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DECOMPOSITIONS OF CUBIC TRACEABLE GRAPHS

Wenzhong Liu

and

Panpan Li

Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

e-mail: wzhiu7502@nuaa.edu.cn
252366887@qq.com

Abstract

A *traceable graph* is a graph with a Hamilton path. The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching. We prove the conjecture for cubic traceable graphs.

Keywords: decomposition, cubic traceable graph, spanning tree, matching, 2-regular graph.

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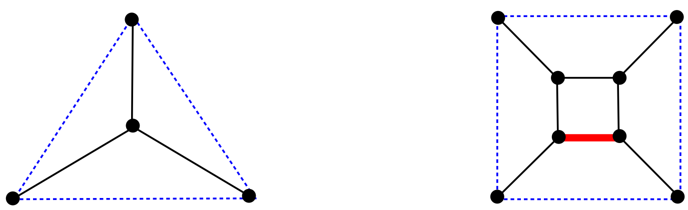
1. Introduction

In the paper all graphs are finite and simple. The reader can refer to [3, 18] for concepts not defined here. A graph G is *cubic* if every vertex in G is of degree 3. A *spanning tree* of G is an acyclic connected subgraph containing all vertices of G. A graph that consists of pairwise disjoint edges is called a *matching*. A k-regular spanning subgraph of G is called a *k-factor*. A 1-factor of G is also called a *perfect matching*. An edge e of G is called a *chord* of a cycle C in G if the two endpoints of e are on C but e is not itself an edge of C. A cycle C is *separating* in a cubic graph G if either C has a chord, or $G - V(C)$ is disconnected; otherwise, *non-separating*. A *Hamilton cycle* is a cycle in G containing all vertices of G. A graph with a Hamilton cycle is called a *Hamiltonian graph*. A *Hamilton path* is a path in G containing all vertices of G. A graph with a Hamilton path is called a *traceable graph*. Assume that H is a Hamilton path in G. Each edge $e \in E(G) \setminus E(H)$ is called a *chord* of H. For every chord $e = vu$ of H, there exists a unique cycle C_e consisting of e and the subpath vHu. We call C_e the *associated cycle* of e. A chord $e = st$ of H is *minimal* if there is no other chord of H whose two endpoints are on the subpath sHt.

A *decomposition* of a graph G consists of pairwise edge-disjoint subgraphs whose union is G. It is a canonical problem in structural graph theory to decompose cubic graphs into subgraphs with certain properties. Such a problem can be traced back to the Petersen Theorem [16] that every bridgeless cubic graph has a 1-factor, which implies that each bridgeless cubic graph can be decomposed into a 1-factor and a 2-factor. The Vizing Theorem [17] on proper edge-coloring shows that every cubic graph admits a decomposition consisting of four matchings.

Decompositions of cubic graphs into paths are related to the Fan-Raspaud conjecture [9] that every 2-edge-connected cubic graph contains three perfect matchings with empty intersection. It is interesting to decompose a cubic graph into a spanning tree and other subgraphs. Malkevitch [14] asked which cubic graphs admit a decomposition into a spanning tree and a 2-regular subgraph, that is, a decomposition with a HIST (a *homeomorphically irreducible spanning tree* is a spanning tree without a 2-degree vertex). Many researchers characterized graphs with a HIST (see [1, 2, 5, 6, 7]). Douglas [8] proved that it is NP-complete to decide whether a graph with maximum degree 3 contains a HIST, which positively solves the problem presented by Albertson, Berman, Hutchinson and Thomassen [2]. It is clear that the complete graph K_4 can be decomposed into a HIST (a star) and a 2-regular subgraph (a triangle) while the cube Q_3 has no HIST. However, we can decompose Q_3 into a spanning tree (with two 2-degree vertices), a 2-regular subgraph (a 4-cycle) and a matching (an edge). See Figure 1. Relaxing the restriction that the spanning tree does not contain a vertex of degree 2, Hoffmann-Ostenhof presented the following conjecture.



(a) (b)

Figure 1. A decomposition of K_4 with a star (thin line) and a triangle (dot line) in (a) while a decomposition of Q_3 with a spanning tree (thin line), a 4-cycle (dot line) and a matching (thick line) in (b).

Conjecture 1 (3-Decomposition Conjecture)**.** *Every connected cubic graph can be decomposed into a spanning tree, a* 2*-regular graph and a matching.*

Conjecture 1 was first posed in [10] (see also [4, Problem BCC 22.12] and [13]). Ozeki and Ye [15] showed that Conjecture 1 holds for 3-connected cubic graphs on the plane and the projective plane. Hoffmann-Ostenhof, Kaiser and Ozeki [12] proved that Conjecture 1 holds for all connected planar cubic graphs. In [1, 11] it was proved that a cubic Hamiltonian graph admits such a desired decomposition. It was informed that Ye [19] showed Conjecture 1 for 3-connected cubic graphs on the Klein bottle and the torus. In the paper, we prove Conjecture 1 for traceable cubic graphs.

Theorem 2. *Every traceable cubic graph can be decomposed into a spanning tree, a* 2*-regular graph and a matching.*

The proof of Theorem 2 consists of four cases (see Section 2). The first case discusses cubic Hamiltonian graphs. The second and third cases are more extensive analyses than the first case. A new technique is used to deal with the fourth case.

2. Proof of Theorem 2

Assume that G is a cubic graph with a Hamilton path H. Let the vertices v_1 and v_6 be the two endpoints of H. Then v_1 and v_6 are incident with two chords of H, every other vertex on H is incident with only one chord. If v_1 is adjacent to v_6 by a chord of H, then let the vertex v_2 be a neighbor of v_1 and v_5 be a neighbor of v_6 such that the two pairs of vertices are jointed by chords of H, respectively. Otherwise, let the vertices v_2, v_3 be two neighbors of v_1 jointed by chords of H such that these vertices are ordered as v_1, v_2, v_3 on H, and let the vertices v_4, v_5 be two neighbors of v_6 jointed by chords of H with the order as v_4, v_5, v_6 on H.

Lemma 3. *Assume that C is a* 2*-regular non-separating subgraph of G that is the union of associated cycles of chords of H, and assume that each of v_1 and v_6 is joined by a chord of H to at least one vertex of $V(C) \cup \{v_1, v_6\}$. Then there is a decomposition of $G - E(C)$ into a spanning tree of G and a matching.*

Proof. Since C is a 2-regular non-separating subgraph of G, $G - E(C)$ is connected and has a spanning tree. Let T be a spanning tree of $G - E(C)$ that contains the forest $H - E(C)$, and let M be the subgraph of G induced by $E(G - E(C \cup T))$. Then M is a matching of $G - E(C)$. Thus $G - E(C)$ admits a decomposition consisting of the spanning tree T and the matching M. ∎

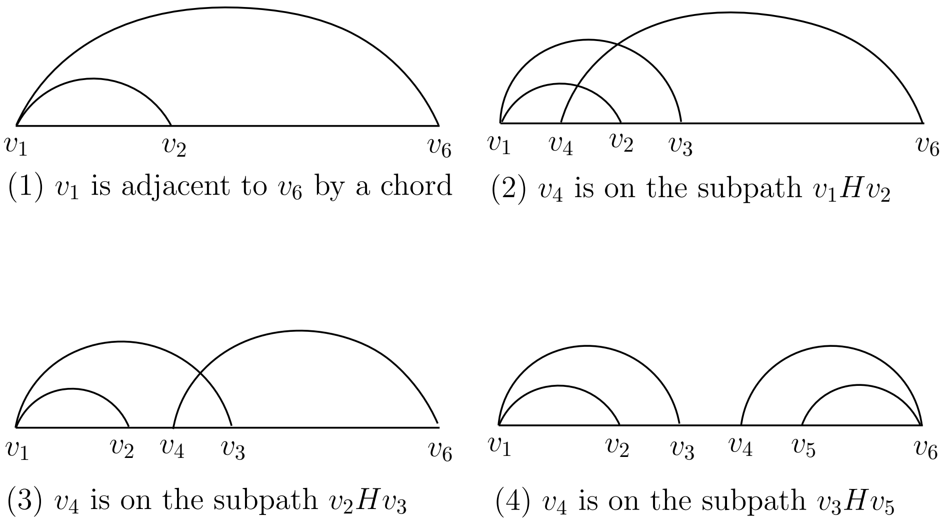
Proof of Theorem 2. Let G and H be defined as above. Considering the symmetry of the position of the vertex v_i $(i = 1, 2, \ldots, 6)$ on H, we have the following four cases.

Case 1. v_1 is adjacent to v_6 by a chord of H.

Case 2. v_4 is on the subpath v_1Hv_2.

Case 3. v_4 is on the subpath v_2Hv_3.

Case 4. v_4 is on the subpath v_3Hv_5.



(1) v_1 is adjacent to v_6 by a chord (2) v_4 is on the subpath v_1Hv_2

(3) v_4 is on the subpath v_2Hv_3 (4) v_4 is on the subpath v_3Hv_5

Figure 2. The four cases are illustrated.

It is sufficient to show that each case admits a desired decomposition of G. See Figure 2.

Case 1. *v_1 is adjacent to v_6 by a chord of H.* In this case, G is a Hamiltonian cubic graph. For completeness we give a proof similar to [1, 11].

Since G is a simple cubic graph, there are other chords of H besides the chord v_1v_6. Then there exists a minimal chord e of H. Let C_e be the associated cycle of e. Then C_e is a non-separating cycle. From Lemma 3, $G - E(C_e)$ admits a decomposition consisting of a spanning tree T and a matching M. So there is a decomposition of G with the 2-regular subgraph C_e, the spanning tree T and the matching M.

Case 2. *v_4 is on the subpath v_1Hv_2.* Let $C_1^2 = v_1Hv_4v_6Hv_3v_1$ and $C_2^2 = v_1v_2Hv_3v_1$ be the cycles (see (2) of Figure 2).

Suppose that C_1^2 is a non-separating cycle of G. From Lemma 3, $G - E(C_1^2)$ admits a decomposition consisting of a spanning tree T and a matching M. Thus we can decompose G into the spanning tree T, the 2-regular subgraph C_1^2 and the matching M. Otherwise, C_1^2 is a separating cycle of G. Then there is at least one chord of C_1^2 (and of H also) locating on the subpath v_1Hv_4, locating on the subpath v_3Hv_6, or linking the subpaths v_1Hv_4 and v_3Hv_6.

Further suppose that C_2^2 is a non-separating cycle of G. Let M be a set of all chords of H whose two endpoints are not both on C_2^2 except the chord v_4v_6, and let $T = G - E(C_2^2) - M$. Then M and T are a matching and a spanning tree of G respectively. $T \cup M \cup C_2^2$ forms a desired decomposition of G. Otherwise, C_2^2 is a separating cycle of G. Then there is at least one chord of C_2^2 on the subpath v_2Hv_3. Now, we discuss three subcases as follows.

Subcase 2.1. *There is at least one chord of C_1^2 on the subpath v_1Hv_4.* Since there is at least one chord of C_1^2 on the subpath v_1Hv_4, we can pick a minimal chord $e_1 = u_1u_2$ of H such that the right endpoint u_2 of e_1 is the closest to the vertex v_4 among all minimal chords of H on the subpath v_1Hv_4. Let C_{e_1} be the associated cycle of e_1. Similarly, since there is at least one chord of C_2^2 on the subpath v_2Hv_3, we have a minimal chord $e_2 = u_3u_4$ of H such that the left endpoint u_3 of e_2 is the closest to the vertex v_2 among all minimal chords of H on the subpath v_2Hv_3. Let C_{e_2} be the associated cycle of e_2. Suppose that there is no chord of H which links C_{e_1} and C_{e_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{e_1} and on C_{e_2} except the chords v_1v_3 and v_4v_6. Thus M becomes a matching of G. Let $T = G - E\left(C_{e_1} \cup C_{e_2}\right) - E(M)$. Then T is a spanning tree of G. We can give a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{e_1} \cup C_{e_2}$, and the matching M. Otherwise, there is at least one chord of H which links the cycles C_{e_1} and C_{e_2}. Let $e_3 = u_5u_6$ be such a chord of H, and let C_{e_3} be the associated cycle of e_3. Suppose that C_{e_3} is a non-separating cycle of G.From Lemma 3, $G - E(C_{e_3})$ has a decomposition consisting of a spanning tree T and a matching M. We can decompose G into the spanning tree T, the 2-regular subgraph C_{e_3} and the matching M. Otherwise, C_{e_3} is a separating cycle of G. Then, there is at least one chord of H on the subpath u_5Hu_6 other than e_3. So there must be a minimal chord of H on the subpath u_5Hu_6.

Let e_4 be a minimal chord of H on the subpath u_5Hu_6, and let C_{e_4} be the associated cycle of e_4. If the vertex v_4 is on C_{e_4}, let M be the set of all chords of H none of whose two endpoints are on C_{e_4} except the chord v_1v_3. Let $T = G - E(C_{e_4}) - E(M)$. So we obtain a desired decomposition of G with the spanning tree T, the 2-regular subgraph C_{e_4}, and the matching M. If the vertex v_2 is on C_{e_4} and the vertex v_4 not on C_{e_4}, let M be the set of all chords of H none of whose two endpoints are on C_{e_4} except the chord v_4v_6. Let $T = G - E(C_{e_4}) - E(M)$. Thus we can decompose G into the spanning tree T, the 2-regular subgraph C_{e_4}, and the matching M. See Figure 3.

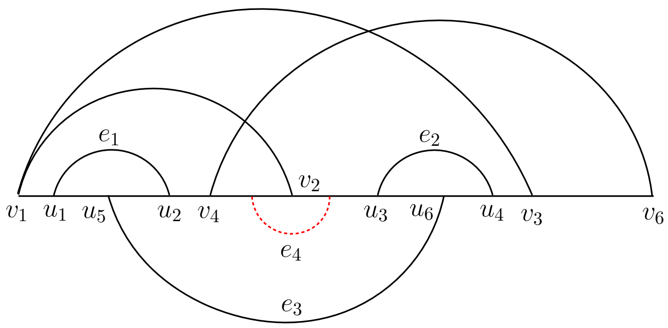


Figure 3. v_2 is on C_{e_4} and v_4 not on C_{e_4}.

So we suppose that neither v_2 nor v_4 is on C_{e_4}. According to the choices of e_1 and e_2, we deduce that e_4 must locate on the subpath v_4Hv_2. Thus there is at least one minimal chord on the subpath v_4Hv_2 (for example, the minimal chord e_4). We pick up a minimal chord, denoted by e_4^*, on the subpath v_4Hv_2 such that the right endpoint u^* of e_4^* is the closed to the vertex v_2 among all minimal chords of H on the subpath v_4Hv_2. Let $C_{e_4^*}$ be the associated cycle of e_4^*. Further suppose that there is no chord of H which links $C_{e_4^*}$ and C_{e_2}. Let M be the set of all chords of H none of whose two endpoints are on $C_{e_4^*}$ and on C_{e_2} except the chords v_1v_2 and v_4v_6. Let $T = G - E(C_{e_4^*} \cup C_{e_2}) - E(M)$. So we obtain a desired decomposition of G with T, $C_{e_4^*} \cup C_{e_2}$, and M. Otherwise, there is at least one chord of H which links $C_{e_4^*}$ and C_{e_2}. Since neither the subpath u^*Hv_2 nor the subpath v_2Hu_3 has any chord, there must exist a minimal chord e_5 of H such that its associated cycle C_{e_5} contains the vertex v_2. We can employ the same means to get a desired decomposition of G as the case that v_2 is on C_{e_4} and v_4 not on C_{e_4}. See Figure 4.

Subcase 2.2. *There is at least one chord of C_1^2 on the subpath v_3Hv_6.* Since there is at least one chord of C_1^2 on the subpath v_3Hv_6, we choose a minimal chord $e_1' = u_1'u_2'$ of H such that the left endpoint u_1' of e_1' is the closest to the vertex v_3 among all minimal chords of H on the subpath v_3Hv_6. Let $C_{e_1'}$ be the associated cycle of e_1'. Similarly, since there is at least one chord of C_2^2 on the subpath v_2Hv_3, there exists a minimal chord $e_2' = u_3'u_4'$ of H such that the right endpoint u_4' of e_2' is the closest to the vertex v_3 among all minimal chords of H on the subpath v_2Hv_3. Let $C_{e_2'}$ be the associated cycle of e_2'. Suppose that there is no chord of H which links $C_{e_1'}$ and $C_{e_2'}$. Let M be the set of all chords of H none of whose two endpoints are on $C_{e_1'}$ and on $C_{e_2'}$ except the chords v_1v_3 and v_4v_6. Thus M becomes a matching of G. Let $T = G - E(C_{e_1'} \cup C_{e_2'}) - E(M)$. Then T is a spanning tree of G. We obtain a desired decomposition of G with

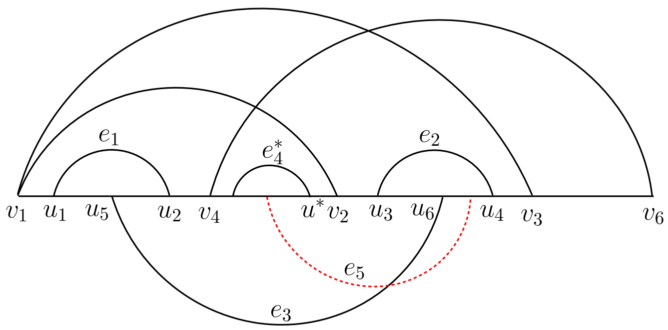


Figure 4. The minimal chord e_5 of H links $C_{e'_4}$ and C_{e_2}, and its associated cycle C_{e_5} contains v_2.

T, $C_{e'_1} \cup C_{e'_2}$, and M. Otherwise, there is at least one chord of H which links the cycles $C_{e'_1}$ and $C_{e'_2}$. Let $e'_3 = u'_5u'_6$ be such a chord of H, and let $C_{e'_3}$ be the associated cycle of e'_3. Suppose that $C_{e'_3}$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on $C_{e'_3}$ except the chord v_4v_6, and let $T = G - E(C_{e'_3}) - E(M)$. We can decompose G into the spanning tree T, the 2-regular subgraph $C_{e'_3}$ and the matching M. Otherwise, $C_{e'_3}$ is a separating cycle of G. Then, there is at least one chord of H on the subpath $u'_5Hu'_6$ other than e'_3. So there must be a minimal chord of H on the subpath $u'_5Hu'_6$. Let e'_4 be a minimal chord of H on the subpath $u'_5Hu'_6$, and let $C_{e'_4}$ be the associated cycle of e'_4. According to the definitions of e'_1 and e'_2, we deduce that e'_4 is incident with the vertex v_3. Let M be the set of all chords of H none of whose two endpoints are on $C_{e'_4}$ except the chord v_4v_6, and let $T = G - E(C_{e'_4}) - E(M)$. So G has the decomposition with the spanning tree T, 2-regular subgraph $C_{e'_4}$ and the matching M.

Subcase 2.3. *There exists at least one chord of C_1^2 which links the subpaths v_1Hv_4 and v_3Hv_6.* From Subcase 2.1 and Subcase 2.2, we only need to consider that neither the subpath v_1Hv_4 nor the subpath v_3Hv_6 has any chord of C_1^2 in the subcase. Since there exists at least one chord of C_1^2 which links the subpaths v_1Hv_4 and v_3Hv_6, we can pick a chord $e_6 = u_7u_8$ whose left endpoint u_7 is the closest to the vertex v_1 among all chords of C_1^2 which link the subpaths v_1Hv_4 and v_3Hv_6. Since neither the subpath v_1Hv_4 nor the subpath v_3Hv_6 has any chord of C_1^2, so do the subpaths v_1Hu_7 and v_3Hu_8. Then, we can deduce the cycle $C_3^2 = v_1Hu_7u_8Hv_3v_1$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on C_3^2 except the chord v_4v_6. Let $T = G - E\left(C_3^2\right) - E(M)$. Then M and T are a matching and a spanning tree of G, respectively. So we get a desired decomposition of G with T, C_3^2, and M, see Figure 5.

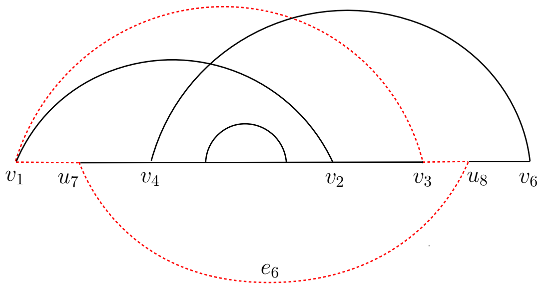


Figure 5. The cycle $C_3^2 = v_1 H u_7 u_8 H v_3 v_1$ is a non-separating cycle of G.

Case 3. *v_4 is on the subpath $v_2 H v_3$.* Suppose that there exists a minimal chord f of H on the subpath $v_1 H v_3$ such that its associated cycle C_f contains the vertex v_4. Let M be the set of all chords of H none of whose two endpoints are on C_f except the chord $v_1 v_3$, and let $T = G - E(C_f) - E(M)$. Then M and T are a matching and a spanning tree of G, respectively. Thus we have a desired decomposition of G with T, C_f and M. Otherwise it is sufficient to consider that

(3.0) *the associated cycle of any minimal chord of H on the subpath $v_1 H v_3$ does not contain v_4.*

Since there is a chord of H on the subpath $v_1 H v_2$ (for example, the chord $v_1 v_2$), we can pick a minimal chord $f_1 = t_1 t_2$ of H such that the right endpoint t_2 is the closest to the vertex v_2 among all minimal chords of H on the subpath $v_1 H v_2$. Note if f_1 is the chord $v_1 v_2$, then let $t_i = v_i$ $(i = 1, 2)$. Let C_{f_1} be the associated cycle of f_1. Similar to the subpath $v_5 H v_6$, we can pick a minimal chord $f_2 = t_3 t_4$ of H such that the left endpoint t_3 is the closest to the vertex v_5 among all minimal chords of H on the subpath $v_5 H v_6$. If f_2 is the chord $v_5 v_6$, then let $t_3 = v_5$ and $t_4 = v_6$. Let C_{f_2} be the associated cycle of f_2. Suppose that there is no chord of H which links the cycles C_{f_1} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_1} and on C_{f_2} except the chords $v_1 v_3$ and $v_4 v_6$. Then M is a matching of G. Let $T = G - E(C_{f_1} \cup C_{f_2}) - E(M)$. T is a spanning tree of G. So we can decompose G into the spanning tree T, the 2-regular subgraph $C_{f_1} \cup C_{f_2}$, and the matching M. Otherwise, there exists at least one chord of H which links C_{f_1} and C_{f_2}. We can assume that a chord $f_3 = t_5 t_6$ of H links C_{f_1} and C_{f_2} and t_5 is the left endpoint of f_3. Let $C_1^3 = v_1 v_2 H v_3 v_1$. If C_1^3 is a non-separating cycle of G, then let M be the set of all chords of H none of whose two endpoints are on C_1^3 except the chord f_3, and $T = G - E\left(C_1^3\right) - E(M)$. It is clear that M and T are a matching and a spanning tree of G, respectively. Thus we obtain a desired decomposition of G with T, C_1^3 and M. Otherwise, C_1^3 is a separating cycle of G. Then, there is at least one chord of H on the subpath $v_2 H v_3$. Let f_4 be any minimal chord of H on the subpath $v_2 H v_3$, and let C_{f_4} be the associated cycle of f_4. From (3.0), C_{f_4} does not contain the vertex v_4.

Suppose that there is not any chord of H which links the cycles C_{f_4} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_4} and on C_{f_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{f_4} \cup C_{f_2}) - E(M)$. Then G has the desired decomposition $\{T, C_{f_4} \cup C_{f_2}, M\}$. Otherwise, there is a chord of H which links C_{f_4} and C_{f_2}. Of course, there is at least one chord of H which links the subpath t_5Hv_3 and C_{f_2}. Let $f_5 = t_7t_8$ be a chord of H linking the subpath t_5Hv_3 and C_{f_2} such that the left endpoint t_7 is the closest to the vertex t_5 among all chords of H linking the subpath t_5Hv_3 and C_{f_2}. Let $C_2^3 = t_5Ht_7t_8Ht_6t_5$. If C_2^3 is a non-separating cycle of G, then let M be the set of all chords of H none of whose two endpoints are on C_2^3 except the chords v_1v_3 and v_5v_6. Let $T = G - E\left(C_2^3\right) - E(M)$. So we get a desired decomposition of G with T, C_2^3 and M. Otherwise, C_2^3 is a separating cycle of G. Then there must be at least one chord of H on the subpath t_5Ht_7. Let f_6 be a minimal chord of H on the subpath t_5Ht_7, and let C_{f_6} be the associated cycle of f_6. From (3.0), we have that C_{f_6} does not contain the vertex v_4. According to the choice of f_5, there is no chord of H which links C_{f_6} and C_{f_2}. Let M be the set of all chords of H none of whose two endpoints are on C_{f_6} and on C_{f_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{f_6} \cup C_{f_2}) - E(M)$. Thus we have a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{f_6} \cup C_{f_2}$, and the matching M, see Figure 6.

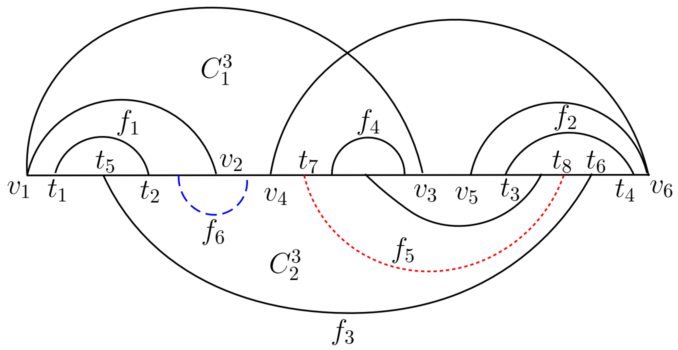


Figure 6. Case 3 is illustrated.

Case 4. *v_4 is on the subpath v_3Hv_5.* Since there are chords of H on the subpath v_1Hv_3 (for example, the chords v_1v_2 and v_1v_3), we can choose a minimal chord $g_1 = s_1s_2$ of H on the subpath v_1Hv_3. If g_1 is the chord v_1v_2, then $s_i = v_i$ $(i = 1, 2)$. Let C_{g_1} be the associated cycle of g_1. Similarly, let $g_2 = s_3s_4$ be a minimal chord of H on the subpath v_4Hv_6. If g_2 is the chord v_5v_6, then $s_3 = v_5$ and $s_4 = v_6$. Let C_{g_2} be the associated cycle of g_2. If there is no chord of H

which links the cycles C_{g_1} and C_{g_2}, then let M be the set of all chords of H none of whose two endpoints are on C_{g_1} and on C_{g_2} except the chords v_1v_3 and v_4v_6. Let $T = G - E(C_{g_1} \cup C_{g_2}) - E(M)$. Thus we have a desired decomposition of G with the spanning tree T, the 2-regular subgraph $C_{g_1} \cup C_{g_2}$ and the matching M. Otherwise, we suppose that

(4.0) *the associated cycle of any minimal chord of H on the subpath v_1Hv_3 is linked by a chord of H with the associated cycle of each minimal chord of H on the subpath v_4Hv_6.*

Since the subpath v_1Hv_2 has at least one chord of H, there is a minimal chord g_3 of H. If the subpath v_1Hv_2 only has the chord v_1v_2, then $g_3 = v_1v_2$. Let C_{g_3} be the associated cycle of g_3. Similarly, there exists a minimal chord g_4 of H on the subpath v_5Hv_6. If the subpath v_5Hv_6 only has the chord v_5v_6, then $g_4 = v_5v_6$. Let C_{g_4} be the associated cycle of g_4. From (4.0), there is at least one chord of H which links C_{g_3} and C_{g_4}. Let $g_5 = s_5s_6$ be such a chord of H. We discuss the following two subcases.

Subcase 4.1. *There are at least two chords of H which link the subpaths v_1Hv_3 and v_4Hv_6.* Let $g_6 = s_7s_8$ be a chord of H linking the subpaths v_1Hv_3 and v_4Hv_6 different from g_5 such that the left endpoint s_7 is the closest to the vertex s_5 among all chords of H linking such two subpaths. Suppose that the cycle $C_1^4 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of G. Let M be the set of all chords of H none of whose two endpoints are on C_1^4 except the chords v_1v_3 and v_4v_6, and let $T = G - E\left(C_1^4\right) - E(M)$. Thus G can be decomposed into the spanning tree T, the 2-regular subgraph C_1^4 and the matching M, see Figure 7.

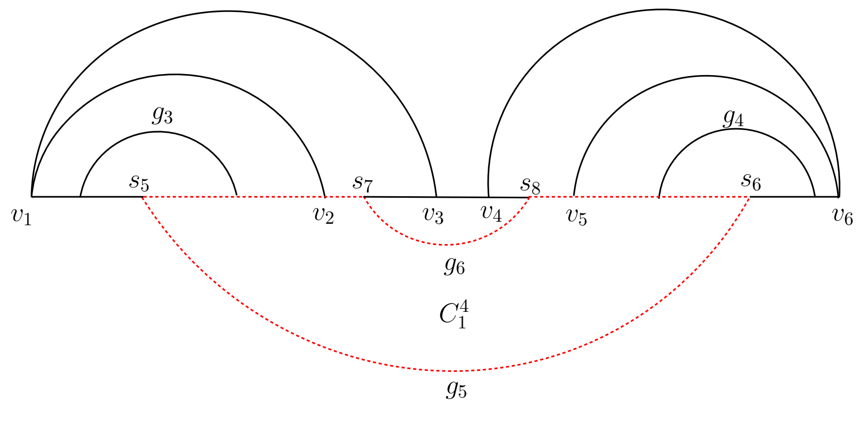


Figure 7. The cycle $C_1^4 = s_5Hs_7s_8Hs_6s_5$ is a non-separating cycle of G (dot line).

Otherwise, C_1^4 is a separating cycle of G. From (4.0) and the choice of g_6, we can deduce that there exists at least one chord of H on the subpath s_8Hs_6.

Then we pick a minimal chord g_7 of H on the subpath s_8Hs_6 such that the right endpoint of g_7 is the closest to the vertex s_6 among all chords of H on the subpath s_8Hs_6. Let C_{g_7} be the associated cycle of g_7. According to (4.0), there is at least one chord of H which links the cycles C_{g_3} and C_{g_7}. Let $g_8 = s_9s_{10}$ be a chord of H linking C_{g_3} and C_{g_7} such that the right endpoint s_{10} is the closest to the vertex s_6 among such all chords of H. Then, the cycle $C_2^4 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle. Since both s_5 and s_9 are on the associated cycle C_{g_3} of the minimal chord g_3, there is no chord of H on the subpath s_9Hs_5. Let M be the set of all chords of H none of whose two endpoints are on C_2^4 except the chords v_1v_3 and v_4v_6, and let $T = G - E\left(C_2^4\right) - E(M)$. So we have a desired decomposition of G with T, C_2^4 and M, see Figure 8.

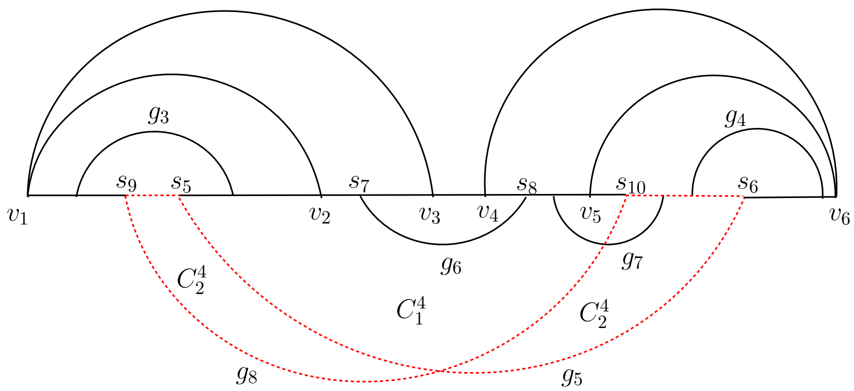


Figure 8. The cycle $C_2^4 = s_9Hs_5s_6Hs_{10}s_9$ is a non-separating cycle of G (dot line).

Subcase 4.2. *The chord g_5 is the only one chord of H which links the subpaths v_1Hv_3 and v_4Hv_6.* According to (4.0), it can be deduced that the vertex s_5 locates between two endpoints of any minimal chord of H on the subpath v_1Hv_3. If not, there is a minimal chord of H such that its associate cycle is not incident with s_5. Then there is a chord of H different from g_5 which links the associated cycle of this minimal chord and C_{g_4}, contradiction. Further we can obtain that s_5 locates between two endpoints of each chord of H on the subpath v_1Hv_3.

Only for convenience, we give a drawing of the graph G here. Except that the chord g_5 is arranged on one side of H, all chords of H are arranged on the other side of H. We discuss two cases as follows.

Subcase 4.2.1. *There exists a chord g of H such that g intersects at least two chords of H on the subpath v_1Hv_3.* Let $g_9 = s_{11}s_{12}$ and $g_{10} = s_{13}s_{14}$ be two chords of H intersecting g such that the left endpoint s_{11} of g_9 is the closest to the left endpoint s_{13} of g_{10} among such all chords of H on the subpath v_1Hv_3. Let the cycle $C_3^4 = s_{11}s_{12}Hs_{14}s_{13}Hs_{11}$. Then C_3^4 is a non-separating cycle of G. If none of g_9 and g_{10} is the chord v_1v_3, then let M be the set of all chords of H none of whose two endpoints are on C_3^4 and on C_{g_4} except the chords v_1v_3, v_4v_6, and g; otherwise, let M be the set of all chords of H none of whose two endpoints are on C_3^4 and on C_{g_4} except the chords v_4v_6 and g. Let $T = G - E\left(C_3^4 \cup C_{g_4}\right) - E(M)$. Thus the graph G can be decomposed into the spanning tree T, the 2-regular subgraph $C_3^4 \cup C_{g_4}$ and the matching M, see Figure 9.

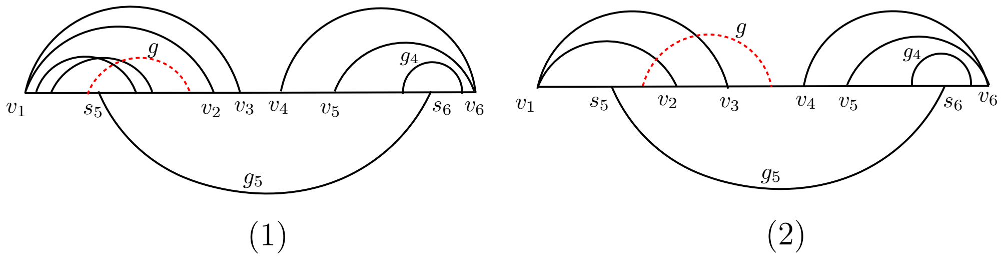


Figure 9. (1) g intersects the chords g_9 and g_{10} on the subpath v_1Hv_3, and none of g_9 and g_{10} is the chord v_1v_3;
(2) g intersects the chords g_9 and g_{10} on the subpath v_1Hv_3, and one of g_9 and g_{10} is the chord v_1v_3.

Subcase 4.2.2. *There is not any chord of H that intersects two chords on the subpath v_1Hv_3.* Suppose that there is a chord $g^1 = s^1s^2$ of H such that the endpoint s^1 is on the subpath v_1Hs_5 and the endpoint s^2 is on the subpath v_2Hv_3 (*the former case* for short). On the subpath v_1Hs^2, we start from the second edge and choose every other edge along the direction from v_1 to s^2. Otherwise, there is not any chord of H one of which endpoints is on the subpath v_1Hs_5 and the other on the subpath v_2Hv_3 (*the latter case* for short). On the subpath v_1Hv_2, we start from the second edge and choose every other edge along the direction from v_1 to v_2. Let M_0 be the set of the chosen edges in both cases. Then M_0 is a matching of G. Let V be the set of vertices on the subpath v_1Hs^2 for the former case or the set of vertices on the subpath v_1Hv_2 for the latter case. We first prove the following claim.

Claim. *Let M_0, V, the former case, and the latter case be defined as above. Then the subgraph $G[V]–E(M_0)$ is a path, where $G[V]$ is a subgraph of G induced by V.*

Proof. Let $G_1 = G[V] - E(M_0)$. Since the vertex s_5 locates between the two endpoints of each chord of H on the subpath v_1Hv_3, V insists of s_5 and the union of the two endpoints of each chord on the subpath v_1Hs^2 for the former case or on the subpath v_1Hv_2 for the latter case. $|V|$ is odd. Both the subpath v_1Hs^2 and the subpath v_1Hv_2 have an even number of edges. According to the choice of M_0, all vertices of G_1 are of degree 2 except two 1-degree vertices s_5 and s^2 for

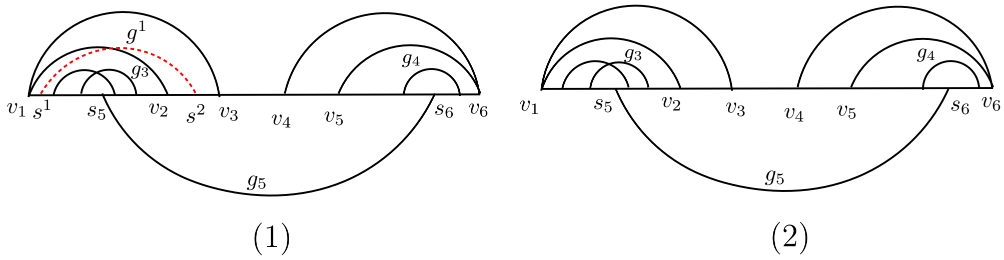


Figure 10. (1) the former case that there exists a chord $g^1 = s^1 s^2$ of H such that s^1 is on the subpath $v_1 H s_5$ and s^2 is on the subpath $v_2 H v_3$;
(2) the latter case that there is not any chord of H like g^1.

the former case or two 1-degree vertices s_5 and v_2 for the latter case. It suffices to prove that G_1 is connected.

Suppose that G_1 is disconnected. The components of G_1 consist of one path and some cycles according to the degree condition of G_1. Let C be a component of G_1 which is a cycle. In G_1, s_5 is not incident with C since s_5 is of degree 1. Let t_1 and t_2 be two vertices of C such that t_1 is the closest to s_5 among all vertices of C which locate on the left side of s_5 and t_2 is the closest to s_5 among all vertices of C which locate on the right side of s_5. Let the path $P = t_1 H s_5 H t_2$. Then the edges incident with t_1 and t_2 on P are edges of M_0. So P has an odd number of edges and an even number of vertices according to the choice of M_0. We can deduce that there is a chord g^* of H such that it only has one endpoint on P. The endpoint of g^* not on P can not be on C according to the choice of M_0 and P. Then g^* intersects at least two edges of C which are chords of H on the subpath $v_1 H v_3$, contraction with assumptions in Subcase 4.2.2. So G_1 is connected, and is a path. □

Let the subpath $P_1 = s^2 H v_6$ for the former case or $P_1 = v_2 H v_6$ for the latter case. Let M_1 be the set of all chords of H on P_1 none of whose two endpoints are on C_{g_4} except the chord $v_4 v_6$. Let $M = M_0 \cup M_1 \cup v_1 v_3$, and let $T = G - E(C_{g_4}) - E(M)$. Thus we get a desired decomposition of G with the spanning tree T, the 2-regular subgraph C_{g_4} and the matching M, see Figure 10. ■

From Theorem 2, we have the following corollary.

Corollary 4. *Let G be a connected cubic graph with n vertices and girth at least $(n-1)$. Then G can be decomposed into a spanning tree, a 2-regular graph and a matching.*

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