-increasing, it follows that $\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t} d(v_i)$. Thus,

$$(1) \qquad\qquad t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n.$$

Since $\mathrm{sub}_k(G)$ is the smallest index for which (1) holds, we must have $\mathrm{sub}_k(G) \leq t = \gamma_k(G)$.

When $k = 1$, note that $\mathrm{sub}(K_{n-1,1}) = 1 = \gamma(K_{n-1,1})$. When $k > 1$, let G be a complete bipartite graph with a perfect matching removed where each part of the vertex partition is of size $k + 1$. Then $\mathrm{sub}_k(G) = \min\left\{t : t + \frac{1}{k}\sum_{i=1}^{t} k \geq n\right\} = k + 1 = \gamma_k(G)$. Thus, the bound is sharp for all k. ∎

We conclude this section by showing that sub-k-domination is subadditive with respect to disjoint unions of graphs.

Lemma 3. *If G and H are disjoint graphs and $k \geq 1$ is an integer, then* $\mathrm{sub}_k(G) + \mathrm{sub}_k(H) \geq \mathrm{sub}_k(G \cup H)$.

Proof. Let G and H be disjoint graphs with degree sequences $D(G) = \left\{d_1^G \geq \cdots \geq d_{n(G)}^G\right\}$, $D(H) = \left\{d_1^H \geq \cdots \geq d_{n(H)}^H\right\}$, and $D(G \cup H) = \left\{d_1^{G \cup H} \geq \cdots \geq d_{n(G \cup H)}^{G \cup H}\right\}$. Since the degree sequences are non-increasing, it follows that

$$(2) \qquad \sum_{i=1}^{\mathrm{sub}_k(G)+\mathrm{sub}_k(H)} d_i^{G \cup H} \geq \sum_{i=1}^{\mathrm{sub}_k(G)} d_i^G + \sum_{i=1}^{\mathrm{sub}_k(H)} d_i^H.$$

Moreover, by definition of sub-k-domination,

$$\mathrm{sub}_k(G) + \frac{1}{k}\sum_{i=1}^{\mathrm{sub}_k(G)} d_i^G + \mathrm{sub}_k(H) + \frac{1}{k}\sum_{i=1}^{\mathrm{sub}_k(H)} d_i^H \geq n(G) + n(H) = n(G \cup H).$$

Combining (2) and the above inequality, we obtain

$$\mathrm{sub}_k(G) + \mathrm{sub}_k(H) + \frac{1}{k}\sum_{i=1}^{\mathrm{sub}_k(G)+\mathrm{sub}_k(H)} d_i^{G \cup H} \geq n(G \cup H).$$

Thus, $\mathrm{sub}_k(G) + \mathrm{sub}_k(H)$ is an integer which satisfies

$$(3) \qquad t + \frac{1}{k}\sum_{i=1}^{t} d_i^{G \cup H} \geq n(G \cup H),$$

but by definition, $\mathrm{sub}_k(G \cup H)$ is the smallest integer which satisfies (3), so it follows that $\mathrm{sub}_k(G) + \mathrm{sub}_k(H) \geq \mathrm{sub}_k(G \cup H)$. ∎

Since k-domination is additive with respect to disjoint unions of graphs; that is $\gamma_k(G \cup H) = \gamma_k(G) + \gamma_k(H)$, for any disjoint graphs G and H, the following theorem is an immediate consequence of Lemma 3 and gives slight improvements to Theorem 2.

Theorem 4. *If G and H are disjoint graphs and $k \geq 1$ is an integer, then*

$$\gamma_k(G \cup H) \geq \mathrm{sub}_k(G) + \mathrm{sub}_k(H) \geq \mathrm{sub}_k(G \cup H).$$

In the next section, we compute $\mathrm{sub}_k(G)$ for several families of graphs and investigate graphs for which $\mathrm{sub}_k(G) = \gamma_k(G)$.

3.   Graphs for Which $\mathrm{sub}_k(G) = \gamma_k(G)$

In this section we explore the case of equality for Theorem 2. First, note that $\mathrm{sub}(G) = \gamma(G) = n$ for an empty graph G. We therefore exclude empty graphs from the following discussion; that is, assume $\Delta \geq 1$. We begin with two propositions for the case $k = 1$.

Proposition 1. *If G is a graph with maximum degree $\Delta \geq n-2$, then $\mathrm{sub}(G) = \gamma(G)$.*

Proof. If $\Delta = n-1$ then $\gamma(G) = 1$ and thus $\mathrm{sub}(G) = \gamma(G)$, since by Theorem 2, $1 \leq \mathrm{sub}(G) \leq \gamma(G) = 1$. If $\Delta = n-2$, then $\gamma(G) = 2$ since no single vertex can dominate the graph, but a maximum degree vertex and its non-neighbor is a dominating set. Moreover, $\mathrm{sub}(G) \neq 1$ since $1 + (n-2) < n$; thus, $2 \leq \mathrm{sub}(G) \leq \gamma(G) = 2$. ∎

If G is a graph with maximum degree $\Delta \leq n-3$, then $\mathrm{sub}(G)$ may not be equal to $\gamma(G)$. For example, let G be the graph obtained by appending a pendant vertex to two leaves of $K_{1,3}$; it can be verified that $\gamma(G) = 3$ and $\mathrm{sub}(G) = 2$.

Proposition 2. *If G is a graph with $\gamma(G) \leq 2$, then $\mathrm{sub}(G) = \gamma(G)$.*

Proof. From Theorem 2, if $\gamma(G) = 1$ then $\mathrm{sub}(G) = 1$. Conversely, if $\mathrm{sub}(G) = 1$, then $1 + d_1 \geq n$ and hence from Proposition 1, $\gamma(G) = 1$. Similarly, if $\gamma(G) = 2$ then $\mathrm{sub}(G) \leq 2$; however, since $\mathrm{sub}(G) = 1$ if and only if $\gamma(G) = 1$, it follows that $\mathrm{sub}(G) = 2$. ∎

If G is a graph with $\gamma(G) \geq 3$, then $\mathrm{sub}(G)$ may not be equal to $\gamma(G)$. For example, let G be the graph obtained by appending two pendants to each vertex of K_3; it can be verified that $\gamma(G) = 3$ and $\mathrm{sub}(G) = 2$.

We next determine the sub-k-domination number of regular graphs. This will reveal some families of graphs for which $\mathrm{sub}_k(G) = \gamma_k(G)$ for $k \geq 2$.

Theorem 5. *If G is an r-regular graph and $k \geq 1$ is an integer, then $\mathrm{sub}_k(G) = \left\lceil \frac{kn}{r+k} \right\rceil$.*

Proof. Let G be an r-regular graph and let $k \geq 1$ be an integer. Since G is r-regular, $d_i = r$ for $1 \leq i \leq n$. Then, from the definition of sub-k-domination, we have

$$(4) \qquad \mathrm{sub}_k(G) + \frac{\mathrm{sub}_k(G)r}{k} = \mathrm{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{\mathrm{sub}_k(G)} d_i \geq n.$$

Rearranging (4), we obtain

$$(5) \qquad \frac{kn}{r+k} \leq \mathrm{sub}_k(G).$$

Since $\mathrm{sub}_k(G)$ is the smallest integer that satisfies (5), it follows that $\mathrm{sub}_k(G) = \left\lceil \frac{kn}{r+k} \right\rceil$. ∎

Note that $\gamma_k(G) = n$ whenever $k > \Delta(G)$. We therefore restrict ourselves to the more interesting case of $k \leq \Delta$. The next example shows an infinite family of graphs for which the sub-k-domination number equals the k-domination number for all $k \leq \Delta$.

Observation 2. *If C_n is a cycle with order n and $k \leq 2$ is a positive integer, then $\mathrm{sub}_k(C_n) = \gamma_k(C_n)$.*

Proof. When $k = 1$, it is known that $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Since cycles are 2-regular, Theorem 5 gives $\mathrm{sub}(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Hence, $\gamma(C_n) = \mathrm{sub}(C_n)$ for all n. When $k = 2$, Theorem 5 gives $\left\lceil \frac{n}{2} \right\rceil = \mathrm{sub}_2(C_n)$. Since we can produce a 2-dominating set for C_n by first picking any vertex v and adding all vertices whose distance from v is even, it follows that $\gamma_2(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$. Thus $\mathrm{sub}_2(C_n) = \gamma_2(C_n)$. ∎

As another example, from Proposition 1 and Theorem 5, we see that $\gamma(K_n) = \mathrm{sub}(K_n) = 1$ and $\gamma_2(K_n) = \mathrm{sub}_2(K_n) = 2$ for all n. When $k \geq 3$, $\gamma_k(K_n)$ does not equal $\mathrm{sub}_k(K_n)$ for all n (for example, $\mathrm{sub}_3(K_4) = 2$ but $\gamma_3(K_4) = 3$); however, our next result shows that equality does hold when n is large enough.

Proposition 3. *If K_n is a complete graph with order n and $k \leq n - 1$ is a positive integer, then*

$$\mathrm{sub}_k(K_n) = \gamma_k(K_n) = k,$$

if and only if $n > (k-1)^2$.

Proof. First, note that $\gamma_k(K_n) = k$ for $k \leq n - 1$, since any set of k vertices of K_n is k-dominating, while any set with at most $k - 1$ vertices is at most $(k-1)$-dominating. Next, since K_n is regular of degree $n - 1$ it follows from Theorem 5 that

$$\mathrm{sub}_k(K_n) = \left\lceil \frac{kn}{n-1+k} \right\rceil \leq k = \gamma_k(K_n).$$

If $\mathrm{sub}_k(K_n) = k$, we must have

$$\frac{kn}{n-1+k} > k - 1.$$

Rearranging, we obtain that $n > (k-1)^2$. ∎

We next consider sub-k-domination in certain trees. A *perfect j-ary tree with height h*, denoted $T(j, h)$ is a rooted tree for which every vertex has either 0 or j children, and where every leaf is at distance h from the root.

Theorem 6. *For any integers $h \geq 1$ and $j \geq k \geq 3$, $\mathrm{sub}_k(T(j,h)) = \gamma_k(T(j,h))$ if and only if $h = 1$ and $k = n - 1$, where n is the order of $T(j, h)$.*

Proof. If $h = 1$, then $T := T(j, h) \cong K_{1,n-1}$ and $\gamma_k(T) = n - 1$. For $k < n - 1$, $(n - 2) + \frac{1}{k}((n - 1) + (n - 3)) = (n - 2 + \frac{2n-4}{k}) \geq (n - 2 + \frac{2n-4}{n-2}) \geq n$, so $\mathrm{sub}_k(T) \leq n - 2 < \gamma_k(T)$. For $k = n - 1$, $\mathrm{sub}_k(T) = n - 1 = \gamma_k(T)$. Thus, assume henceforth that $h \geq 2$. When $h \geq 2$, the order of T is $n = \sum_{i=1}^{h} j^h$ and the degree sequence of T is $\{(j + 1)^{n-1-j^h}, j^1, 1^{j^h}\}$. Since $k \geq 3$, no leaf can be excluded from a k-dominating set; moreover, since $h \geq 2$, there is at least one vertex which is not adjacent to any leaf, so $\gamma_k(T) \geq j^h + 1$.

By definition, $\mathrm{sub}_k(T)$ is the smallest integer t such that $t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n$. Suppose that $t \leq n - j^h$. Then,

(6)
$$\left(n - j^h\right) + \frac{1}{k} \sum_{i=1}^{(n-j^h)} d_i \geq n$$
$$\left(n - j^h\right) + \left(n - j^h - 1\right)(j + 1) + j \geq kn$$
$$(j + 1 + k)\left(n - j^h\right) - 1 \geq kn$$
$$\frac{j^h(1 + j + k - jk) - 2j}{j - 1} \geq 0.$$

The last inequality (6) follows by substituting $n = 1 + j + j^2 + \cdots + j^h = \frac{j^{h+1}-1}{j-1}$. Since $j \geq 3$, (6) is satisfied only when $1 + j + k - jk > 0$, i.e., when $k = 1$ or $j = k = 2$; this contradicts $k \geq 3$. Thus, $t > n - j^h$, and we have

(7)
$$t + \frac{1}{k}\left(\left(n - j^h - 1\right)(j + 1) + j + \left(t - \left(n - j^h - 1\right) - 1\right)\right) \geq n$$
$$kt + \left(n - j^h - 1\right)j + j + t - 1 \geq kn$$
$$t \geq \frac{j^{h+1} - n(j - k) + 1}{k + 1}.$$

Since $\mathrm{sub}_k(T)$ is the smallest positive integer t which satisfies (7), it follows that

$$\mathrm{sub}_k(T) = \left\lceil \frac{j^{h+1} - n(j - k) + 1}{k + 1} \right\rceil$$
$$= \left\lceil \frac{j^{h+1} - (j^h + j^{h-1} + \cdots + j + 1)(j - k) + 1}{k + 1} \right\rceil$$

$$= \left\lceil \frac{j^{h+1} - (j^h + 1)(j - k) - (j^{h-1} + \cdots + j)(j - k) + 1}{k + 1} \right\rceil$$

$$\leq \left\lceil \frac{j^{h+1} - (j^h + 1)(j - k) + 1}{k + 1} \right\rceil = \left\lceil \frac{kj^h + k + 1 - j}{k + 1} \right\rceil$$

$$< \left\lceil j^h + \frac{k + 1 - j}{k} \right\rceil \leq j^h + 1 \leq \gamma_k(T).$$ ∎

Our last focus in this section is on the sub-k-domination number and k-domination number of 3-regular, or *cubic*, graphs. First, we recall an upper bound for the k-domination number due to Caro and Roditty [2].

Theorem 7 [2]**.** *If G is a graph with order n, and $k, r \geq 1$ are integers such that $\delta \geq \frac{r+1}{r}k - 1$, then $\gamma_k(G) \leq \frac{r}{r+1}n$.*

In particular, for cubic graphs, Theorem 5 and the Caro-Roditty bound (with r taken to be the smallest positive integer satisfying $3 \geq \frac{r+1}{r}k - 1$) imply the following intervals for the k-domination number.

Corollary 4. *If G is a cubic graph with order n, then*

1. $\left\lceil \frac{n}{4} \right\rceil \leq \gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$,
2. $\left\lceil \frac{2n}{5} \right\rceil \leq \gamma_2(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$,
3. $\left\lceil \frac{n}{2} \right\rceil \leq \gamma_3(G) \leq \left\lfloor \frac{3n}{4} \right\rfloor$.

We see from Corollary 4 that $\mathrm{sub}_k(G) = \gamma_k(G)$ for some cubic graphs with small values of n; for example, $\mathrm{sub}(G) = \gamma(G)$ when $n \leq 6$ and $\mathrm{sub}_2(G) = \gamma_2(G)$ when $n \leq 8$.

4. COMPARISON TO KNOWN BOUNDS ON $\gamma_k(G)$

A well-known lower bound on the domination number of a graph is $\frac{n}{\Delta+1}$. This bound is not difficult to derive *a priori*, but it immediately follows from the definition of $\mathrm{sub}(G)$ and Theorem 2. In [9], Fink and Jacobson generalized this bound by showing that $\frac{kn}{\Delta+k} \leq \gamma_k(G)$; this also follows from a result of Hansberg and Pepper in [13]. In the following theorem, we show that $\mathrm{sub}_k(G)$ is an improvement on this bound.

Theorem 8. *If G is a graph and $k \leq \Delta$ is an integer, then*

$$\frac{kn}{\Delta + k} \leq \mathrm{sub}_k(G) \leq \gamma_k(G).$$

Proof. The right hand side of the inequality in the theorem follows from Theorem 2. Thus, in order to prove the theorem, it suffices to show the left hand side of the inequality. Fix k and let $t = \mathrm{sub}_k(G)$. By definition, $t + \frac{1}{k}\sum_{i=1}^{t} d_i \geq n$. Since $\Delta \geq d_i$ for $1 \leq i \leq n$, it follows that

$$t + \frac{t\Delta}{k} = t + \frac{1}{k}\sum_{i=1}^{t}\Delta \geq t + \frac{1}{k}\sum_{i=1}^{t} d_i \geq n.$$

Rearranging the above inequality gives

$$\frac{kn}{\Delta + k} \leq t = \mathrm{sub}_k(G). \qquad\blacksquare$$

Recall from Theorem 5 that if G is regular of degree r, then $\mathrm{sub}_k(G) = \left\lceil \frac{kn}{r+k} \right\rceil$. Thus, from Theorem 8, we see that regular graphs minimize the sub-k-domination number over all graphs with n vertices and maximum degree Δ. This suggests that in order to maximize the sub-k-domination number, we might consider graphs which are, in some sense, highly irregular with respect to vertex degrees. This motivates the following theorem and its corollary, where the number of vertices with the t largest distinct degrees in the graph (s_t) and the $(t+1)$-largest distinct degree in the graph (Δ_t) are leveraged to bound $\mathrm{sub}_k(G)$.

Theorem 9. *Let G be a graph; for $1 \leq t \leq \Delta$ let n_t be the number of vertices of G with degree t, let $s_t = \sum_{i=1}^{t} n_{\Delta+1-i}$, and let $\Delta_t = d_{s_t+1}$. If $s_t + \sum_{i=1}^{s_t} d_i < n$ for some t, then*

$$\frac{kn - \sum_{i=1}^{t}(\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}}{k + \Delta_t} \leq \mathrm{sub}_k(G).$$

Proof. From the definition of $\mathrm{sub}_k(G)$, we have

$$(8)\qquad\qquad n \leq \mathrm{sub}_k(G) + \frac{1}{k}\sum_{i=1}^{\mathrm{sub}_k(G)} d_i.$$

Since $s_t + \sum_{i=1}^{s_t} d_i < n$, it follows that $s_t < \mathrm{sub}_1(G) \leq \mathrm{sub}_k(G)$, and thus

$$(9)\qquad\qquad \sum_{i=1}^{\mathrm{sub}_k(G)} d_i = \sum_{i=1}^{s_t} d_i + \sum_{i=s_t+1}^{\mathrm{sub}_k(G)} d_i.$$

Since $s_t = n_\Delta + n_{\Delta-1} + \cdots + n_{\Delta-t+1}$ and since the degree sequence of G is non-increasing and has n_j elements with value j, we have

$$\sum_{i=1}^{s_t} d_i = \Delta n_\Delta + (\Delta - 1)n_{\Delta-1} + \cdots + (\Delta - t + 1)n_{\Delta-t+1}$$

$$(10)\qquad\qquad = \sum_{i=1}^{t}(\Delta + 1 - i)n_{\Delta+1-i}.$$

Again since $D(G)$ is non-decreasing, we have that $\Delta_t = d_{s_t+1} \geq d_{s_t+2} \geq \cdots \geq d_{\text{sub}_k(G)}$. Thus, it follows that

$$(11) \qquad \sum_{i=s_t+1}^{\text{sub}_k(G)} d_i \leq \sum_{i=s_t+1}^{\text{sub}_k(G)} \Delta_t = (\text{sub}_k(G) - s_t)\Delta_t.$$

Substituting (9), (10), and (11) into the right-hand-side of (8) yields

$$n \leq \text{sub}_k(G) + \frac{1}{k}\sum_{i=1}^{t}(\Delta + 1 - i)n_{\Delta+1-i} + \frac{1}{k}\big(\text{sub}_k(G) - s_t\big)\Delta_t.$$

By expanding $(\text{sub}_k(G) - s_t)\Delta_t$ and substituting $s_t = \sum_{i=1}^{t} n_{\Delta+1-i}$, the above inequality can be rewritten as

$$n \leq \text{sub}_k(G)\left(1 + \frac{\Delta_t}{k}\right) + \frac{1}{k}\sum_{i=1}^{t}(\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}.$$

Rearranging the last inequality gives

$$\frac{kn - \sum_{i=1}^{t}(\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}}{k + \Delta_t} \leq \text{sub}_k(G). \qquad \blacksquare$$

We note that the bound in Theorem 9 is optimal when t is taken to be the maximum positive integer for which $s_t + \sum_{i=1}^{s_t} d_i < n$. Theorem 9 can be used to give simple lower bounds for the k-domination number of a graph when certain restrictions on the order and maximum degree are met. These bounds also improve on the lower bound given in Theorem 8.

Corollary 5. *Let G be a graph, let n_Δ denote the number of maximum degree vertices of G, and let Δ' denote the second-largest degree of G. If k is a positive integer and $n_\Delta + \frac{\Delta n_\Delta}{k} < n$, then*

$$(12) \qquad \frac{kn - n_\Delta(\Delta - \Delta')}{\Delta' + k} \leq \text{sub}_k(G) \leq \gamma_k(G).$$

Proof. Take $t = 1$ in the bound from Theorem 9 and note that $s_1 = n_\Delta$ and $\Delta_1 = d_{n_\Delta+1} = \Delta'$. Since $n_\Delta + \frac{\Delta n_\Delta}{k} < n$, we have that $s_1 + \frac{1}{k}\sum_{i=1}^{s_1} d_i = n_\Delta + \frac{1}{k}\sum_{i=1}^{n_\Delta} d_i = n_\Delta + \frac{\Delta n_\Delta}{k} < n$. Thus, the condition of Theorem 9 is satisfied, and we obtain the first inequality in (12); the second inequality in (12) follows from Theorem 2. $\blacksquare$

We see from Corollary 5 that if G has a unique maximum degree vertex, then

$$\frac{kn - \Delta + \Delta'}{\Delta' + k} \leq \gamma_k(G).$$

Corollary 5 gives significant improvements on the lower bound in Theorem 8 whenever the difference between Δ and Δ' is large. For example, consider the subdivided star $S(K_{1,n-1})$ $(n \geq 3)$ which is obtained by subdividing each edge in $K_{1,n-1}$. The degree sequence of this graph is $\{n-1, 2^{n-1}, 1^{n-1}\}$ and its order is $2n-1$. This graph meets the conditions of Corollary 5, and the bound given in the corollary simplifies to $\frac{(2k-1)n-(k-3)}{2+k}$, whereas the bound given by Theorem 8 is $\frac{k(2n-1)}{n-1+k}$. To compare these two bounds, we first compute the difference between them

$$\frac{(2k-1)n-(k-2)}{2+k} - \frac{k(2n-1)}{n-1+k} = \frac{(2k-1)n^2 + (4-6k)n + 8k - k^2 - 3}{(2+k)(n-1+k)}.$$

When k is fixed, the difference between these two bounds approaches ∞ as $n \to \infty$. In particular, the bound given by Theorem 8 approaches a constant, $2k$, as n grows large, while the bound given by Corollary 5 is approximately $2n$.

5. Critical Graphs

There are three natural ways to consider critical graphs in the context of sub-k-domination: graphs which are critical with respect to edge-deletion, edge-addition, and vertex-deletion.

Definition 3. If G is a graph and $k \geq 1$ is an integer, we shall define

1. G is *edge-deletion-$\mathrm{sub}_k(G)$-critical* if for any $e \in E(G)$, $\mathrm{sub}_k(G-e) > \mathrm{sub}_k(G)$.
2. G is *edge-addition-$\mathrm{sub}_k(G)$-critical* if for any $e \in E(\overline{G})$, $\mathrm{sub}_k(G+e) < \mathrm{sub}_k(G)$.
3. G is *vertex-deletion-$\mathrm{sub}_k(G)$-critical* if for any $v \in V(G)$, $\mathrm{sub}_k(G-v) > \mathrm{sub}_k(G)$.

These properties will respectively be abbreviated as $\mathrm{sub}_k(G)$-*ED-critical*, $\mathrm{sub}_k(G)$-*EA-critical*, and $\mathrm{sub}_k(G)$-*VD-critical*.

In this section, we present several structural results about sub-k-domination critical graphs, including connections to other graph parameters. Throughout the section, we will assume that given a graph G with $V(G) = \{v_1, \ldots, v_n\}$ and $D(G) = \{d_1, \ldots, d_n\}$ where $d_1 \geq \cdots \geq d_n$, it holds that $d_i = d(v_i)$ — in other words, the vertices of G are labeled according to a non-increasing ordering of their degrees.

We first present two results about $\mathrm{sub}_k(G)$-ED-critical graphs.

Proposition 6. *If G is a $\mathrm{sub}_k(G)$-ED-critical graph with $\mathrm{sub}_k(G) = t$, then $\{v_{t+1}, \ldots, v_n\}$ is an independent set of G, and $n - \mathrm{sub}_k(G) \leq \alpha(G)$.*

Proof. Suppose for contradiction that $\{v_{t+1}, \ldots, v_n\}$ is not an independent set and let $e = v_x v_y$ be an edge with $v_x, v_y \in \{v_{t+1}, \ldots, v_n\}$. Then, the degree sequence of $G - e$ is $d'_1 \geq \cdots \geq d'_n$, where $d'_i = d_i$ for all $1 \leq i \leq t$. Thus, $t + \frac{1}{k} \sum_{i=1}^{t} d'_i = t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n$, which implies that $\mathrm{sub}_k(G - e) \leq t$; this contradicts the assumption that G is $\mathrm{sub}_k(G)$-ED-critical. Thus, $\{v_{t+1}, \ldots, v_n\}$ is an independent set, so $\alpha(G) \geq n - t$. ∎

Proposition 7. *If G is a $\mathrm{sub}_k(G)$-ED-critical graph with no isolates and $\mathrm{sub}_k(G) = t$, then $k(n - t) = \sum_{i=1}^{t} d_i$, and for any $e \in E(G)$, $\mathrm{sub}_k(G - e) = \mathrm{sub}_k(G) + 1$.*

Proof. By definition of $\mathrm{sub}_k(G)$ and since n is an integer, we have that $\lfloor t + \frac{1}{k} \sum_{i=1}^{t} d_i \rfloor \geq n$. Suppose for contradiction that $\lfloor t + \frac{1}{k} \sum_{i=1}^{t} d_i \rfloor > n$. More precisely, $t + \frac{1}{k} \sum_{i=1}^{t} d_i \geq n + \frac{1}{k}$ since $t + \frac{1}{k} \sum_{i=1}^{t} d_i$ cannot be any number between n and $n + \frac{1}{k}$. Since by Proposition 6, $\{v_{t+1}, \ldots, v_n\}$ is an independent set of G and since G has no isolates, we can choose an edge e incident to exactly one vertex in $\{v_1, \ldots, v_t\}$. The degree sequence of $G - e$ is $d'_1 \geq \cdots \geq d'_n$, where $\sum_{i=1}^{t} d'_i = \left(\sum_{i=1}^{t} d_i \right) - 1$. Thus,

$$t + \frac{1}{k} \sum_{i=1}^{t} d'_i = t + \frac{1}{k} \left(\sum_{i=1}^{t} d_i - 1 \right) = \left(t + \frac{1}{k} \sum_{i=1}^{t} d_i \right) - \frac{1}{k} \geq \left(n + \frac{1}{k} \right) - \frac{1}{k} = n,$$

meaning $\mathrm{sub}_k(G - e) = t$, which contradicts G being $\mathrm{sub}_k(G)$-ED-critical. By rearranging $t + \frac{1}{k} \sum_{i=1}^{t} d_i = n$, we get $k(n - t) = \sum_{i=1}^{t} d_i$.

Now let e be any edge of G and $d'_1 \geq \cdots \geq d'_n$ be the degree sequence of $G - e$. The deletion of e decreases $\sum_{i=1}^{t+1} d_i$ by at most 2, i.e., $\sum_{i=1}^{t+1} d'_i \geq \left(\sum_{i=1}^{t+1} d_i \right) - 2$. Thus,

$$(t + 1) + \frac{1}{k} \sum_{i=1}^{t+1} d'_i \geq (t + 1) + \frac{1}{k} \sum_{i=1}^{t+1} d_i - \frac{2}{k} = t + \frac{1}{k} \left(\sum_{i=1}^{t} d_i \right) + \frac{d_{t+1} - 2}{k} + 1 \geq n,$$

where in the last inequality $d_{t+1} \geq 1$ since G has no isolates; this implies $\mathrm{sub}_k(G - e) = t + 1 = \mathrm{sub}_k(G) + 1$. ∎

Next, we present two analogous results about $\mathrm{sub}_k(G)$-EA-critical graphs.

Proposition 8. *If G is a $\mathrm{sub}_k(G)$-EA-critical graph with $\mathrm{sub}_k(G) = t$, then the vertices in $\{v \in V(G) : d(v) < d_t\}$ form a clique.*

Proof. Suppose on the contrary that there are two non-adjacent vertices v_x and v_y with $d_t > d_x \geq d_y$. Then, the degree sequence of $G + v_x v_y$ is $d'_1 \geq \cdots \geq d'_n$, where $d'_i = d_i$ for all $1 \leq i \leq t$. This implies that $\mathrm{sub}_k(G + e) = \mathrm{sub}_k(G)$, a contradiction. ∎

Proposition 9. *If G is a $\mathrm{sub}_k(G)$-EA-critical graph with no isolates and $\mathrm{sub}_k(G)$ $= t$, then $k(n - t + 1) - 1 = \sum_{i=1}^{t-1} d_i$ and for each $e \in E(\overline{G})$, $\mathrm{sub}_k(G + e) = \mathrm{sub}_k(G) - 1$.*

Proof. Let e be any edge in $\overline{G}$ and $d_1' \geq \cdots \geq d_n'$ be the degree sequence of $G + e$. Let $\{v_1, \ldots, v_n\}$ be the vertices in G such that $\deg_G(v_i) = d_i$. Let ε denote the change in degree sum to $\{v_1, \ldots, v_{t-1}\}$ by adding the edge e, i.e., $\sum_{i=1}^{t-1} d_i' = \sum_{i=1}^{t-1} d_i + \varepsilon$. Then,

$$n \leq t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i' = t - 1 + \frac{1}{k} \left(\sum_{i=1}^{t-1} d_i + \varepsilon \right) = \left(t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i \right) + \frac{\varepsilon}{k} < n + \frac{\varepsilon}{k}.$$

This means that $n - \frac{\varepsilon}{k} \leq t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i < n$. Since $\varepsilon \leq 2$, it follows that $t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i$ equals $n - \frac{1}{k}$ or $n - \frac{2}{k}$. In the latter case, $\varepsilon = 2$ and the only edges that could be added are edges in which both vertices are contained in $\{v_1, \ldots, v_{t-1}\}$. Therefore, each vertex in $\{v_t, \ldots, v_n\}$ is already adjacent to every other vertex in the graph. However, since these are the vertices of smallest degree, this would imply that the graph is a clique, so no edge could have been added, a contradiction. Thus $t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i = n - \frac{1}{k}$ and by algebraically manipulating this equation, we get the desired result.

We now aim to show $\mathrm{sub}_k(G + e) = \mathrm{sub}_k(G) - 1$. With ε defined as above, and using the fact that $t - 1 + \frac{1}{k} \sum_{i=1}^{t-1} d_i = n - \frac{1}{k}$, we have

$$
\begin{aligned}
(t - 2) + \frac{1}{k} \sum_{i=1}^{t-2} d_i' &= (t - 2) + \frac{1}{k} \left(\sum_{i=1}^{t-1} d_i' \right) - \frac{d_{t-1}'}{k} \\
&= \left((t - 1) + \frac{1}{k} \left(\sum_{i=1}^{t-1} d_i \right) \right) + \frac{1}{k}(\varepsilon - d_{t-1}') - 1 \\
&= \left(n - \frac{1}{k} \right) + \frac{\varepsilon - d_{t-1}'}{k} - 1 = n - \frac{1 + d_{t-1}' + k - \varepsilon}{k} < n,
\end{aligned}
$$

where in the last inequality $1 + d_{t-1}' + k - \varepsilon > 0$ since G has no isolates; this implies $\mathrm{sub}_k(G + e) > t - 2$, so $\mathrm{sub}_k(G + e) = \mathrm{sub}_k(G) - 1$. ∎

Graphs that are $\mathrm{sub}_k(G)$-VD-critical differ from $\mathrm{sub}_k(G)$-ED-critical graphs and $\mathrm{sub}_k(G)$-EA-critical graphs, in the sense that it is possible for $\mathrm{sub}_k(G - v)$ and $\mathrm{sub}_k(G)$ to differ by much more than 1. For example, this is the case for the star $K_{n-1,1}$ when the center of the star is the vertex removed. We now show another result for $\mathrm{sub}_k(G)$-VD-critical graphs.

Proposition 10. *If G is a $\mathrm{sub}_k(G)$-VD-critical graph with $\mathrm{sub}_k(G) = t$, then each vertex in $\{v_{t+1}, \ldots, v_n\}$ is adjacent to at least $k + 1$ vertices in $\{v_1, \ldots, v_t\}$.*

Proof. Suppose that $v_x \in \{v_{t+1}, \ldots, v_n\}$ is adjacent to at most k vertices in $\{v_1, \ldots, v_t\}$. Then $G - v_x$ has degree sequence $d'_1, \ldots, d'_{n-1}$ such that $\sum_{i=1}^{t} d'_i \geq \left(\sum_{i=1}^{t} d_i\right) - k$. Thus, $t + \frac{1}{k} \sum_{i=1}^{t} d'_i \geq t + \frac{1}{k} \sum_{i=1}^{t} d_i - 1 \geq n - 1$, which implies that $\mathrm{sub}_k(G - v_x) \leq t$; this contradicts the assumption that G is $\mathrm{sub}_k(G)$-VD-critical. ∎

6. Conclusion

In this paper, we introduced the sub-k-domination number and showed that it is a computationally efficient lower bound on the k-domination number of a graph. We also showed that the sub-k-domination number improves on several known bounds for the k-domination number, and gave some conditions which assure that $\mathrm{sub}_k(G) = \gamma_k(G)$. This investigation was a step toward the general problem of characterizing graphs for which $\gamma_k(G) = \mathrm{sub}_k(G)$ for each positive integer k. In particular, future work in this direction could focus on generalizing Theorem 6 by characterizing all trees T for which $\gamma_k(T) = \mathrm{sub}_k(T)$. Other work to improve lower bounds on k-domination using the DSI-strategy can be pursued, in the vein of Theorem 9 and Corollary 5.

We also explored critical graphs in the context of sub_k-domination, and found that adding an edge to a sub_k-EA-critical graph and deleting an edge from a sub_k-ED-critical graph changes the sub_k-domination number by one. It may be interesting to investigate criticality with respect to edge contraction or other operations, and determine, e.g., if $\mathrm{sub}_k(G/e)$ must differ from $\mathrm{sub}_k(G)$ by one.

As another direction for future work, it would be interesting to define and study an analogue of sub-k-domination which is an upper bound to the k-domination number, or explore degree sequence based invariants which bound the connected domination number or the independent domination number of a graph.

Acknowledgements

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