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LONGER CYCLES IN ESSENTIALLY 4-CONNECTED PLANAR GRAPHS

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Abstract

A planar 3-connected graph G is called *essentially* 4*-connected* if, for every 3-separator S, at least one of the two components of $G - S$ is an isolated vertex. Jackson and Wormald proved that the length $\mathrm{circ}(G)$ of a longest cycle of any essentially 4-connected planar graph G on n vertices is at least $\frac{2n+4}{5}$ and Fabrici, Harant and Jendroľ improved this result to $\mathrm{circ}(G) \geq \frac{1}{2}(n + 4)$. In the present paper, we prove that an essentially 4-connected planar graph on n vertices contains a cycle of length at least $\frac{3}{5}(n + 2)$ and that such a cycle can be found in time $O(n^2)$.

Keywords: essentially 4-connected planar graph, longest cycle, circumference, shortness coefficient.

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For a finite and simple graph G with vertex set $V(G)$ and edge set $E(G)$, let $N(x)$ and $d(x) = |N(x)|$ denote the neighborhood and the degree of any $x \in V(G)$ in G, respectively. The *circumference* $\mathrm{circ}(G)$ of a graph G is the length of a longest cycle of G. A subset $S \subseteq V(G)$ is an *s-separator* of G if $|S| = s$ and $G - S$ is disconnected. From now on, let G be a 3-connected planar graph. For every 3-separator S of G, it is well-known that $G - S$ has exactly two components. We call S *trivial* if at least one component of $G - S$ is a single vertex. If every 3-separator S of G is trivial, we call the 3-connected planar graph G *essentially* 4*-connected*. In the present paper, we are interested in lower bounds on the circumference of essentially 4-connected planar graphs.

Jackson and Wormald [4] proved that $\mathrm{circ}(G) \geq \frac{2n+4}{5}$ for every essentially 4-connected planar graph on n vertices and presented an infinite family of essentially 4-connected planar graphs G such that $\mathrm{circ}(G) \leq c{\cdot}n$ for each real constant $c > \frac{2}{3}$. Moreover, there is a construction of infinitely many essentially 4-connected planar graphs with $\mathrm{circ}(G) = \frac{2}{3}(n+4)$ (for example see [2]). It is open whether there exists an essentially 4-connected planar graph G on n vertices with $\mathrm{circ}(G) < \frac{2}{3}(n+4)$. Further results on the length of longest cycles in essentially 4-connected planar graphs can be found in [2, 3, 7].

Fabrici, Harant and Jendrol' [2] extended the result of Jackson and Wormald by proving that $\mathrm{circ}(G) \geq \frac{1}{2}(n+4)$ for every essentially 4-connected planar graph G on n vertices.

Our result is presented in the following theorem.

Theorem 1. *For any essentially* 4*-connected planar graph G on n vertices,*
$$\mathrm{circ}(G) \geq \tfrac{3}{5}(n+2).$$

We remark that the assertion of the theorem can be improved to $\mathrm{circ}(G) \geq \frac{3}{5}(n+4)$ if $n \geq 16$. This follows from using Lemma 5 in [2] and a more special version of the forthcoming inequality (i). We will also show how cycles of G of length at least $\frac{3}{5}(n+2)$ can be found in quadratic time.

Let C be a plane cycle and let B be a set disjoint from $V(C)$. A plane graph H is called a *(B,C)-graph* if $B \cup V(C)$ is the vertex set of H, the cycle C is an induced subgraph of H, the subgraph of H induced by B is edgeless, and each vertex of B has degree 3 in H. The vertices in B are called *outer vertices* of C.

A face f of H is called *minor* (*major*) if it is incident with at most one (at least two) outer vertices. Note that f is incident with no outer vertex if and only if C is the facial cycle of f.

For every (B,C)-graph H, let $\mu(H)$ denote the number of minor faces of H. Then

$$\text{(i)} \qquad \mu(H) \geq |V(H)| - |V(C)| + 2.$$

***Proof of* (i).** Let H be a smallest counterexample. Since $B = \emptyset$ implies $|V(H)| = |V(C)|$ and $\mu(H) = 2$, which satisfies the inequality (i), we may assume that B is non-empty. For each vertex $y \in B$, the three neighbors of y divide C into three internally disjoint paths $P_1(y)$, $P_2(y)$, and $P_3(y)$ with endvertices in $N(y)$. We may assume that $|V(P_1(y))| \leq |V(P_2(y))| \leq |V(P_3(y))|$ and define $\phi(y) = |V(P_1(y))| + |V(P_2(y))| - 1$ in this case.

Let $x \in B$ be chosen such that $\phi(x) = \min\{\phi(y) \mid y \in B\}$. Consider the two cycles A_1 and A_2 induced by $V(P_1(x)) \cup \{x\}$ and $V(P_2(x)) \cup \{x\}$, respectively. We claim that the interior of A_1 as well as the interior of A_2 is a face of H and hence, both are minor faces. Suppose that there is a vertex z in the interior of A_i for $i \in \{1, 2\}$. Then $\phi(z) = |V(P_1(z))| + |V(P_2(z))| - 1 \leq \max\{\, |V(P_1(x))|, |V(P_2(x))| \,\} < |V(P_1(x))| + |V(P_2(x))| - 1 = \phi(x)$, which contradicts the choice of x.

Let $H' = H - x$. Note that H' is a $((B \setminus \{x\}), C)$-graph and has fewer vertices than H. Then $|V(H')| = |V(H)| - 1$, $\mu(H') \leq \mu(H) - 1$, and $\mu(H') \geq |V(H')| - |V(C)| + 2$, hence $\mu(H) \geq 1 + \mu(H') \geq 1 + |V(H')| - |V(C)| + 2 = |V(H)| - |V(C)| + 2$. ∎

***Proof of Theorem* 1.** Let G be an essentially 4-connected plane graph on n vertices. If G has at most 10 vertices, then it is well known that G is Hamiltonian [1]. In this case, we are done, since $n \geq \frac{3}{5}(n + 2)$ for $n \geq 3$. Thus, we assume $n \geq 11$. A cycle C of G is called an *outer-independent-3-cycle* (OI3-cycle) if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for every $x \in V(G) \setminus V(C)$. An edge $a = xy \in E(C)$ of a cycle C is called an *extendable edge* of C if x and y have a common neighbor in $V(G) \setminus V(C)$.

In [2], it is shown that every essentially 4-connected planar graph G on $n \geq 11$ vertices contains an OI3-cycle. In this proof, let C be a longest OI3-cycle of G, let $c = |V(C)|$, and let H be the graph obtained from G by removing all *chords* of C, i. e. by removing all edges in $E(G) \setminus E(C)$ that connect vertices of C. Clearly, C does not contain an extendable edge. Obviously, H is a (B, C)-graph, with $B = V(H) \setminus V(C)$.

For the number μ of minor faces of H, we have by (i) $\mu \geq n - c + 2$.

Moreover, we will show

$$6\,\mu \leq 4\,c \tag{ii}$$

and then, the theorem follows immediately.

***Proof of* (ii).** An edge e of C is incident with exactly two faces f_1 and f_2 of H. In this case, we say f_1 is *opposite* to f_2 with respect to e. A face f of H is called *j-face* if it is incident with exactly j edges of C and the edges of C incident with f are called *C-edges* of f. Because C does not contain an extendable edge, we have $j \geq 2$ for every minor j-face of H.

We define a weight function w_0 on the set $F(H)$ of faces of H, by setting weight $w_0(f) = 6$ for every minor face f of H and weight $w_0(f) = 0$ for every major face f of H. Then $\sum_{f \in F(H)} w_0(f) = 6\,\mu$. Next, we redistribute the weights of faces of H by the rules **R1** and **R2**.

Rule R1. A minor 2-face f of H sends weight 1 through both C-edges to the opposite (possibly identical) faces.

Rule R2. A minor 3-face f of H with C-edges ux, xy, and yz sends weight 1 through its *middle C-edge xy* to the opposite face.
Let w_1 denote the new weight function; clearly, $\sum_{f \in F(H)} w_1(f) = 6\,\mu$ still holds.

For the proof of (ii), we will show

(iii) $\quad w_1(f) \leq 2\,j$ for each j-face f of H.

To see that (ii) is a consequence of (iii), let each j-face f of H satisfying $j \geq 1$ send the weight $\frac{w_1(f)}{j}$ to each of its C-edges. Note that each 0-face f is major, thus $w_1(f) = 0$. Hence, the total weight of all minor and major faces is moved to the edges of C. Since every edge of C gets weight at most 4, we obtain $6\,\mu = \sum_{f \in F(H)} w_1(f) \leq 4\,c$, and (ii) follows.

***Proof of* (iii).** Next we distinguish several cases. In most of them, we construct a cycle $\tilde{C}$ that is obtained from C by replacing a subpath of C with another path. In every case, $\tilde{C}$ will be an OI3-cycle of G that is longer than C. This contradicts the choice of C and therefore shows that the considered case cannot occur. Note that all vertices of C in the following figures are different, because the length of the longest OI3-cycle C in a planar graph on $n \geq 11$ vertices is at least 8 [2, Lemma 4(ii)].

Case 1. *f is a major j-face.* Because $w_0(f) = 0$ and f gets weight ≤ 1 through each of its C-edges, we have $w_1(f) \leq j$.

Case 2. *f is a minor* 2*-face* (*see Figure* 1)*.* We will show that f does not get any new weight by **R1** or by **R2**; this implies $w_1(f) = w_0(f) - (1 + 1) = 4$. Let xy and yz be the C-edges of f and a be the outer vertex incident with f (see Figure 1).

If f gets new weight by **R1** or by **R2** from a face f' opposite to f with respect to a C-edge of f, then f' is a minor 2-face or a minor 3-face of H. Without loss of generality, we may assume that f' is opposite to f with respect to the edge yz. Then yz is a common C-edge of f and f' and we distinguish the following subcases.

Case 2a. *f' is a* 2*-face and xy is a C-edge of f'.* Then $\{x, z\}$ is the neighborhood of y in G, which contradicts the 3-connectedness of G.

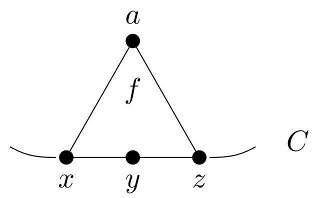

Figure 1

Case 2b. *f' is a* 2*-face and xy is not a C-edge of f'* (*see Figure* 2). Then a longer OI3-cycle $\tilde{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, b, u), which gives a contradiction.

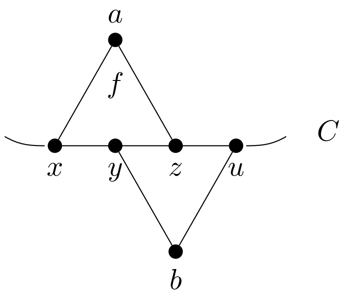

Figure 2

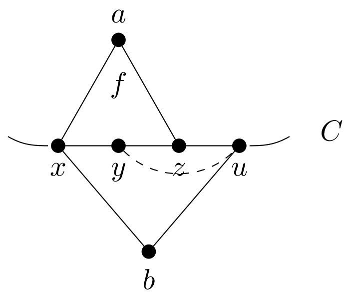

Figure 3

Case 2c. *f' is a* 3*-face.* Since f' sends weight to f, then, by rule **R2**, a C-edge of f is the middle C-edge of f'. It follows that both C-edges of f are also C-edges of f' and the situation as shown in Figure 3 occurs. The edge yu exists in G, because otherwise $d(y) = 2$ and G would not be 3-connected. Then $\tilde{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, u).

Case 3. *f is a minor* 3*-face* (*see Figure* 4). Since f loses weight 1 by rule **R2** and possibly gets weight w by **R1** or by **R2**, we have $w_1(f') = 5 + w$.

If $w \leq 1$, then we are done.

If $w \geq 2$, then f does not get any weight through the edge xy from the opposite face f'. Otherwise, if f' is a 2-face, then we have the situation as in Case 2c and if f' is a 3-face, then $w = 1$, with contradiction in both cases. Hence, f gets weight 1 through vx from the opposite face f_1 and weight 1 through yz from the opposite face f_2. Clearly, $f_1 \neq f_2$ and they are not simultaneously 3-faces.

Case 3a. *Both f_1 and f_2 are* 2*-faces.* The situation is as illustrated in Figure 5 and $\tilde{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path $(w, b, x, v, a, z, y, c, u)$. Note that $b \neq c$, because $d(b) = d(c) = 3$.

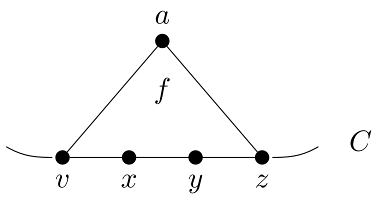

Figure 4

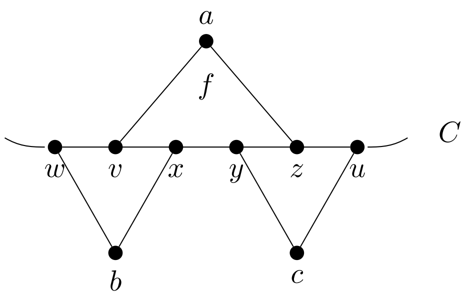

Figure 5

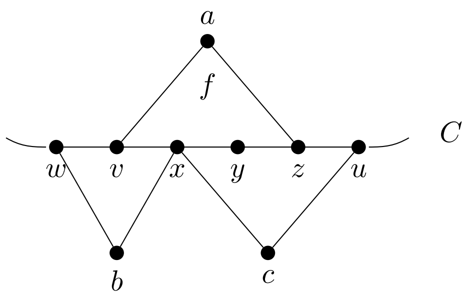

Figure 6

Case 3b. f_1 *is a* 2*-face and* f_2 *is a* 3*-face.* Then $e_2 = yz$ is the middle C-edge of f_2, as shown in Figure 6, and $\tilde{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path (w, v, a, z, y, x, c, u).

Case 4. f *is a minor* 4*-face* (*see Figure* 7).

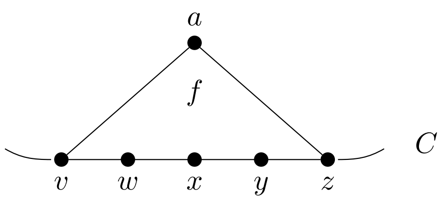

Figure 7

If $w_1(f) = w_0(f) + w = 6 + w$ and $w \leq 2$, then we are done.

If otherwise $w \geq 3$, there are at least three edges e_1, e_2, and e_3 among the four C-edges vw, wx, xy, and yz of f such that f gets weight from minor faces which are opposite to f with respect to e_1, e_2, and e_3, respectively.

Case 4a. $w = 3$ *and* $\{e_1, e_2, e_3\} = \{vw, wx, xy\}$. Then no edge of $\{e_1, e_2, e_3\}$ is the middle C-edge of a minor 3-face and yz is not a C-edge of a minor 2-face. We have the situation of Figure 8 and one of the edges vx or xz exists in G, because otherwise x would have degree 2 in G.

Then $\tilde{C}$ is obtained again from C by replacing the path (v, w, x, y, z) with the path (v, x, w, c, y, z) or with the path (v, w, c, y, x, z), respectively.

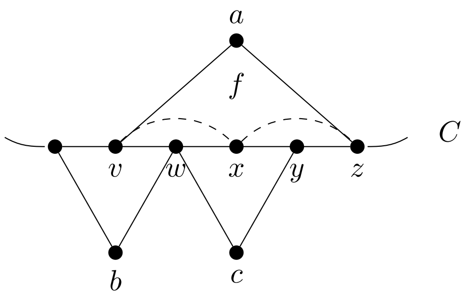

Figure 8

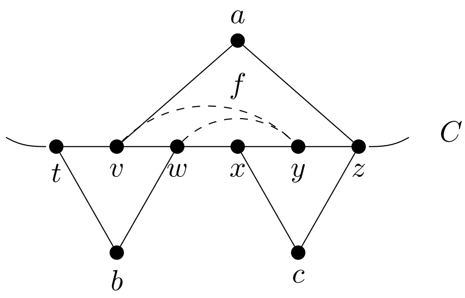

Figure 9

Case 4b. $w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ *and* wx *is not a* C*-edge of a minor* 3*-face.* Then vw is not the middle C-edge of a minor 3-face opposite to f. We have the situation of Figure 9 and one of the edges vy or wy exists in G, because otherwise y would have degree 2 in G.

Note that $b \neq c$, because $d(b) = d(c) = 3$. Then $\tilde{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with the path (t, b, w, v, y, x, c, z) or with the path (t, v, w, y, x, c, z).

Case 4c. $w = 3$, $\{e_1, e_2, e_3\} = \{vw, xy, yz\}$ *and* wx *is a* C*-edge of a minor* 3*-face.* Then vw is the middle C-edge of a minor 3-face opposite to f (see Figure 10).

Then at least one of the edges vy or wy exists, because otherwise y would have degree 2 in G, and $\tilde{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with the path (t, b, x, w, v, y, z) or with the path (t, v, w, y, x, c, z).

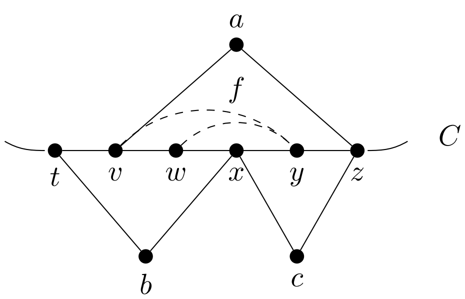

Figure 10

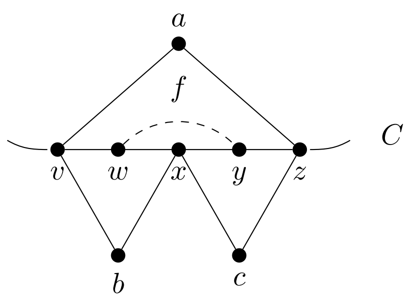

Figure 11

Case 4d. $w = 4$. Then the edges vw, wx, xy, and yz are C-edges of minor 2-faces of H. Either a situation similar to Case 4a occurs, a contradiction, or the situation of Figure 11 follows.

Then the edge wy exists in G, because otherwise $d(w) = 2$ or $d(y) = 2$ in G, and $\tilde{C}$ is obtained from C by replacing the path (v, w, x, y, z) with the path (v, w, y, x, c, z).

Case 5. *f is a minor* 5*-face.* Let $w_1(f) = w_0(f) + w = 6 + w$. If $w \leq 4$, then $w_1(f) \leq 10$ and we are done. If $w = 5$, then all five C-edges of f are also C-edges of minor 2-faces and we have the situation of Figure 12.

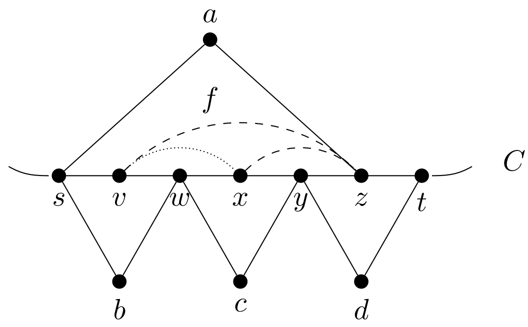

Figure 12

If the edge vx exists, then $\tilde{C}$ is obtained from C by replacing the path (s, v, w, x) with the path (s, b, w, v, x).

If vx does not exist, then, because $d(v) \geq 3$, y or z is a neighbor of v. If the edge vy exists, we get $d(x) = 2$, a contradiction. Hence, vz exists and, since $d(x) \geq 3$, xz exists as well. In this case, $\tilde{C}$ is obtained from C by replacing the path (w, x, y, z) with the path (w, c, y, x, z).

The remaining case completes the proof of (iii) and therefore the proof of (ii).

Case 6. *f is a minor j-face with $j \geq 6$.* Then $w_1(f) = w_0(f) + w = 6 + w \leq 6 + j \leq 2j$. ∎

Algorithm. We now show that a cycle of length at least $\frac{3}{5}(n+2)$ in any essentially 4-connected planar graph G on n vertices can be computed in time $O(n^2)$. For $n \leq 10$, we may compute even a longest cycle in constant time, so assume $n \geq 11$. The existential proof of the theorem proceeds by using a longest not extendable OI3-cycle of G. However, it is straightforward to observe that the proof is still valid when we replace this cycle by an OI3-cycle C that is not extendable and for which none of the local replacements described in the Cases 1–6 can be applied to increase its length (as argued, all these replacements preserve an OI3-cycle).

It suffices to describe how such a cycle C can be computed efficiently; the desired length of C is then implied by the theorem. In [2, Lemma 3], an OI3-cycle of G is obtained by constructing a special Tutte cycle with the aid of Sander's result on Tutte paths [5]. Using the recent result in [6], we can compute such Tutte paths and, by prescribing its end vertices accordingly, also the desired Tutte cycle in time $O(n^2)$. This gives an OI3-cycle C_i of G.

If C_i is extendable, we compute an extendable edge of C_i and extend C_i to a longer cycle C_{i+1}; this takes time $O(n)$ and preserves that C_{i+1} is an OI3-cycle. Otherwise, if there is no extendable edge of C_i (in this case, the length of C_i is at least 8 due to $n \geq 11$ and [2, Lemma 4(ii)]), we decide in time $O(n)$ whether one of the local replacements of the Cases 1–6 can be applied to C_i. If so, we apply any such case and obtain the longer OI3-cycle C_{i+1} (which however may be extendable); since all replacements modify only subgraphs of constant size, this can be done in constant time. Iterating this implies a total running time of $O(n^2)$, as the length of the cycle is increased at most $O(n)$ times.

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