# DECOMPOSITIONS OF CUBIC TRACEABLE GRAPHS 

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#### Abstract

A traceable graph is a graph with a Hamilton path. The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching. We prove the conjecture for cubic traceable graphs.


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## 1. Introduction

In the paper all graphs are finite and simple. The reader can refer to [3, 18] for concepts not defined here. A graph $G$ is cubic if every vertex in $G$ is of degree 3. A spanning tree of $G$ is an acyclic connected subgraph containing all vertices of $G$. A graph that consists of pairwise disjoint edges is called a matching. A $k$ regular spanning subgraph of $G$ is called a $k$-factor. A 1-factor of $G$ is also called a perfect matching. An edge $e$ of $G$ is called a chord of a cycle $C$ in $G$ if the two endpoints of $e$ are on $C$ but $e$ is not itself an edge of $C$. A cycle $C$ is separating in a cubic graph $G$ if either $C$ has a chord, or $G-V(C)$ is disconnected; otherwise, non-separating. A Hamilton cycle is a cycle in $G$ containing all vertices of $G$. A graph with a Hamilton cycle is called a Hamiltonian graph. A Hamilton path
is a path in $G$ containing all vertices of $G$. A graph with a Hamilton path is called a traceable graph. Assume that $H$ is a Hamilton path in $G$. Each edge $e \in E(G) \backslash E(H)$ is called a chord of $H$. For every chord $e=v u$ of $H$, there exists a unique cycle $C_{e}$ consisting of $e$ and the subpath $v H u$. We call $C_{e}$ the associated cycle of $e$. A chord $e=$ st of $H$ is minimal if there is no other chord of $H$ whose two endpoints are on the subpath sHt.

A decomposition of a graph $G$ consists of pairwise edge-disjoint subgraphs whose union is $G$. It is a canonical problem in structural graph theory to decompose cubic graphs into subgraphs with certain properties. Such a problem can be traced back to the Petersen Theorem [16] that every bridgeless cubic graph has a 1 -factor, which implies that each bridgeless cubic graph can be decomposed into a 1 -factor and a 2 -factor. The Vizing Theorem [17] on proper edge-coloring shows that every cubic graph admits a decomposition consisting of four matchings.

Decompositions of cubic graphs into paths are related to the Fan-Raspaud conjecture [9] that every 2 -edge-connected cubic graph contains three perfect matchings with empty intersection. It is interesting to decompose a cubic graph into a spanning tree and other subgraphs. Malkevitch [14] asked which cubic graphs admit a decomposition into a spanning tree and a 2 -regular subgraph, that is, a decomposition with a HIST (a homeomorphically irreducible spanning tree is a spanning tree without a 2-degree vertex). Many researchers characterized graphs with a HIST (see [1, 2, 5, 6, 7]). Douglas [8] proved that it is NPcomplete to decide whether a graph with maximum degree 3 contains a HIST, which positively solves the problem presented by Albertson, Berman, Hutchinson and Thomassen [2]. It is clear that the complete graph $K_{4}$ can be decomposed into a HIST (a star) and a 2-regular subgraph (a triangle) while the cube $Q_{3}$ has no HIST. However, we can decompose $Q_{3}$ into a spanning tree (with two 2-degree vertices), a 2 -regular subgraph (a 4 -cycle) and a matching (an edge). See Figure 1. Relaxing the restriction that the spanning tree does not contain a vertex of degree 2, Hoffmann-Ostenhof presented the following conjecture.

(a)

(b)

Figure 1. A decomposition of $K_{4}$ with a star (thin line) and a triangle (dot line) in (a) while a decomposition of $Q_{3}$ with a spanning tree (thin line), a 4-cycle (dot line) and a matching (thick line) in (b).

Conjecture 1 (3-Decomposition Conjecture). Every connected cubic graph can be decomposed into a spanning tree, a 2 -regular graph and a matching.

Conjecture 1 was first posed in [10] (see also [4, Problem BCC 22.12] and [13]). Ozeki and Ye [15] showed that Conjecture 1 holds for 3-connected cubic graphs on the plane and the projective plane. Hoffmann-Ostenhof, Kaiser and Ozeki [12] proved that Conjecture 1 holds for all connected planar cubic graphs. In $[1,11]$ it was proved that a cubic Hamiltonian graph admits such a desired decomposition. It was informed that Ye [19] showed Conjecture 1 for 3-connected cubic graphs on the Klein bottle and the torus. In the paper, we prove Conjecture 1 for traceable cubic graphs.

Theorem 2. Every traceable cubic graph can be decomposed into a spanning tree, a 2 -regular graph and a matching.

The proof of Theorem 2 consists of four cases (see Section 2). The first case discusses cubic Hamiltonian graphs. The second and third cases are more extensive analyses than the first case. A new technique is used to deal with the fourth case.

## 2. Proof of Theorem 2

Assume that $G$ is a cubic graph with a Hamilton path $H$. Let the vertices $v_{1}$ and $v_{6}$ be the two endpoints of $H$. Then $v_{1}$ and $v_{6}$ are incident with two chords of $H$, every other vertex on $H$ is incident with only one chord. If $v_{1}$ is adjacent to $v_{6}$ by a chord of $H$, then let the vertex $v_{2}$ be a neighbor of $v_{1}$ and $v_{5}$ be a neighbor of $v_{6}$ such that the two pairs of vertices are jointed by chords of $H$, respectively. Otherwise, let the vertices $v_{2}, v_{3}$ be two neighbors of $v_{1}$ jointed by chords of $H$ such that these vertices are ordered as $v_{1}, v_{2}, v_{3}$ on $H$, and let the vertices $v_{4}, v_{5}$ be two neighbors of $v_{6}$ jointed by chords of $H$ with the order as $v_{4}, v_{5}, v_{6}$ on $H$.

Lemma 3. Assume that $C$ is a 2 -regular non-separating subgraph of $G$ that is the union of associated cycles of chords of $H$, and assume that each of $v_{1}$ and $v_{6}$ is joined by a chord of $H$ to at least one vertex of $V(C) \cup\left\{v_{1}, v_{6}\right\}$. Then there is a decomposition of $G-E(C)$ into a spanning tree of $G$ and a matching.

Proof. Since $C$ is a 2-regular non-separating subgraph of $G, G-E(C)$ is connected and has a spanning tree. Let $T$ be a spanning tree of $G-E(C)$ that contains the forest $H-E(C)$, and let $M$ be the subgraph of $G$ induced by $E(G-E(C \cup T))$. Then $M$ is a matching of $G-E(C)$. Thus $G-E(C)$ admits a decomposition consisting of the spanning tree $T$ and the matching $M$.

Proof of Theorem 2. Let $G$ and $H$ be defined as above. Considering the symmetry of the position of the vertex $v_{i}(i=1,2, \ldots, 6)$ on $H$, we have the following four cases.

Case 1. $v_{1}$ is adjacent to $v_{6}$ by a chord of $H$.
Case 2. $v_{4}$ is on the subpath $v_{1} H v_{2}$.
Case 3. $v_{4}$ is on the subpath $v_{2} H v_{3}$.
Case 4. $v_{4}$ is on the subpath $v_{3} H v_{5}$.


Figure 2. The four cases are illustrated.

It is sufficient to show that each case admits a desired decomposition of $G$. See Figure 2.

Case 1. $v_{1}$ is adjacent to $v_{6}$ by a chord of $H$. In this case, $G$ is a Hamiltonian cubic graph. For completeness we give a proof similar to $[1,11]$.

Since $G$ is a simple cubic graph, there are other chords of $H$ besides the chord $v_{1} v_{6}$. Then there exists a minimal chord $e$ of $H$. Let $C_{e}$ be the associated cycle of $e$. Then $C_{e}$ is a non-separating cycle. From Lemma $3, G-E\left(C_{e}\right)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. So there is a decomposition of $G$ with the 2-regular subgraph $C_{e}$, the spanning tree $T$ and the matching $M$.

Case 2. $v_{4}$ is on the subpath $v_{1} H v_{2}$. Let $C_{1}^{2}=v_{1} H v_{4} v_{6} H v_{3} v_{1}$ and $C_{2}^{2}=$ $v_{1} v_{2} H v_{3} v_{1}$ be the cycles (see (2) of Figure 2).

Suppose that $C_{1}^{2}$ is a non-separating cycle of $G$. From Lemma 3, $G-E\left(C_{1}^{2}\right)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. Thus we can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{1}^{2}$ and the matching $M$. Otherwise, $C_{1}^{2}$ is a separating cycle of $G$. Then there is at least one chord of $C_{1}^{2}$ (and of $H$ also) locating on the subpath $v_{1} H v_{4}$, locating on the subpath $v_{3} H v_{6}$, or linking the subpaths $v_{1} H v_{4}$ and $v_{3} H v_{6}$.

Further suppose that $C_{2}^{2}$ is a non-separating cycle of $G$. Let $M$ be a set of all chords of $H$ whose two endpoints are not both on $C_{2}^{2}$ except the chord $v_{4} v_{6}$, and let $T=G-E\left(C_{2}^{2}\right)-M$. Then $M$ and $T$ are a matching and a spanning tree of $G$ respectively. $T \cup M \cup C_{2}^{2}$ forms a desired decomposition of $G$. Otherwise, $C_{2}^{2}$ is a separating cycle of $G$. Then there is at least one chord of $C_{2}^{2}$ on the subpath $v_{2} H v_{3}$. Now, we discuss three subcases as follows.

Subcase 2.1. There is at least one chord of $C_{1}^{2}$ on the subpath $v_{1} H v_{4}$. Since there is at least one chord of $C_{1}^{2}$ on the subpath $v_{1} H v_{4}$, we can pick a minimal chord $e_{1}=u_{1} u_{2}$ of $H$ such that the right endpoint $u_{2}$ of $e_{1}$ is the closest to the vertex $v_{4}$ among all minimal chords of $H$ on the subpath $v_{1} H v_{4}$. Let $C_{e_{1}}$ be the associated cycle of $e_{1}$. Similarly, since there is at least one chord of $C_{2}^{2}$ on the subpath $v_{2} H v_{3}$, we have a minimal chord $e_{2}=u_{3} u_{4}$ of $H$ such that the left endpoint $u_{3}$ of $e_{2}$ is the closest to the vertex $v_{2}$ among all minimal chords of $H$ on the subpath $v_{2} \mathrm{Hv}_{3}$. Let $C_{e_{2}}$ be the associated cycle of $e_{2}$. Suppose that there is no chord of $H$ which links $C_{e_{1}}$ and $C_{e_{2}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{1}}$ and on $C_{e_{2}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Thus $M$ becomes a matching of $G$. Let $T=G-E\left(C_{e_{1}} \cup C_{e_{2}}\right)-E(M)$. Then $T$ is a spanning tree of $G$. We can give a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_{1}} \cup C_{e_{2}}$, and the matching $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{e_{1}}$ and $C_{e_{2}}$. Let $e_{3}=u_{5} u_{6}$ be such a chord of $H$, and let $C_{e_{3}}$ be the associated cycle of $e_{3}$. Suppose that $C_{e_{3}}$ is a non-separating cycle of $G$.From Lemma 3, $G-E\left(C_{e_{3}}\right)$ has a decomposition consisting of a spanning tree $T$ and a matching $M$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{e_{3}}$ and the matching $M$. Otherwise, $C_{e_{3}}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u_{5} H u_{6}$ other than $e_{3}$. So there must be a minimal chord of $H$ on the subpath $u_{5} H u_{6}$.

Let $e_{4}$ be a minimal chord of $H$ on the subpath $u_{5} H u_{6}$, and let $C_{e_{4}}$ be the associated cycle of $e_{4}$. If the vertex $v_{4}$ is on $C_{e_{4}}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{4}}$ except the chord $v_{1} v_{3}$. Let $T=G-E\left(C_{e_{4}}\right)-E(M)$. So we obtain a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_{4}}$, and the matching $M$. If the vertex $v_{2}$ is on $C_{e_{4}}$ and the vertex $v_{4}$ not on $C_{e_{4}}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{4}}$ except the chord $v_{4} v_{6}$. Let $T=G-E\left(C_{e_{4}}\right)-E(M)$. Thus we can decompose $G$ into the spanning tree $T$,
the 2 -regular subgraph $C_{e_{4}}$, and the matching $M$. See Figure 3.


Figure 3. $v_{2}$ is on $C_{e_{4}}$ and $v_{4}$ not on $C_{e_{4}}$.

So we suppose that neither $v_{2}$ nor $v_{4}$ is on $C_{e_{4}}$. According to the choices of $e_{1}$ and $e_{2}$, we deduce that $e_{4}$ must locate on the subpath $v_{4} H v_{2}$. Thus there is at least one minimal chord on the subpath $v_{4} H v_{2}$ (for example, the minimal chord $e_{4}$ ). We pick up a minimal chord, denoted by $e_{4}^{*}$, on the subpath $v_{4} H v_{2}$ such that the right endpoint $u^{*}$ of $e_{4}^{*}$ is the closed to the vertex $v_{2}$ among all minimal chords of $H$ on the subpath $v_{4} H v_{2}$. Let $C_{e_{4}^{*}}$ be the associated cycle of $e_{4}^{*}$. Further suppose that there is no chord of $H$ which links $C_{e_{4}^{*}}$ and $C_{e_{2}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{4}^{*}}$ and on $C_{e_{2}}$ except the chords $v_{1} v_{2}$ and $v_{4} v_{6}$. Let $T=G-E\left(C_{e_{4}^{*}} \cup C_{e_{2}}\right)-E(M)$. So we obtain a desired decomposition of $G$ with $T, C_{e_{4}^{*}} \cup C_{e_{2}}$, and $M$. Otherwise, there is at least one chord of $H$ which links $C_{e_{4}^{*}}$ and $C_{e_{2}}$. Since neither the subpath $u^{*} H v_{2}$ nor the subpath $v_{2} H u_{3}$ has any chord, there must exist a minimal chord $e_{5}$ of $H$ such that its associated cycle $C_{e_{5}}$ contains the vertex $v_{2}$. We can employ the same means to get a desired decomposition of $G$ as the case that $v_{2}$ is on $C_{e_{4}}$ and $v_{4}$ not on $C_{e_{4}}$. See Figure 4.

Subcase 2.2. There is at least one chord of $C_{1}^{2}$ on the subpath $v_{3} H v_{6}$. Since there is at least one chord of $C_{1}^{2}$ on the subpath $v_{3} H v_{6}$, we choose a minimal chord $e_{1}^{\prime}=u_{1}^{\prime} u_{2}^{\prime}$ of $H$ such that the left endpoint $u_{1}^{\prime}$ of $e_{1}^{\prime}$ is the closest to the vertex $v_{3}$ among all minimal chords of $H$ on the subpath $v_{3} H v_{6}$. Let $C_{e_{1}^{\prime}}$ be the associated cycle of $e_{1}^{\prime}$. Similarly, since there is at least one chord of $C_{2}^{2}$ on the subpath $v_{2} H v_{3}$, there exists a minimal chord $e_{2}^{\prime}=u_{3}^{\prime} u_{4}^{\prime}$ of $H$ such that the right endpoint $u_{4}^{\prime}$ of $e_{2}^{\prime}$ is the closest to the vertex $v_{3}$ among all minimal chords of $H$ on the subpath $v_{2} H v_{3}$. Let $C_{e_{2}^{\prime}}$ be the associated cycle of $e_{2}^{\prime}$. Suppose that there is no chord of $H$ which links ${ }_{e_{1}^{\prime}}$ and $C_{e_{2}^{\prime}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{1}^{\prime}}$ and on $C_{e_{2}^{\prime}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Thus $M$ becomes a matching of $G$. Let $T=G-E\left(C_{e_{1}^{\prime}} \cup C_{e_{2}^{\prime}}\right)-E(M)$. Then $T$ is a spanning tree of $G$. We obtain a desired decomposition of $G$ with


Figure 4. The minimal chord $e_{5}$ of $H$ links $C_{e_{4}^{\prime}}$ and $C_{e_{2}}$, and its associated cycle $C_{e_{5}}$ contains $v_{2}$.
$T, C_{e_{1}^{\prime}} \cup C_{e_{2}^{\prime}}$, and $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{e_{1}^{\prime}}$ and $C_{e_{2}^{\prime}}$. Let $e_{3}^{\prime}=u_{5}^{\prime} u_{6}^{\prime}$ be such a chord of $H$, and let $C_{e_{3}^{\prime}}$ be the associated cycle of $e_{3}^{\prime}$. Suppose that $C_{e_{3}^{\prime}}$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{3}^{\prime}}$ except the chord $v_{4} v_{6}$, and let $T=G-E\left(C_{e_{3}^{\prime}}\right)-E(M)$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{e_{3}^{\prime}}$ and the matching $M$. Otherwise, $C_{e_{3}^{\prime}}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u_{5}^{\prime} H u_{6}^{\prime}$ other than $e_{3}^{\prime}$. So there must be a minimal chord of $H$ on the subpath $u_{5}^{\prime} H u_{6}^{\prime}$. Let $e_{4}^{\prime}$ be a minimal chord of $H$ on the subpath $u_{5}^{\prime} H u_{6}^{\prime}$, and let $C_{e_{4}^{\prime}}$ be the associated cycle of $e_{4}^{\prime}$. According to the definitions of $e_{1}^{\prime}$ and $e_{2}^{\prime}$, we deduce that $e_{4}^{\prime}$ is incident with the vertex $v_{3}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_{4}^{\prime}}$ except the chord $v_{4} v_{6}$, and let $T=G-E\left(C_{e_{4}^{\prime}}\right)-E(M)$. So $G$ has the decomposition with the spanning tree $T$, 2-regular subgraph $C_{e_{4}^{\prime}}$ and the matching $M$.

Subcase 2.3. There exists at least one chord of $C_{1}^{2}$ which links the subpaths $v_{1} H v_{4}$ and $v_{3} H v_{6}$. From Subcase 2.1 and Subcase 2.2, we only need to consider that neither the subpath $v_{1} H v_{4}$ nor the subpath $v_{3} H v_{6}$ has any chord of $C_{1}^{2}$ in the subcase. Since there exists at least one chord of $C_{1}^{2}$ which links the subpaths $v_{1} H v_{4}$ and $v_{3} H v_{6}$, we can pick a chord $e_{6}=u_{7} u_{8}$ whose left endpoint $u_{7}$ is the closest to the vertex $v_{1}$ among all chords of $C_{1}^{2}$ which link the subpaths $v_{1} H v_{4}$ and $v_{3} H v_{6}$. Since neither the subpath $v_{1} H v_{4}$ nor the subpath $v_{3} H v_{6}$ has any chord of $C_{1}^{2}$, so do the subpaths $v_{1} H u_{7}$ and $v_{3} H u_{8}$. Then, we can deduce the cycle $C_{3}^{2}=v_{1} H u_{7} u_{8} H v_{3} v_{1}$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{3}^{2}$ except the chord $v_{4} v_{6}$. Let $T=G-E\left(C_{3}^{2}\right)-E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. So we get a desired decomposition of $G$ with $T, C_{3}^{2}$, and $M$, see Figure 5 .


Figure 5. The cycle $C_{3}^{2}=v_{1} H u_{7} u_{8} H v_{3} v_{1}$ is a non-separating cycle of $G$.

Case 3. $v_{4}$ is on the subpath $v_{2} \mathrm{Hv}_{3}$. Suppose that there exists a minimal chord $f$ of $H$ on the subpath $v_{1} H v_{3}$ such that its associated cycle $C_{f}$ contains the vertex $v_{4}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f}$ except the chord $v_{1} v_{3}$, and let $T=G-E\left(C_{f}\right)-E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we have a desired decomposition of $G$ with $T, C_{f}$ and $M$. Otherwise it is sufficient to consider that
(3.0) the associated cycle of any minimal chord of $H$ on the subpath $v_{1} H v_{3}$ does not contain $v_{4}$.

Since there is a chord of $H$ on the subpath $v_{1} H v_{2}$ (for example, the chord $v_{1} v_{2}$ ), we can pick a minimal chord $f_{1}=t_{1} t_{2}$ of $H$ such that the right endpoint $t_{2}$ is the closest to the vertex $v_{2}$ among all minimal chords of $H$ on the subpath $v_{1} H v_{2}$. Note if $f_{1}$ is the chord $v_{1} v_{2}$, then let $t_{i}=v_{i}(i=1,2)$. Let $C_{f_{1}}$ be the associated cycle of $f_{1}$. Similar to the subpath $v_{5} H v_{6}$, we can pick a minimal chord $f_{2}=t_{3} t_{4}$ of $H$ such that the left endpoint $t_{3}$ is the closest to the vertex $v_{5}$ among all minimal chords of $H$ on the subpath $v_{5} H v_{6}$. If $f_{2}$ is the chord $v_{5} v_{6}$, then let $t_{3}=v_{5}$ and $t_{4}=v_{6}$. Let $C_{f_{2}}$ be the associated cycle of $f_{2}$. Suppose that there is no chord of $H$ which links the cycles $C_{f_{1}}$ and $C_{f_{2}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_{1}}$ and on $C_{f_{2}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Then $M$ is a matching of $G$. Let $T=G-E\left(C_{f_{1}} \cup C_{f_{2}}\right)-E(M)$. $T$ is a spanning tree of $G$. So we can decompose $G$ into the spanning tree $T$, the 2 regular subgraph $C_{f_{1}} \cup C_{f_{2}}$, and the matching $M$. Otherwise, there exists at least one chord of $H$ which links $C_{f_{1}}$ and $C_{f_{2}}$. We can assume that a chord $f_{3}=t_{5} t_{6}$ of $H$ links $C_{f_{1}}$ and $C_{f_{2}}$ and $t_{5}$ is the left endpoint of $f_{3}$. Let $C_{1}^{3}=v_{1} v_{2} H v_{3} v_{1}$. If $C_{1}^{3}$ is a non-separating cycle of $G$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{1}^{3}$ except the chord $f_{3}$, and $T=G-E\left(C_{1}^{3}\right)-E(M)$. It is clear that $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we obtain a desired decomposition of $G$ with $T, C_{1}^{3}$ and $M$. Otherwise, $C_{1}^{3}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $v_{2} H v_{3}$. Let $f_{4}$ be any minimal chord of $H$ on the subpath $v_{2} H v_{3}$, and let $C_{f_{4}}$ be
the associated cycle of $f_{4}$. From (3.0), $C_{f_{4}}$ does not contain the vertex $v_{4}$.
Suppose that there is not any chord of $H$ which links the cycles $C_{f_{4}}$ and $C_{f_{2}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_{4}}$ and on $C_{f_{2}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Let $T=G-E\left(C_{f_{4}} \cup C_{f_{2}}\right)-E(M)$. Then $G$ has the desired decomposition $\left\{T, C_{f_{4}} \cup C_{f_{2}}, M\right\}$. Otherwise, there is a chord of $H$ which links $C_{f_{4}}$ and $C_{f_{2}}$. Of course, there is at least one chord of $H$ which links the subpath $t_{5} H v_{3}$ and $C_{f_{2}}$. Let $f_{5}=t_{7} t_{8}$ be a chord of $H$ linking the subpath $t_{5} H v_{3}$ and $C_{f_{2}}$ such that the left endpoint $t_{7}$ is the closest to the vertex $t_{5}$ among all chords of $H$ linking the subpath $t_{5} H v_{3}$ and $C_{f_{2}}$. Let $C_{2}^{3}=t_{5} \mathrm{Ht}_{7} t_{8} H t_{6} t_{5}$. If $C_{2}^{3}$ is a non-separating cycle of $G$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{2}^{3}$ except the chords $v_{1} v_{3}$ and $v_{5} v_{6}$. Let $T=G-E\left(C_{2}^{3}\right)-E(M)$. So we get a desired decomposition of $G$ with $T, C_{2}^{3}$ and $M$. Otherwise, $C_{2}^{3}$ is a separating cycle of $G$. Then there must be at least one chord of $H$ on the subpath $t_{5} H_{7}$. Let $f_{6}$ be a minimal chord of $H$ on the subpath $t_{5} H t_{7}$, and let $C_{f_{6}}$ be the associated cycle of $f_{6}$. From (3.0), we have that $C_{f_{6}}$ does not contain the vertex $v_{4}$. According to the choice of $f_{5}$, there is no chord of $H$ which links $C_{f_{6}}$ and $C_{f_{2}}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_{6}}$ and on $C_{f_{2}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Let $T=G-E\left(C_{f_{6}} \cup C_{f_{2}}\right)-E(M)$. Thus we have a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{f_{6}} \cup C_{f_{2}}$, and the matching $M$, see Figure 6.


Figure 6. Case 3 is illustrated.

Case 4. $v_{4}$ is on the subpath $v_{3} H v_{5}$. Since there are chords of $H$ on the subpath $v_{1} H v_{3}$ (for example, the chords $v_{1} v_{2}$ and $v_{1} v_{3}$ ), we can choose a minimal chord $g_{1}=s_{1} s_{2}$ of $H$ on the subpath $v_{1} H v_{3}$. If $g_{1}$ is the chord $v_{1} v_{2}$, then $s_{i}=v_{i}$ $(i=1,2)$. Let $C_{g_{1}}$ be the associated cycle of $g_{1}$. Similarly, let $g_{2}=s_{3} s_{4}$ be a minimal chord of $H$ on the subpath $v_{4} H v_{6}$. If $g_{2}$ is the chord $v_{5} v_{6}$, then $s_{3}=v_{5}$ and $s_{4}=v_{6}$. Let $C_{g_{2}}$ be the associated cycle of $g_{2}$. If there is no chord of $H$
which links the cycles $C_{g_{1}}$ and $C_{g_{2}}$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{g_{1}}$ and on $C_{g_{2}}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$. Let $T=G-E\left(C_{g_{1}} \cup C_{g_{2}}\right)-E(M)$. Thus we have a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_{1}} \cup C_{g_{2}}$ and the matching $M$. Otherwise, we suppose that
(4.0) the associated cycle of any minimal chord of $H$ on the subpath $v_{1} H v_{3}$ is linked by a chord of $H$ with the associated cycle of each minimal chord of $H$ on the subpath $v_{4} H v_{6}$.

Since the subpath $v_{1} H v_{2}$ has at least one chord of $H$, there is a minimal chord $g_{3}$ of $H$. If the subpath $v_{1} H v_{2}$ only has the chord $v_{1} v_{2}$, then $g_{3}=v_{1} v_{2}$. Let $C_{g_{3}}$ be the associated cycle of $g_{3}$. Similarly, there exists a minimal chord $g_{4}$ of $H$ on the subpath $v_{5} H v_{6}$. If the subpath $v_{5} H v_{6}$ only has the chord $v_{5} v_{6}$, then $g_{4}=v_{5} v_{6}$. Let $C_{g_{4}}$ be the associated cycle of $g_{4}$. From (4.0), there is at least one chord of $H$ which links $C_{g_{3}}$ and $C_{g_{4}}$. Let $g_{5}=s_{5} s_{6}$ be such a chord of $H$. We discuss the following two subcases.

Subcase 4.1. There are at least two chords of $H$ which link the subpaths $v_{1} H v_{3}$ and $v_{4} H v_{6}$. Let $g_{6}=s_{7} s_{8}$ be a chord of $H$ linking the subpaths $v_{1} H v_{3}$ and $v_{4} H v_{6}$ different from $g_{5}$ such that the left endpoint $s_{7}$ is the closest to the vertex $s_{5}$ among all chords of $H$ linking such two subpaths. Suppose that the cycle $C_{1}^{4}=s_{5} H s_{7} s_{8} H s_{6} s_{5}$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{1}^{4}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$, and let $T=G-E\left(C_{1}^{4}\right)-E(M)$. Thus $G$ can be decomposed into the spanning tree $T$, the 2 -regular subgraph $C_{1}^{4}$ and the matching $M$, see Figure 7 .


Figure 7. The cycle $C_{1}^{4}=s_{5} H s_{7} s_{8} H s_{6} s_{5}$ is a non-separating cycle of $G$ (dot line).
Otherwise, $C_{1}^{4}$ is a separating cycle of $G$. From (4.0) and the choice of $g_{6}$, we can deduce that there exists at least one chord of $H$ on the subpath $s_{8} H s_{6}$.

Then we pick a minimal chord $g_{7}$ of $H$ on the subpath $s_{8} H s_{6}$ such that the right endpoint of $g_{7}$ is the closest to the vertex $s_{6}$ among all chords of $H$ on the subpath $s_{8} H s_{6}$. Let $C_{g_{7}}$ be the associated cycle of $g_{7}$. According to (4.0), there is at least one chord of $H$ which links the cycles $C_{g_{3}}$ and $C_{g_{7}}$. Let $g_{8}=s_{9} s_{10}$ be a chord of $H$ linking $C_{g_{3}}$ and $C_{g_{7}}$ such that the right endpoint $s_{10}$ is the closest to the vertex $s_{6}$ among such all chords of $H$. Then, the cycle $C_{2}^{4}=s_{9} H s_{5} s_{6} H s_{10} s_{9}$ is a non-separating cycle. Since both $s_{5}$ and $s_{9}$ are on the associated cycle $C_{g_{3}}$ of the minimal chord $g_{3}$, there is no chord of $H$ on the subpath $s_{9} H s_{5}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{2}^{4}$ except the chords $v_{1} v_{3}$ and $v_{4} v_{6}$, and let $T=G-E\left(C_{2}^{4}\right)-E(M)$. So we have a desired decomposition of $G$ with $T, C_{2}^{4}$ and $M$, see Figure 8.


Figure 8. The cycle $C_{2}^{4}=s_{9} H s_{5} s_{6} H s_{10} s_{9}$ is a non-separating cycle of $G$ (dot line).

Subcase 4.2. The chord $g_{5}$ is the only one chord of $H$ which links the subpaths $v_{1} H v_{3}$ and $v_{4} H v_{6}$. According to (4.0), it can be deduced that the vertex $s_{5}$ locates between two endpoints of any minimal chord of $H$ on the subpath $v_{1} H v_{3}$. If not, there is a minimal chord of $H$ such that its associate cycle is not incident with $s_{5}$. Then there is a chord of $H$ different from $g_{5}$ which links the associated cycle of this minimal chord and $C_{g_{4}}$, contradiction. Further we can obtain that $s_{5}$ locates between two endpoints of each chord of $H$ on the subpath $v_{1} H v_{3}$.

Only for convenience, we give a drawing of the graph $G$ here. Except that the chord $g_{5}$ is arranged on one side of $H$, all chords of $H$ are arranged on the other side of $H$. We discuss two cases as follows.

Subcase 4.2.1. There exists a chord $g$ of $H$ such that $g$ intersects at least two chords of $H$ on the subpath $v_{1} H v_{3}$. Let $g_{9}=s_{11} s_{12}$ and $g_{10}=s_{13} s_{14}$ be two chords of $H$ intersecting $g$ such that the left endpoint $s_{11}$ of $g_{9}$ is the closest to the left endpoint $s_{13}$ of $g_{10}$ among such all chords of $H$ on the subpath $v_{1} H v_{3}$. Let the cycle $C_{3}^{4}=s_{11} s_{12} H s_{14} s_{13} H s_{11}$. Then $C_{3}^{4}$ is a non-separating cycle of $G$. If
none of $g_{9}$ and $g_{10}$ is the chord $v_{1} v_{3}$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{3}^{4}$ and on $C_{g_{4}}$ except the chords $v_{1} v_{3}, v_{4} v_{6}$, and $g$; otherwise, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{3}^{4}$ and on $C_{g_{4}}$ except the chords $v_{4} v_{6}$ and $g$. Let $T=G-E\left(C_{3}^{4} \cup C_{g_{4}}\right)-E(M)$. Thus the graph $G$ can be decomposed into the spanning tree $T$, the 2-regular subgraph $C_{3}^{4} \cup C_{g_{4}}$ and the matching $M$, see Figure 9 .


Figure 9. (1) $g$ intersects the chords $g_{9}$ and $g_{10}$ on the subpath $v_{1} H v_{3}$, and none of $g_{9}$ and $g_{10}$ is the chord $v_{1} v_{3}$;
(2) $g$ intersects the chords $g_{9}$ and $g_{10}$ on the subpath $v_{1} H v_{3}$, and one of $g_{9}$ and $g_{10}$ is the chord $v_{1} v_{3}$.

Subcase 4.2.2. There is not any chord of $H$ that intersects two chords on the subpath $v_{1} H v_{3}$. Suppose that there is a chord $g^{1}=s^{1} s^{2}$ of $H$ such that the endpoint $s^{1}$ is on the subpath $v_{1} H s_{5}$ and the endpoint $s^{2}$ is on the subpath $v_{2} H v_{3}$ (the former case for short). On the subpath $v_{1} H s^{2}$, we start from the second edge and choose every other edge along the direction from $v_{1}$ to $s^{2}$. Otherwise, there is not any chord of $H$ one of which endpoints is on the subpath $v_{1} H s_{5}$ and the other on the subpath $v_{2} H v_{3}$ (the latter case for short). On the subpath $v_{1} H v_{2}$, we start from the second edge and choose every other edge along the direction from $v_{1}$ to $v_{2}$. Let $M_{0}$ be the set of the chosen edges in both cases. Then $M_{0}$ is a matching of $G$. Let $V$ be the set of vertices on the subpath $v_{1} H s^{2}$ for the former case or the set of vertices on the subpath $v_{1} H v_{2}$ for the latter case. We first prove the following claim.
Claim. Let $M_{0}, V$, the former case, and the latter case be defined as above. Then the subgraph $G[V]-E\left(M_{0}\right)$ is a path, where $G[V]$ is a subgraph of $G$ induced by $V$.

Proof. Let $G_{1}=G[V]-E\left(M_{0}\right)$. Since the vertex $s_{5}$ locates between the two endpoints of each chord of $H$ on the subpath $v_{1} H v_{3}, V$ insists of $s_{5}$ and the union of the two endpoints of each chord on the subpath $v_{1} H s^{2}$ for the former case or on the subpath $v_{1} H v_{2}$ for the latter case. $|V|$ is odd. Both the subpath $v_{1} H s^{2}$ and the subpath $v_{1} H v_{2}$ have an even number of edges. According to the choice of $M_{0}$, all vertices of $G_{1}$ are of degree 2 except two 1-degree vertices $s_{5}$ and $s^{2}$ for


Figure 10. (1) the former case that there exists a chord $g^{1}=s^{1} s^{2}$ of $H$ such that $s^{1}$ is on the subpath $v_{1} H s_{5}$ and $s^{2}$ is on the subpath $v_{2} H v_{3}$;
(2) the latter case that there is not any chord of $H$ like $g^{1}$.
the former case or two 1-degree vertices $s_{5}$ and $v_{2}$ for the latter case. It suffices to prove that $G_{1}$ is connected.

Suppose that $G_{1}$ is disconnected. The components of $G_{1}$ consist of one path and some cycles according to the degree condition of $G_{1}$. Let $C$ be a component of $G_{1}$ which is a cycle. In $G_{1}, s_{5}$ is not incident with $C$ since $s_{5}$ is of degree 1. Let $t_{1}$ and $t_{2}$ be two vertices of $C$ such that $t_{1}$ is the closest to $s_{5}$ among all vertices of $C$ which locate on the left side of $s_{5}$ and $t_{2}$ is the closest to $s_{5}$ among all vertices of $C$ which locate on the right side of $s_{5}$. Let the path $P=t_{1} H s_{5} H t_{2}$. Then the edges incident with $t_{1}$ and $t_{2}$ on $P$ are edges of $M_{0}$. So $P$ has an odd number of edges and an even number of vertices according to the choice of $M_{0}$. We can deduce that there is a chord $g^{*}$ of $H$ such that it only has one endpoint on $P$. The endpoint of $g^{*}$ not on $P$ can not be on $C$ according to the choice of $M_{0}$ and $P$. Then $g^{*}$ intersects at least two edges of $C$ which are chords of $H$ on the subpath $v_{1} H v_{3}$, contraction with assumptions in Subcase 4.2.2. So $G_{1}$ is connected, and is a path.

Let the subpath $P_{1}=s^{2} H v_{6}$ for the former case or $P_{1}=v_{2} H v_{6}$ for the latter case. Let $M_{1}$ be the set of all chords of $H$ on $P_{1}$ none of whose two endpoints are on $C_{g_{4}}$ except the chord $v_{4} v_{6}$. Let $M=M_{0} \cup M_{1} \cup v_{1} v_{3}$, and let $T=G-E\left(C_{g_{4}}\right)-E(M)$. Thus we get a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_{4}}$ and the matching $M$, see Figure 10.

From Theorem 2, we have the following corollary.
Corollary 4. Let $G$ be a connected cubic graph with $n$ vertices and girth at least $(n-1)$. Then $G$ can be decomposed into a spanning tree, a 2 -regular graph and a matching.

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