Discussiones Mathematicae
Graph Theory 40 (2020) 35–49
doi:10.7151/dmgt.2132

DECOMPOSITIONS OF CUBIC TRACEABLE GRAPHS

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Abstract

A *traceable graph* is a graph with a Hamilton path. The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching. We prove the conjecture for cubic traceable graphs.

Keywords: decomposition, cubic traceable graph, spanning tree, matching, 2-regular graph.

2010 Mathematics Subject Classification: 05C70, 05C75.

1. Introduction

In the paper all graphs are finite and simple. The reader can refer to [3, 18] for concepts not defined here. A graph G is *cubic* if every vertex in G is of degree 3. A *spanning tree* of G is an acyclic connected subgraph containing all vertices of G. A graph that consists of pairwise disjoint edges is called a *matching*. A k-regular spanning subgraph of G is called a *k-factor*. A 1-factor of G is also called a *perfect matching*. An edge e of G is called a *chord* of a cycle C in G if the two endpoints of e are on C but e is not itself an edge of C. A cycle C is *separating* in a cubic graph G if either C has a chord, or $G - V(C)$ is disconnected; otherwise, *non-separating*. A *Hamilton cycle* is a cycle in G containing all vertices of G. A graph with a Hamilton cycle is called a *Hamiltonian graph*. A *Hamilton path* is a path in G containing all vertices of G. A graph with a Hamilton path is called a *traceable graph.* Assume that H is a Hamilton path in G. Each edge $e \in E(G) \setminus E(H)$ is called a *chord* of H. For every chord $e = vu$ of H, there exists a unique cycle C_e consisting of e and the subpath vHu. We call C_e the *associated cycle* of e. A chord $e = st$ of H is *minimal* if there is no other chord of H whose two endpoints are on the subpath sHt.

A *decomposition* of a graph G consists of pairwise edge-disjoint subgraphs whose union is G. It is a canonical problem in structural graph theory to decompose cubic graphs into subgraphs with certain properties. Such a problem can be traced back to the Petersen Theorem [16] that every bridgeless cubic graph has a 1-factor, which implies that each bridgeless cubic graph can be decomposed into a 1-factor and a 2-factor. The Vizing Theorem [17] on proper edge-coloring shows that every cubic graph admits a decomposition consisting of four matchings.

Decompositions of cubic graphs into paths are related to the Fan-Raspaud conjecture [9] that every 2-edge-connected cubic graph contains three perfect matchings with empty intersection. It is interesting to decompose a cubic graph into a spanning tree and other subgraphs. Malkevitch [14] asked which cubic graphs admit a decomposition into a spanning tree and a 2-regular subgraph, that is, a decomposition with a HIST (a *homeomorphically irreducible spanning tree* is a spanning tree without a 2-degree vertex). Many researchers characterized graphs with a HIST (see [1, 2, 5, 6, 7]). Douglas [8] proved that it is NP-complete to decide whether a graph with maximum degree 3 contains a HIST, which positively solves the problem presented by Albertson, Berman, Hutchinson and Thomassen [2]. It is clear that the complete graph K_4 can be decomposed into a HIST (a star) and a 2-regular subgraph (a triangle) while the cube Q_3 has no HIST. However, we can decompose Q_3 into a spanning tree (with two 2-degree vertices), a 2-regular subgraph (a 4-cycle) and a matching (an edge). See Figure 1. Relaxing the restriction that the spanning tree does not contain a vertex of degree 2, Hoffmann-Ostenhof presented the following conjecture.

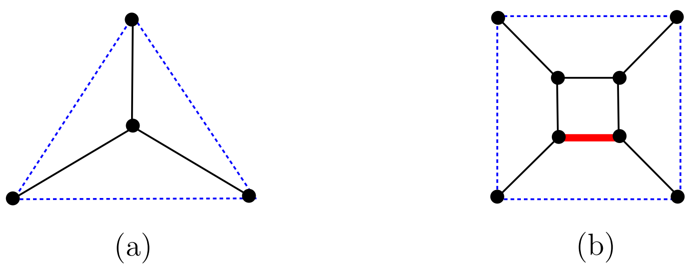

Figure 1. A decomposition of K_4 with a star (thin line) and a triangle (dot line) in (a) while a decomposition of Q_3 with a spanning tree (thin line), a 4-cycle (dot line) and a matching (thick line) in (b).

Conjecture 1 (3-Decomposition Conjecture)**.** *Every connected cubic graph can be decomposed into a spanning tree, a* 2*-regular graph and a matching.*

Conjecture 1 was first posed in [10] (see also [4, Problem BCC 22.12] and [13]). Ozeki and Ye [15] showed that Conjecture 1 holds for 3-connected cubic graphs on the plane and the projective plane. Hoffmann-Ostenhof, Kaiser and Ozeki [12] proved that Conjecture 1 holds for all connected planar cubic graphs. In [1, 11] it was proved that a cubic Hamiltonian graph admits such a desired decomposition. It was informed that Ye [19] showed Conjecture 1 for 3-connected cubic graphs on the Klein bottle and the torus. In the paper, we prove Conjecture 1 for traceable cubic graphs.

Theorem 2. *Every traceable cubic graph can be decomposed into a spanning tree, a* 2*-regular graph and a matching.*

The proof of Theorem 2 consists of four cases (see Section 2). The first case discusses cubic Hamiltonian graphs. The second and third cases are more extensive analyses than the first case. A new technique is used to deal with the fourth case.

2. Proof of Theorem 2

Assume that G is a cubic graph with a Hamilton path H. Let the vertices v_1 and v_6 be the two endpoints of H. Then v_1 and v_6 are incident with two chords of H, every other vertex on H is incident with only one chord. If v_1 is adjacent to v_6 by a chord of H, then let the vertex v_2 be a neighbor of v_1 and v_5 be a neighbor of v_6 such that the two pairs of vertices are jointed by chords of H, respectively. Otherwise, let the vertices v_2, v_3 be two neighbors of v_1 jointed by chords of H such that these vertices are ordered as v_1, v_2, v_3 on H, and let the vertices v_4, v_5 be two neighbors of v_6 jointed by chords of H with the order as v_4, v_5, v_6 on H.

Lemma 3. *Assume that C is a* 2*-regular non-separating subgraph of G that is the union of associated cycles of chords of H, and assume that each of v_1 and v_6 is joined by a chord of H to at least one vertex of $V(C) \cup \{v_1, v_6\}$. Then there is a decomposition of $G - E(C)$ into a spanning tree of G and a matching.*

Proof. Since C is a 2-regular non-separating subgraph of G, $G - E(C)$ is connected and has a spanning tree. Let T be a spanning tree of $G - E(C)$ that contains the forest $H - E(C)$, and let M be the subgraph of G induced by $E(G - E(C \cup T))$. Then M is a matching of $G - E(C)$. Thus $G - E(C)$ admits a decomposition consisting of the spanning tree T and the matching M. ■

Proof of Theorem 2. Let G and H be defined as above. Considering the symmetry of the position of the vertex v_i $(i = 1, 2, \ldots, 6)$ on H, we have the following four cases.

Case 1. v_1 is adjacent to v_6 by a chord of H.

Case 2. v_4 is on the subpath v_1Hv_2.

Case 3. v_4 is on the subpath v_2Hv_3.

Case 4. v_4 is on the subpath v_3Hv_5.

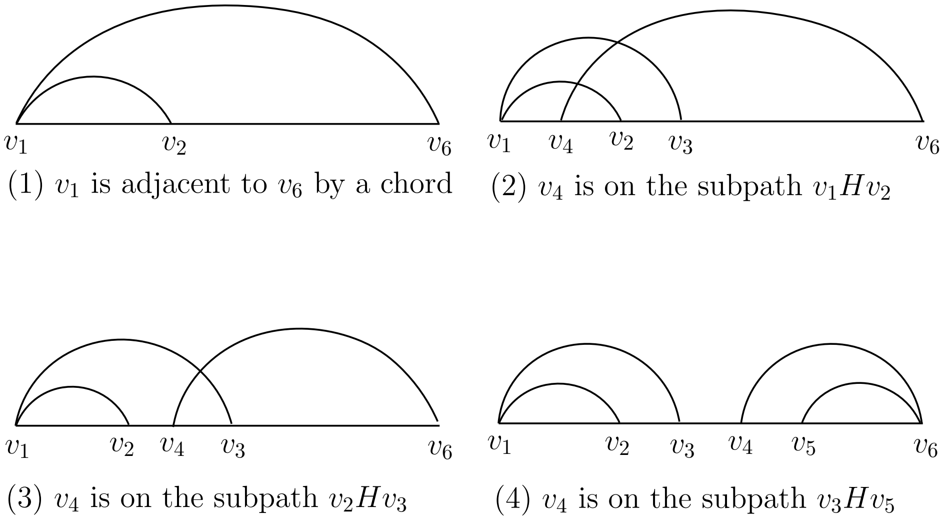

Figure 2. The four cases are illustrated.

It is sufficient to show that each case admits a desired decomposition of G. See Figure 2.

Case 1. *v_1 is adjacent to v_6 by a chord of H.* In this case, G is a Hamiltonian cubic graph. For completeness we give a proof similar to [1, 11].

Since G is a simple cubic graph, there are other chords of H besides the chord v_1v_6. Then there exists a minimal chord e of H. Let C_e be the associated cycle of e. Then C_e is a non-separating cycle. From Lemma 3, $G - E(C_e)$ admits a decomposition consisting of a spanning tree T and a matching M. So there is a decomposition of G with the 2-regular subgraph C_e, the spanning tree T and the matching M.

Case 2. *v_4 is on the subpath v_1Hv_2.* Let $C_1^2 = v_1Hv_4v_6Hv_3v_1$ and $C_2^2 = v_1v_2Hv_3v_1$ be the cycles (see (2) of Figure 2).

Suppose that C_1^2 is a non-separating cycle of G. From Lemma 3, $G - E(C_1^2)$ admits a decomposition consisting of a spanning tree T and a matching M. Thus we can decompose G into the spanning tree T, the 2-regular subgraph C_1^2 and the matching M. Otherwise, C_1^2 is a separating cycle of G. Then there is at least one chord of C_1^2 (and of H also) locating on the subpath v_1Hv_4, locating on the subpath v_3Hv_6, or linking the subpaths v_1Hv_4 and v_3Hv_6.

Further suppose that C_2^2 is a non-separating cycle of G. Let M be a set of all chords of H whose two endpoints are not both on C_2^2 except the chord v_4v_6, and let $T = G - E(C_2^2) - M$. Then M and T are a matching and a spanning tree of G respectively. $T \cup M \cup C_2^2$ forms a desired decomposition of G. Otherwise, C_2^2 is a separating cycle of G. Then there is at least one chord of C_2^2 on the subpath v_2Hv_3. Now, we discuss three subcases as follows.

Subcase 2.1. *There is at least one chord of C_1^2 on the subpath v_1Hv_4.* Since there is at least one chord of C_1^2 on the subpath v_1Hv_4, we can pick a minimal chord $e_1 = u_1u_2$ of H such that the right endpoint u_2 of e_1 is the closest to the vertex v_4 among all minimal chords of H on the subpath v_1Hv_4. Let C_{e_1} be the associated cycle of e_1. Similarly, since there is at least one chord of C_2^2 on the subpath v_2Hv_3, we have a minimal chord $e_2 = u_3u_4$ of H such that the left endpoint u_3 of e_2 is the closest to the vertex v_2 among all minimal chords of H on the subpath v_2Hv_3. Let C_{e_2} be the associated cycle of e_2. Suppose that there is no chord of H which links C_{e_1} and C_{e_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{e_1} and on C_{e_2} except the chords v_1v_3 and v_4v_6. Thus M becomes a matching of G. Let $T = G - E\left(C_{e_1} \cup C_{e_2}\right) - E(M)$. Then T is a spanning tree of G. We can give a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{e_1} \cup C_{e_2}$, and the matching M. Otherwise, there is at least one chord of H which links the cycles C_{e_1} and C_{e_2}. Let $e_3 = u_5u_6$ be such a chord of H, and let C_{e_3} be the associated cycle of e_3. Suppose that C_{e_3} is a non-separating cycle of G.From Lemma 3, $G - E(C_{e_3})$ has a decomposition consisting of a spanning tree T and a matching M. We can decompose G into the spanning tree T, the 2-regular subgraph C_{e_3} and the matching M. Otherwise, C_{e_3} is a separating cycle of G. Then, there is at least one chord of H on the subpath u_5Hu_6 other than e_3. So there must be a minimal chord of H on the subpath u_5Hu_6.

Let e_4 be a minimal chord of H on the subpath u_5Hu_6, and let C_{e_4} be the associated cycle of e_4. If the vertex v_4 is on C_{e_4}, let M be the set of all chords of H none of whose two endpoints are on C_{e_4} except the chord v_1v_3. Let $T = G - E(C_{e_4}) - E(M)$. So we obtain a desired decomposition of G with the spanning tree T, the 2-regular subgraph C_{e_4}, and the matching M. If the vertex v_2 is on C_{e_4} and the vertex v_4 not on C_{e_4}, let M be the set of all chords of H none of whose two endpoints are on C_{e_4} except the chord v_4v_6. Let $T = G - E(C_{e_4}) - E(M)$. Thus we can decompose G into the spanning tree T, the 2-regular subgraph C_{e_4}, and the matching M. See Figure 3.

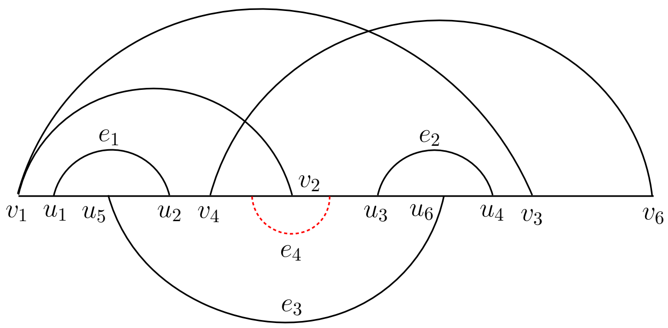

Figure 3. v_2 is on C_{e_4} and v_4 not on C_{e_4}.

So we suppose that neither v_2 nor v_4 is on C_{e_4}. According to the choices of e_1 and e_2, we deduce that e_4 must locate on the subpath v_4Hv_2. Thus there is at least one minimal chord on the subpath v_4Hv_2 (for example, the minimal chord e_4). We pick up a minimal chord, denoted by e_4^*, on the subpath v_4Hv_2 such that the right endpoint u^* of e_4^* is the closed to the vertex v_2 among all minimal chords of H on the subpath v_4Hv_2. Let $C_{e_4^*}$ be the associated cycle of e_4^*. Further suppose that there is no chord of H which links $C_{e_4^*}$ and C_{e_2}. Let M be the set of all chords of H none of whose two endpoints are on $C_{e_4^*}$ and on C_{e_2} except the chords v_1v_2 and v_4v_6. Let $T = G - E(C_{e_4^*} \cup C_{e_2}) - E(M)$. So we obtain a desired decomposition of G with T, $C_{e_4^*} \cup C_{e_2}$, and M. Otherwise, there is at least one chord of H which links $C_{e_4^*}$ and C_{e_2}. Since neither the subpath u^*Hv_2 nor the subpath v_2Hu_3 has any chord, there must exist a minimal chord e_5 of H such that its associated cycle C_{e_5} contains the vertex v_2. We can employ the same means to get a desired decomposition of G as the case that v_2 is on C_{e_4} and v_4 not on C_{e_4}. See Figure 4.

Subcase 2.2. *There is at least one chord of C_1^2 on the subpath v_3Hv_6.* Since there is at least one chord of C_1^2 on the subpath v_3Hv_6, we choose a minimal chord $e_1' = u_1'u_2'$ of H such that the left endpoint u_1' of e_1' is the closest to the vertex v_3 among all minimal chords of H on the subpath v_3Hv_6. Let $C_{e_1'}$ be the associated cycle of e_1'. Similarly, since there is at least one chord of C_2^2 on the subpath v_2Hv_3, there exists a minimal chord $e_2' = u_3'u_4'$ of H such that the right endpoint u_4' of e_2' is the closest to the vertex v_3 among all minimal chords of H on the subpath v_2Hv_3. Let $C_{e_2'}$ be the associated cycle of e_2'. Suppose that there is no chord of H which links $C_{e_1'}$ and $C_{e_2'}$. Let M be the set of all chords of H none of whose two endpoints are on $C_{e_1'}$ and on $C_{e_2'}$ except the chords v_1v_3 and v_4v_6. Thus M becomes a matching of G. Let $T = G - E(C_{e_1'} \cup C_{e_2'}) - E(M)$. Then T is a spanning tree of G. We obtain a desired decomposition of G with

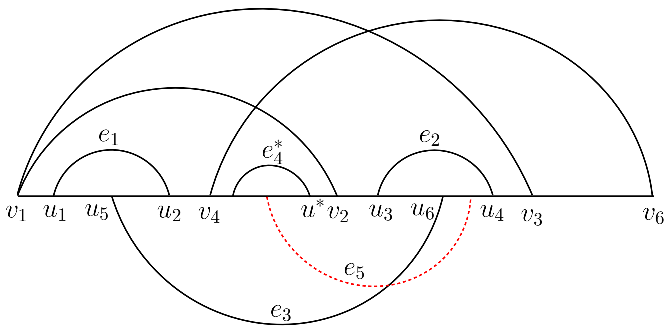

Figure 4. The minimal chord e_5 of H links $C_{e'_4}$ and C_{e_2}, and its associated cycle C_{e_5} contains v_2.

T, $C_{e'_1} \cup C_{e'_2}$, and M. Otherwise, there is at least one chord of H which links the cycles $C_{e'_1}$ and $C_{e'_2}$. Let $e'_3 = u'_5 u'_6$ be such a chord of H, and let $C_{e'_3}$ be the associated cycle of e'_3. Suppose that $C_{e'_3}$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on $C_{e'_3}$ except the chord v_4v_6, and let $T = G - E(C_{e'_3}) - E(M)$. We can decompose G into the spanning tree T, the 2-regular subgraph $C_{e'_3}$ and the matching M. Otherwise, $C_{e'_3}$ is a separating cycle of G. Then, there is at least one chord of H on the subpath $u'_5 H u'_6$ other than e'_3. So there must be a minimal chord of H on the subpath $u'_5 H u'_6$. Let e'_4 be a minimal chord of H on the subpath $u'_5 H u'_6$, and let $C_{e'_4}$ be the associated cycle of e'_4. According to the definitions of e'_1 and e'_2, we deduce that e'_4 is incident with the vertex v_3. Let M be the set of all chords of H none of whose two endpoints are on $C_{e'_4}$ except the chord v_4v_6, and let $T = G - E(C_{e'_4}) - E(M)$. So G has the decomposition with the spanning tree T, 2-regular subgraph $C_{e'_4}$ and the matching M.

Subcase 2.3. *There exists at least one chord of* C_1^2 *which links the subpaths* v_1Hv_4 *and* v_3Hv_6. From Subcase 2.1 and Subcase 2.2, we only need to consider that neither the subpath v_1Hv_4 nor the subpath v_3Hv_6 has any chord of C_1^2 in the subcase. Since there exists at least one chord of C_1^2 which links the subpaths v_1Hv_4 and v_3Hv_6, we can pick a chord $e_6 = u_7u_8$ whose left endpoint u_7 is the closest to the vertex v_1 among all chords of C_1^2 which link the subpaths v_1Hv_4 and v_3Hv_6. Since neither the subpath v_1Hv_4 nor the subpath v_3Hv_6 has any chord of C_1^2, so do the subpaths v_1Hu_7 and v_3Hu_8. Then, we can deduce the cycle $C_3^2 = v_1Hu_7u_8Hv_3v_1$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on C_3^2 except the chord v_4v_6. Let $T = G - E\left(C_3^2\right) - E(M)$. Then M and T are a matching and a spanning tree of G, respectively. So we get a desired decomposition of G with T, C_3^2, and M, see Figure 5.

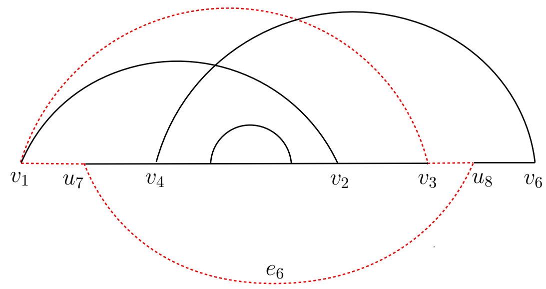

Figure 5. The cycle $C_3^2 = v_1Hu_7u_8Hv_3v_1$ is a non-separating cycle of G.

Case 3. *v_4 is on the subpath v_2Hv_3.* Suppose that there exists a minimal chord f of H on the subpath v_1Hv_3 such that its associated cycle C_f contains the vertex v_4. Let M be the set of all chords of H none of whose two endpoints are on C_f except the chord v_1v_3, and let $T = G - E(C_f) - E(M)$. Then M and T are a matching and a spanning tree of G, respectively. Thus we have a desired decomposition of G with T, C_f and M. Otherwise it is sufficient to consider that

(3.0) *the associated cycle of any minimal chord of H on the subpath v_1Hv_3 does not contain v_4.*

Since there is a chord of H on the subpath v_1Hv_2 (for example, the chord v_1v_2), we can pick a minimal chord $f_1 = t_1t_2$ of H such that the right endpoint t_2 is the closest to the vertex v_2 among all minimal chords of H on the subpath v_1Hv_2. Note if f_1 is the chord v_1v_2, then let $t_i = v_i$ $(i = 1, 2)$. Let C_{f_1} be the associated cycle of f_1. Similar to the subpath v_5Hv_6, we can pick a minimal chord $f_2 = t_3t_4$ of H such that the left endpoint t_3 is the closest to the vertex v_5 among all minimal chords of H on the subpath v_5Hv_6. If f_2 is the chord v_5v_6, then let $t_3 = v_5$ and $t_4 = v_6$. Let C_{f_2} be the associated cycle of f_2. Suppose that there is no chord of H which links the cycles C_{f_1} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_1} and on C_{f_2} except the chords v_1v_3 and v_4v_6. Then M is a matching of G. Let $T = G - E(C_{f_1} \cup C_{f_2}) - E(M)$. T is a spanning tree of G. So we can decompose G into the spanning tree T, the 2-regular subgraph $C_{f_1} \cup C_{f_2}$, and the matching M. Otherwise, there exists at least one chord of H which links C_{f_1} and C_{f_2}. We can assume that a chord $f_3 = t_5t_6$ of H links C_{f_1} and C_{f_2} and t_5 is the left endpoint of f_3. Let $C_1^3 = v_1v_2Hv_3v_1$. If C_1^3 is a non-separating cycle of G, then let M be the set of all chords of H none of whose two endpoints are on C_1^3 except the chord f_3, and $T = G - E\left(C_1^3\right) - E(M)$. It is clear that M and T are a matching and a spanning tree of G, respectively. Thus we obtain a desired decomposition of G with T, C_1^3 and M. Otherwise, C_1^3 is a separating cycle of G. Then, there is at least one chord of H on the subpath v_2Hv_3. Let f_4 be any minimal chord of H on the subpath v_2Hv_3, and let C_{f_4} be the associated cycle of f_4. From (3.0), C_{f_4} does not contain the vertex v_4.

Suppose that there is not any chord of H which links the cycles C_{f_4} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_4} and on C_{f_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{f_4} \cup C_{f_2}) - E(M)$. Then G has the desired decomposition $\{T, C_{f_4} \cup C_{f_2}, M\}$. Otherwise, there is a chord of H which links C_{f_4} and C_{f_2}. Of course, there is at least one chord of H which links the subpath t_5Hv_3 and C_{f_2}. Let $f_5 = t_7t_8$ be a chord of H linking the subpath t_5Hv_3 and C_{f_2} such that the left endpoint t_7 is the closest to the vertex t_5 among all chords of H linking the subpath t_5Hv_3 and C_{f_2}. Let $C_2^3 = t_5Ht_7t_8Ht_6t_5$. If C_2^3 is a non-separating cycle of G, then let M be the set of all chords of H none of whose two endpoints are on C_2^3 except the chords v_1v_3 and v_5v_6. Let $T = G - E\left(C_2^3\right) - E(M)$. So we get a desired decomposition of G with T, C_2^3 and M. Otherwise, C_2^3 is a separating cycle of G. Then there must be at least one chord of H on the subpath t_5Ht_7. Let f_6 be a minimal chord of H on the subpath t_5Ht_7, and let C_{f_6} be the associated cycle of f_6. From (3.0), we have that C_{f_6} does not contain the vertex v_4. According to the choice of f_5, there is no chord of H which links C_{f_6} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_6} and on C_{f_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{f_6} \cup C_{f_2}) - E(M)$. Thus we have a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{f_6} \cup C_{f_2}$, and the matching M, see Figure 6.

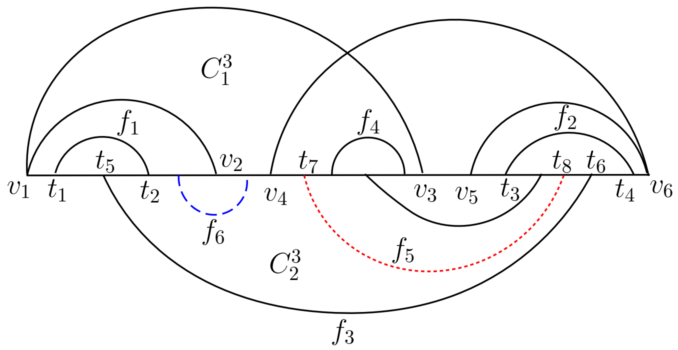

Figure 6. Case 3 is illustrated.

Case 4. *v_4 is on the subpath v_3Hv_5.* Since there are chords of H on the subpath v_1Hv_3 (for example, the chords v_1v_2 and v_1v_3), we can choose a minimal chord $g_1 = s_1s_2$ of H on the subpath v_1Hv_3. If g_1 is the chord v_1v_2, then $s_i = v_i$ $(i = 1, 2)$. Let C_{g_1} be the associated cycle of g_1. Similarly, let $g_2 = s_3s_4$ be a minimal chord of H on the subpath v_4Hv_6. If g_2 is the chord v_5v_6, then $s_3 = v_5$ and $s_4 = v_6$. Let C_{g_2} be the associated cycle of g_2. If there is no chord of H

which links the cycles C_{g_1} and C_{g_2}, then let M be the set of all chords of H none of whose two endpoints are on C_{g_1} and on C_{g_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{g_1} \cup C_{g_2}) - E(M)$. Thus we have a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{g_1} \cup C_{g_2}$ and the matching M. Otherwise, we suppose that

(4.0) *the associated cycle of any minimal chord of H on the subpath v_1Hv_3 is linked by a chord of H with the associated cycle of each minimal chord of H on the subpath v_4Hv_6.*

Since the subpath v_1Hv_2 has at least one chord of H, there is a minimal chord g_3 of H. If the subpath v_1Hv_2 only has the chord v_1v_2, then $g_3 = v_1v_2$. Let C_{g_3} be the associated cycle of g_3. Similarly, there exists a minimal chord g_4 of H on the subpath v_5Hv_6. If the subpath v_5Hv_6 only has the chord v_5v_6, then $g_4 = v_5v_6$. Let C_{g_4} be the associated cycle of g_4. From (4.0), there is at least one chord of H which links C_{g_3} and C_{g_4}. Let $g_5 = s_5s_6$ be such a chord of H. We discuss the following two subcases.

Subcase 4.1. *There are at least two chords of H which link the subpaths v_1Hv_3 and v_4Hv_6.* Let $g_6 = s_7s_8$ be a chord of H linking the subpaths v_1Hv_3 and v_4Hv_6 different from g_5 such that the left endpoint s_7 is the closest to the vertex s_5 among all chords of H linking such two subpaths. Suppose that the cycle $C_1^4 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on C_1^4 except the chords v_1v_3 and v_4v_6, and let $T = G - E\left(C_1^4\right) - E(M)$. Thus G can be decomposed into the spanning tree T, the 2-regular subgraph C_1^4 and the matching M, see Figure 7.

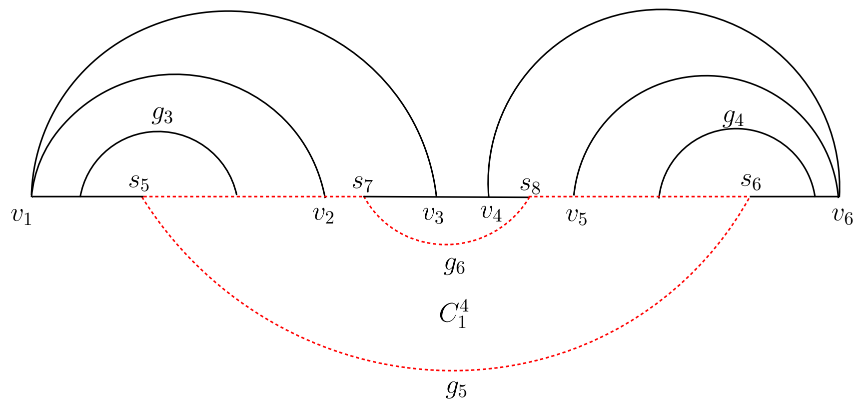

Figure 7. The cycle $C_1^4 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of G (dot line).

Otherwise, C_1^4 is a separating cycle of G. From (4.0) and the choice of g_6, we can deduce that there exists at least one chord of H on the subpath s_8Hs_6.

Then we pick a minimal chord g_7 of H on the subpath s_8Hs_6 such that the right endpoint of g_7 is the closest to the vertex s_6 among all chords of H on the subpath s_8Hs_6. Let C_{g_7} be the associated cycle of g_7. According to (4.0), there is at least one chord of H which links the cycles C_{g_3} and C_{g_7}. Let $g_8 = s_9s_{10}$ be a chord of H linking C_{g_3} and C_{g_7} such that the right endpoint s_{10} is the closest to the vertex s_6 among such all chords of H. Then, the cycle $C_2^4 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle. Since both s_5 and s_9 are on the associated cycle C_{g_3} of the minimal chord g_3, there is no chord of H on the subpath s_9Hs_5. Let M be the set of all chords of H none of whose two endpoints are on C_2^4 except the chords v_1v_3 and v_4v_6, and let $T = G - E\left(C_2^4\right) - E(M)$. So we have a desired decomposition of G with T, C_2^4 and M, see Figure 8.

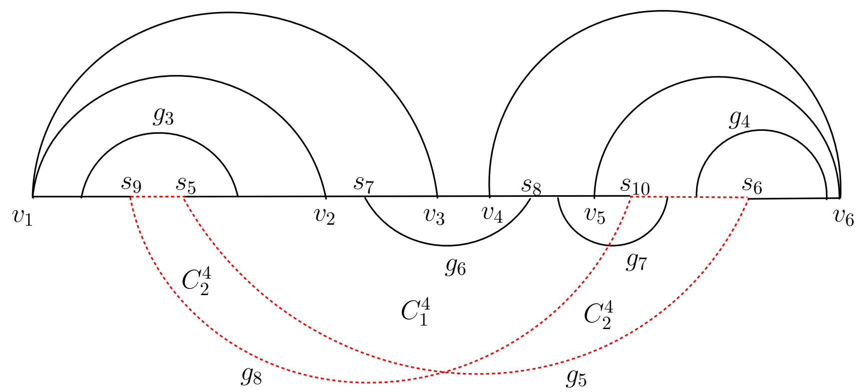

Figure 8. The cycle $C_2^4 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle of G (dot line).

Subcase 4.2. *The chord g_5 is the only one chord of H which links the subpaths v_1Hv_3 and v_4Hv_6.* According to (4.0), it can be deduced that the vertex s_5 locates between two endpoints of any minimal chord of H on the subpath v_1Hv_3. If not, there is a minimal chord of H such that its associate cycle is not incident with s_5. Then there is a chord of H different from g_5 which links the associated cycle of this minimal chord and C_{g_4}, contradiction. Further we can obtain that s_5 locates between two endpoints of each chord of H on the subpath v_1Hv_3.

Only for convenience, we give a drawing of the graph G here. Except that the chord g_5 is arranged on one side of H, all chords of H are arranged on the other side of H. We discuss two cases as follows.

Subcase 4.2.1. *There exists a chord g of H such that g intersects at least two chords of H on the subpath v_1Hv_3.* Let $g_9 = s_{11}s_{12}$ and $g_{10} = s_{13}s_{14}$ be two chords of H intersecting g such that the left endpoint s_{11} of g_9 is the closest to the left endpoint s_{13} of g_{10} among such all chords of H on the subpath v_1Hv_3. Let the cycle $C_3^4 = s_{11}s_{12}Hs_{14}s_{13}Hs_{11}$. Then C_3^4 is a non-separating cycle of G. If none of g_9 and g_{10} is the chord v_1v_3, then let M be the set of all chords of H none of whose two endpoints are on C_3^4 and on C_{g_4} except the chords v_1v_3, v_4v_6, and g; otherwise, let M be the set of all chords of H none of whose two endpoints are on C_3^4 and on C_{g_4} except the chords v_4v_6 and g. Let $T = G - E\left(C_3^4 \cup C_{g_4}\right) - E(M)$. Thus the graph G can be decomposed into the spanning tree T, the 2-regular subgraph $C_3^4 \cup C_{g_4}$ and the matching M, see Figure 9.

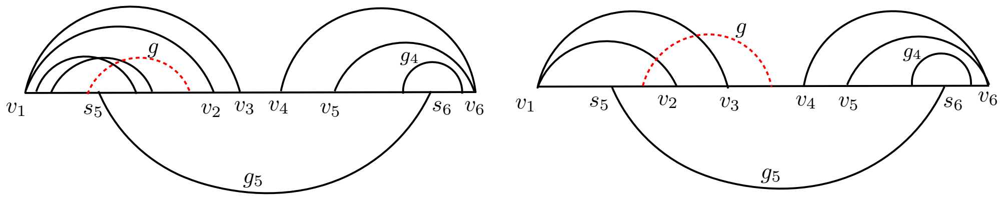

Figure 9. (1) g intersects the chords g_9 and g_{10} on the subpath v_1Hv_3, and none of g_9 and g_{10} is the chord v_1v_3;
(2) g intersects the chords g_9 and g_{10} on the subpath v_1Hv_3, and one of g_9 and g_{10} is the chord v_1v_3.

Subcase 4.2.2. *There is not any chord of H that intersects two chords on the subpath v_1Hv_3.* Suppose that there is a chord $g^1 = s^1s^2$ of H such that the endpoint s^1 is on the subpath v_1Hs_5 and the endpoint s^2 is on the subpath v_2Hv_3 (*the former case* for short). On the subpath v_1Hs^2, we start from the second edge and choose every other edge along the direction from v_1 to s^2. Otherwise, there is not any chord of H one of which endpoints is on the subpath v_1Hs_5 and the other on the subpath v_2Hv_3 (*the latter case* for short). On the subpath v_1Hv_2, we start from the second edge and choose every other edge along the direction from v_1 to v_2. Let M_0 be the set of the chosen edges in both cases. Then M_0 is a matching of G. Let V be the set of vertices on the subpath v_1Hs^2 for the former case or the set of vertices on the subpath v_1Hv_2 for the latter case. We first prove the following claim.

Claim. *Let M_0, V, the former case, and the latter case be defined as above. Then the subgraph $G[V]$–$E(M_0)$ is a path, where $G[V]$ is a subgraph of G induced by V.*

Proof. Let $G_1 = G[V] - E(M_0)$. Since the vertex s_5 locates between the two endpoints of each chord of H on the subpath v_1Hv_3, V insists of s_5 and the union of the two endpoints of each chord on the subpath v_1Hs^2 for the former case or on the subpath v_1Hv_2 for the latter case. $|V|$ is odd. Both the subpath v_1Hs^2 and the subpath v_1Hv_2 have an even number of edges. According to the choice of M_0, all vertices of G_1 are of degree 2 except two 1-degree vertices s_5 and s^2 for

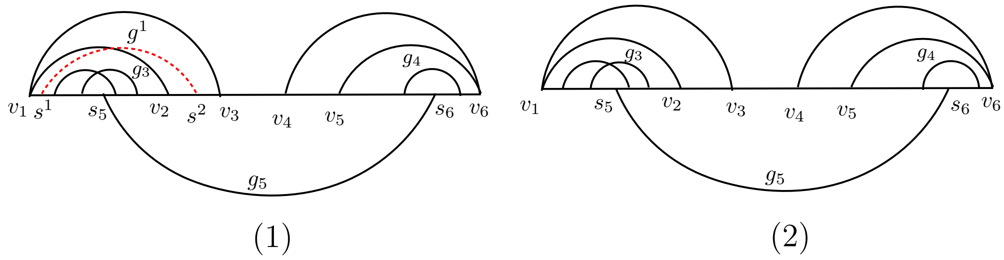

Figure 10. (1) the former case that there exists a chord $g^1 = s^1s^2$ of H such that s^1 is on the subpath v_1Hs_5 and s^2 is on the subpath v_2Hv_3;
(2) the latter case that there is not any chord of H like g^1.

the former case or two 1-degree vertices s_5 and v_2 for the latter case. It suffices to prove that G_1 is connected.

Suppose that G_1 is disconnected. The components of G_1 consist of one path and some cycles according to the degree condition of G_1. Let C be a component of G_1 which is a cycle. In G_1, s_5 is not incident with C since s_5 is of degree 1. Let t_1 and t_2 be two vertices of C such that t_1 is the closest to s_5 among all vertices of C which locate on the left side of s_5 and t_2 is the closest to s_5 among all vertices of C which locate on the right side of s_5. Let the path $P = t_1Hs_5Ht_2$. Then the edges incident with t_1 and t_2 on P are edges of M_0. So P has an odd number of edges and an even number of vertices according to the choice of M_0. We can deduce that there is a chord g^* of H such that it only has one endpoint on P. The endpoint of g^* not on P can not be on C according to the choice of M_0 and P. Then g^* intersects at least two edges of C which are chords of H on the subpath v_1Hv_3, contraction with assumptions in Subcase 4.2.2. So G_1 is connected, and is a path. □

Let the subpath $P_1 = s^2Hv_6$ for the former case or $P_1 = v_2Hv_6$ for the latter case. Let M_1 be the set of all chords of H on P_1 none of whose two endpoints are on C_{g_4} except the chord v_4v_6. Let $M = M_0 \cup M_1 \cup v_1v_3$, and let $T = G - E(C_{g_4}) - E(M)$. Thus we get a desired decomposition of G with the spanning tree T, the 2-regular subgraph C_{g_4} and the matching M, see Figure 10. ■

From Theorem 2, we have the following corollary.

Corollary 4. *Let G be a connected cubic graph with n vertices and girth at least $(n-1)$. Then G can be decomposed into a spanning tree, a 2-regular graph and a matching.*

Acknowledgements

The authors would like to thank the referees for their careful reading and useful suggestions. The first author was supported by Fundamental Research Funds for the Central Universities NO. NZ2015106.

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Received 6 February 2017
Revised 18 January 2018
Accepted 18 January 2018