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MORE ON THE MINIMUM SIZE OF GRAPHS WITH GIVEN RAINBOW INDEX

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Abstract

The concept of k-rainbow index $rx_k(G)$ of a connected graph G, introduced by Chartrand *et al.*, is a natural generalization of the rainbow connection number of a graph. Liu introduced a parameter $t(n, k, \ell)$ to investigate the problems of the minimum size of a connected graph with given order and k-rainbow index at most ℓ and obtained some exact values and upper bounds for $t(n, k, \ell)$. In this paper, we obtain some exact values of $t(n, k, \ell)$ for large ℓ and better upper bounds of $t(n, k, \ell)$ for small ℓ and $k = 3$.

Keywords: Steiner distance, rainbow S-tree, k-rainbow index.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. For a graph, by *size* of it we mean number of its edges. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if every two edges of the path have distinct colors. The graph G is *rainbow connected* if for every two vertices u and v of G, there is a rainbow path connecting u and v. The minimum number of colors for which there is an edge coloring of G such that G is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. Results on the rainbow connectivity can be found in [2, 4, 5, 8].

These concepts were introduced by Chartrand *et al.* in [2]. In [3], they generalized the concept of rainbow path to rainbow tree. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow* S-*tree* is a rainbow tree connecting the vertices of S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* if for every set S of k vertices of G, there exists a rainbow S-tree. In this case, we call G *k-rainbow connected.* The minimum number of colors that are needed in a k-rainbow coloring of G is called the *k-rainbow index*, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is nothing new but the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d_G(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. The *k-Steiner diameter* $\mathrm{sdiam}_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. Then there is a simple upper bound and lower bound for $rx_k(G)$.

Observation 1 [3]**.** *For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k - 1 \leq \mathrm{sdiam}_k(G) \leq rx_k(G) \leq n - 1$.*

It is [3] shown that the tree is a class of graphs whose k-rainbow index attains the upper bound.

Proposition 2 [3]**.** *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n - 1$.*

Chartrand *et al.* also showed that the k-rainbow index of the unicyclic graph is $n - 1$ or $n - 2$.

Theorem 3 [3]**.** *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & \text{if } k = 3 \text{ and } g \geq 4, \\ n-1, & \text{if } g = 3 \text{ or } 4 \leq k \leq n. \end{cases} \tag{1}$$

Schiermeyer [11] introduced a parameter $t(n, d)$ to investigate the rainbow connection. For integers n and d let $t(n, d)$ denote the minimum size (number of edges) in d-rainbow connected graphs of order n. Since a network which satisfies our certain requirements and has as few links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is significant. Later, this parameter was investigated [7, 10] and was solved completely by Lo [10]. Motivated by the parameter $t(n, d)$, Liu [9] introduced a new parameter to study the minimum size of a graph G such that G has a k-rainbow coloring using a fixed number of colors. Let $t(n, k, \ell)$ be the minimum size of a connected graph G of order n with $rx_k(G) \leq \ell$, where $2 \leq \ell \leq n - 1$ and $2 \leq k \leq n$. Clearly, $t(n, k, 1) \geq t(n, k, 2) \geq \cdots \geq t(n, k, n - 1)$. Liu [9] got some exact values and some upper bounds for $t(n, k, \ell)$ when k and ℓ take specific values. In this paper, we obtain some exact values of $t(n, k, \ell)$ for large ℓ and better upper bounds of $t(n, k, \ell)$ for small ℓ and $k = 3$.

2. Preliminaries

Definition. A *rose graph* R_p with p petals (or p-rose graph) is a graph obtained by taking p cycles with just a vertex in common. The common vertex is called the center of R_p. The rose graph with p petals is denoted by $R_p(\ell_1, \ell_2, \ldots, \ell_p)$ if the length of each cycle needs to be specified.

Definition. An edge-colored graph is *rainbow* if no two edges in the graph share the same color.

Definition. To *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. Any graph derived from a graph G by a sequence of edge subdivisions is called a *subdivision* of G.

Lemma 4 [6]**.** *Let H be a connected subgraph of a connected graph G. Then $rx_k(G) \leq rx_k(H)$ for $2 \leq k \leq n-1$.*

Lemma 5 [6]**.** *Let G be a connected graph, and H be a subdivision of G. Then $rx_k(H) \leq rx_k(G) + |H| - |G|$.*

Theorem 6 [6]**.** *For each integer k with $k \geq 3$, $rx_3(K_{k,k}) = 3$.*

Lemma 7. *For each integer k with $k \geq 3$, $rx_k(K_{2,k-1}) = k-1$.*

Proof. Let $G = K_{2,k-1} = G[X, Y]$, where $X = \{u, w\}$ and $Y = \{v_1, v_2, \ldots, v_{k-1}\}$. Define an edge-coloring c as follows: $c(uv_i) = i$ for $1 \leq i \leq k-1$, $c(wv_i) = i+1$ for $1 \leq i \leq k-1$. It is easily checked that there exists a rainbow S-tree for any $S \subseteq V(G)$ and $|S| = k$. Thus $rx_k(K_{2,k-1}) \leq k-1$. Conversely, by Observation 1, we have $rx_k(K_{2,k-1}) \geq k-1$. Therefore, $rx_k(K_{2,k-1}) = k-1$. ∎

3. Some Results for $t(n, k, n-1)$ and $t(n, k, n-2)$

In this section, we first consider the case when ℓ is large. By Proposition 2, we get the exact value of $t(n, k, \ell)$ for $\ell = n-1$ and every k with $3 \leq k \leq n$.

Theorem 8. *Let $n \geq 3$ be an integer. For each integer k with $3 \leq k \leq n$, $t(n, k, n-1) = n-1$.*

For $\ell = n-2$, Liu [9] got $t(n, k, n-2) = n$ when $k = 3$, we get results for $k = 4$ and $k = n-1$.

Theorem 9. *Let $n \geq 4$ be an integer. Then $t(n, 4, n-2) = n+1$.*

Proof. Let H be a graph obtained from $K_{2,3}$ by subdividing $n-5$ edges. Then H has n vertices and $n+1$ edges. Since $rx_4(K_{2,3}) = 3$, it follows that, by Lemma 5, $rx_4(H) \leq n-2$. Thus $t(n, 4, n-2) \leq n+1$. Conversely, if G is a tree or unicyclic, then by Proposition 2 and Theorem 3, $rx_4(G) = n-1$. Thus $t(n, 4, n-2) \geq n+1$. Therefore, $t(n, 4, n-2) = n+1$. ∎

Theorem 10. *Let $n \geq 4$ be an integer. Then $t(n, n-1, n-2) = 2n-4$.*

Proof. since $t(n, n-1, n-2) \leq 2n-4$ has been proved in [9], we need to prove that $t(n, n-1, n-2) \geq 2n-4$. To the contrary, suppose $t(n, n-1, n-2) \leq 2n-5$. Assume that G is a connected graph with $2n-5$ edges and $n-2$ colors. By the drawer principle, at least a color appears exactly once in G. Suppose a c_1-edge is incident to the vertex x. Delete the vertex x from G, and the remaining graph $G-x$ has $n-1$ vertices but at most $n-3$ colors, it follows that $G-x$ has no rainbow tree, a contradiction. ∎

From $t(n, 3, n-2) = n$, $t(n, 4, n-2) = n+1$ and $t(n, n-1, n-2) = 2n-4$, we believe that $t(n, k, n-2) = n+k-3$ for general k. In fact, this is true for general k.

Theorem 11. *Let $n \geq 4$ be an integer. For each integer k with $3 \leq k \leq n-1$, $t(n, k, n-2) = n+k-3$.*

Proof. Let H be a graph obtained from $K_{2,k-1}$ by subdividing $n-k-1$ edges. Since $rx_k(K_{2,k-1}) = k-1$, it follows that, by Lemma 5, $rx_k(H) \leq n-2$. As H has n vertices and $n+k-3$ edges, we have $t(n, k, n-2) \leq n+k-3$. Conversely, we need to prove that $t(n, k, n-2) \geq n+k-3$. To the contrary, suppose $t(n, k, n-2) \leq n+k-4$. Let G be a connected graph with $n+k-4$ edges and $n-2$ colors. Then at least $n-k$ colors appears exactly once in G; otherwise, at most $n-k-1$ colors appear exactly once and at least $k-1$ colors appear at least twice in G, it follows that $e(G) \geq n-k-1+2(k-1) = n+k-3$, a contradiction. Delete $n-k$ vertices incident to the edges colored with the $n-k$ colors which appear exactly once, then the remaining graph has k vertices but at most $n-2-(n-k) = k-2$ colors. Thus the remaining graph has no rainbow spanning tree, a contradiction. ∎

4. Some Results for $t(n, 3, \ell)$

In this section, we first focus on the case when ℓ is large.

Theorem 12. *Let $n \geq 8$ be an integer. Then $t(n, 3, n-4) = n+1$.*

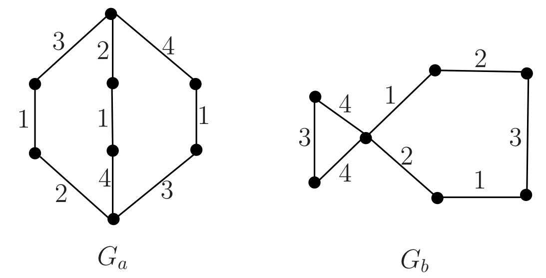

Figure 1. Graphs for Theorem 12 and Corollary 13.

Proof. Consider the graph G_a in Figure 1. Clearly, G_a has 8 vertices, 9 edges and 4 colors. It is easily checked that G_a is 3-rainbow connection, thus $rx_3(G_a) \leq 4$. Let H be a graph obtained from G_a by subdividing $n-8$ edges. Then by Lemma 5, $rx_3(H) \leq 4+(n-8) = n-4$. Since H has n vertices and $n+1$ edges, it follows that $t(n, 3, n-4) \leq n+1$. Conversely, if G is a tree or a unicyclic graph, then by Proposition 2 and Theorem 3, $rx_3(G)$ is $n-1$ or $n-2$. So if G is a graph with $rx_3(G) = n-4$, then G has at least $n+1$ edges and then $t(n, 3, n-4) \geq n+1$. Therefore, $t(n, 3, n-4) = n+1$. ■

Corollary 13. *Let $n \geq 7$ be an integer. Then $t(n, 3, n-3) = n+1$.*

Proof. Consider the graph G_b in Figure 1. Clearly, G_b has 7 vertices, 8 edges and 4 colors. It is easily checked that G_b is 3-rainbow connection, thus $rx_3(G_b) \leq 4$ and $t(7, 3, 4) \leq 8$. Let $n \geq 8$. Since $t(n, 3, n-4) \geq (n, 3, n-3)$, it follows that $t(n, 3, n-3) \leq n+1$ by Theorem 12. Conversely, if G is a tree or a unicyclic graph, then by Proposition 2 and Theorem 3, $rx_3(G)$ is $n-1$ or $n-2$. So if G is a graph with $rx_3(G) = n-3$, then G has at least $n+1$ edges. Thus $t(n, 3, n-3) \geq n+1$. Therefore, $t(n, 3, n-3) = n+1$. ■

Remark 14. For $\ell = n-1$ and $\ell = n-2$, Liu [9] got $t(n, 3, n-1) = n-1$ and $t(n, 3, n-2) = n$. For $\frac{n}{2} \leq \ell \leq n-3$, Liu [9] (see Theorem 2.11) got $t(n, 3, \ell) \leq 2n-\ell-1$, which implies $t(n, 3, n-3) \leq n+2$, $t(n, 3, n-4) \leq n+3$ and $t(n, 3, n-5) \leq n+4$. In fact, we get the exact values of $t(n, 3, n-3)$ and $t(n, 3, n-4)$ in Theorem 12 and Corollary 13, respectively.

Theorem 15. *Let $n \geq 11$ be an integer. For each integer ℓ with $\left\lceil \frac{n+1}{2} \right\rceil \leq \ell \leq n-5$, $t(n, 3, \ell) \leq \left\lfloor \frac{3n-\ell-1}{2} \right\rfloor$.*

Proof. We consider two cases according to the parity of n and ℓ.

Case 1. n and ℓ have the same parity. Let w_0 be the center of $R_{\frac{n-\ell}{2}}(5, 5, \ldots, 5)$ and let $C_i = w_0 u_i x_i y_i v_i w_0$ for $1 \leq i \leq \frac{n-\ell}{2}$. To show that $rx_3\big(R_{\frac{n-\ell}{2}}(5, 5, \ldots, 5)\big) \leq$

$n-\ell+1$, we provide an edge-coloring $c_1 : E\big(R_{\frac{n-\ell}{2}}(5,5,\dots,5)\big) \to \{1,2,\dots,n-\ell+1\}$ defined by

$$c_1(e) = \begin{cases} 2i-1, & e = w_0u_i \text{ or } y_iv_i \ \left(1 \le i \le \frac{n-\ell}{2}\right), \\ 2i, & e = w_0v_i \text{ or } u_ix_i \ \left(1 \le i \le \frac{n-\ell}{2}\right), \\ n-\ell+1, & e = x_iy_i \ \left(1 \le i \le \frac{n-\ell}{2}\right). \end{cases}$$

For any $S \subseteq V\big(R_{\frac{n-\ell}{2}}(5,5,\dots,5)\big)$ and $|S| = 3$, it is easily checked that there exists a rainbow S-tree. Thus, $rx_3\big(R_{\frac{n-\ell}{2}}(5,5,\dots,5)\big) \le n-\ell+1$. Let G be a graph obtained from $R_{\frac{n-\ell}{2}}(5,5,\dots,5)$ by subdividing $2\ell-n-1$ edges arbitrarily. Then $\big|V(G)\big| = 4\cdot\frac{n-\ell}{2}+1+(2\ell-n-1) = n$ and $\big|E(G)\big| = 5\cdot\frac{n-\ell}{2}+(2\ell-n-1) = \frac{3n-\ell-2}{2}$. By Lemma 5, $rx_3(G) \le rx_3\big(R_{\frac{n-\ell}{2}}(5,5,\dots,5)\big) + (2\ell-n-1) \le \ell$. Since G has n vertices and $\frac{3n-\ell-2}{2}$ edges, it follows that $t(n,3,\ell) \le \frac{3n-\ell-2}{2}$. See G_c in Figure 2 for an example with $n = 13, \ell = 7$.

Case 2. n and ℓ have different parities. Let w_0 be the center of $R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)$, $C_1 = w_0x_1y_1w_0$ and let $C_i = w_0u_ix_iy_iv_iw_0$, where $2 \le i \le \frac{n-\ell+1}{2}$. To show that $rx_3\big(R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)\big) \le n-\ell+1$, we provide an edge-coloring $c_2 : E\big(R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)\big) \to \{1,2,\dots,n-\ell+1\}$ defined by

$$c_2(e) = \begin{cases} 1, & e = w_0x_1 \text{ or } w_0y_1, \\ 2i-2, & e = w_0u_i \text{ or } y_iv_i \ \left(2 \le i \le \frac{n-\ell+1}{2}\right), \\ 2i-1, & e = w_0v_i \text{ or } u_ix_i \ \left(2 \le i \le \frac{n-\ell+1}{2}\right), \\ n-\ell+1, & e = x_iy_i \ \left(1 \le i \le \frac{n-\ell+1}{2}\right). \end{cases}$$

For any $S \subseteq V\big(R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)\big)$ and $|S| = 3$, it is easily checked that there exists a rainbow S-tree. Thus, $rx_3\big(R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)\big) \le n-\ell+1$. Let G be a graph obtained from $R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)$ by subdividing $2\ell-n-1$ edges. Then $\big|V(G)\big| = 4\cdot\big(\frac{n-\ell+1}{2}-1\big)+3+(2\ell-n-1) = n$ and $\big|E(G)\big| = 5\cdot\big(\frac{n-\ell+1}{2}-1\big)+3+(2\ell-n-1) = \frac{3n-\ell-1}{2}$. By Lemma 5, $rx_3(G) \le rx_3\big(R_{\frac{n-\ell+1}{2}}(3,5,\dots,5)\big) + (2\ell-n-1) \le \ell$. Since G has n vertices and $\frac{3n-\ell-1}{2}$ edges, it follows that $t(n,3,\ell) \le \frac{3n-\ell-1}{2}$. See G_d in Figure 2 for an example with $n = 11, \ell = 6$.

Combining the above two cases, we have that $t(n,3,\ell) \le \left\lfloor\frac{3n-\ell-1}{2}\right\rfloor$. ■

Remark 16. For $\frac{n}{2} \le \ell \le n-3$, Liu [9] got $t(n,3,\ell) \le 2n-\ell-1$ (see Theorem 2.11). Since $\ell \le n-5$ in Theorem 15, it follows that $2n-\ell-1-\left\lfloor\frac{3n-\ell-1}{2}\right\rfloor \ge 2n-\ell-1-\frac{3n-\ell-1}{2} = \frac{n-\ell-1}{2} \ge 2$. Thus the upper bound in Theorem 15 is better than the one in [9].

Theorem 17. *Let $n \ge 17$ and ℓ be integers with $9 \le \ell \le \left\lfloor\frac{n+1}{2}\right\rfloor$. Then $t(n,3,\ell) \le \ell\left\lceil\frac{3n-3}{2\ell-3}\right\rceil$.*

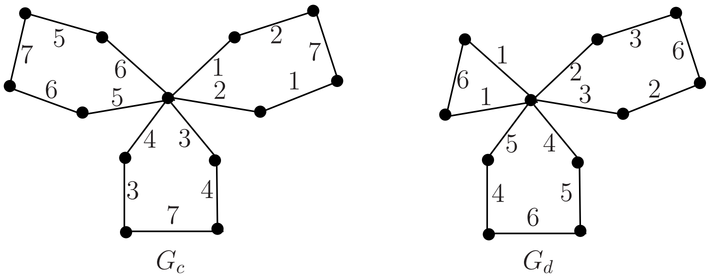

Figure 2. Graphs for Theorem 15.

Proof. We consider three cases according to $\ell \equiv \ell'\ (\mathrm{mod}\ 3)$.

Case 1. $\ell' = 0$. Set $\ell = 3t$. Let H^* be a connected rainbow graph with $2t$ vertices and $3t$ edges, where $V(H^*) = \{v_1, v_2, \ldots, v_{2t}\}$ and $E(H^*) = \left\{v_i v_{i+1}, v_1 v_{2t}, v_j v_{2t+2-j}, v_1 v_{t+1}\right\}$ for $1 \le i \le 2t-1$ and $2 \le j \le t$. Note that, v_1 and v_i $(2 \le i \le 2t, i \ne t+1)$ have three internally-disjoint rainbow paths $P_1 = v_1 v_2 \cdots v_{i-1} v_i$, $P_2 = v_1 v_{t+1} v_t \cdots v_{i+1} v_i$, $P_3 = v_1 v_{2t} v_{2t-1} \cdots v_{2t+3-i} v_{2t+2-i} v_i$; v_1 and v_{t+1} have three internally-disjoint rainbow paths $P_1 = v_1 v_2 \cdots v_t v_{t+1}$, $P_2 = v_1 v_{t+1}$, $P_3 = v_1 v_{2t} v_{2t-1} \cdots v_{t+2} v_{t+1}$ in H^*. Also note that vertices v_1 and v_{t+1} divide the cycle $C_{2t} := v_1 v_2 \cdots v_{2t-1} v_{2t} v_1$ into two segments $C_* = v_1 v_2 \cdots v_t v_{t+1}$ and $C_{**} = v_{t+1} v_{t+2} \cdots v_{2t} v_1$ in H^*.

Take $\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ copies of H^* and denote them by $H^1, H^2, \ldots, H^{\left\lfloor \frac{n-1}{2t-1} \right\rfloor}$ with $V(H^p) = \{v_1^p, v_2^p, \ldots, v_{2t}^p\}$ and $E(H^p) = \{v_i^p v_{i+1}^p, v_1^p v_{2t}^p, v_j^p v_{2t+2-j}^p, v_1^p v_{t+1}^p\}$ for $1 \le i \le 2t-1$ and $2 \le j \le t$, $1 \le p \le \left\lfloor \frac{n-1}{2t-1} \right\rfloor$, and take a subgraph graph of H^*, denoted by $H^{\left\lceil \frac{n-1}{2t-1} \right\rceil}$, with $n-(2t-1)\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ vertices and corresponding edges of H^*. Let G be a graph with n vertices by identifying the vertices v_1^p $\left(1 \le p \le \left\lfloor \frac{n-1}{2t-1} \right\rfloor\right)$ and $v_1^{\left\lceil \frac{n-1}{2t-1} \right\rceil}$ if $H^{\left\lceil \frac{n-1}{2t-1} \right\rceil}$ exists. Clearly, $e(G) \le \ell \left\lceil \frac{n-1}{2t-1} \right\rceil = \ell \left\lceil \frac{3n-3}{2\ell-3} \right\rceil$. See G_e in Figure 3 for an example with $n = 15, \ell = 9$.

Let v_i^x, v_j^y and v_k^z be three vertices in G. Denote the corresponding vertex of v_j^y (v_k^z) in H^x by $v_j^x(v_k^x)$. By the drawer principle, we need to consider two subcases according to the positions of v_i^x, v_j^x, v_k^x.

Subcase 1.1. v_i^x, v_j^x and v_k^x are in the same segment of H^x divided by v_1^x and v_{t+1}^x. If $v_i^x, v_j^x, v_k^x \in C_*^x$ for $i \le j \le k$, then $v_1^x v_2^x \cdots v_{i-1}^x v_i^x \cup v_j^y v_{2t+2-j}^y v_{2t+3-j}^y \cdots v_{2t}^y v_1^y \cup v_k^z v_{k+1}^z \cdots v_t^z v_{t+1}^z v_1^z$ is a $\{v_i^x, v_j^y, v_k^z\}$-rainbow tree. Since other cases are similar, we omit them here.

Subcase 1.2. Two of v_i^x, v_j^x and v_k^x are in one segment of H^x, and the third one is in the other segment of H^x. Suppose $v_i^x, v_j^x \in C_*^x$ for $i \le j$ and $v_k^x \in C_{**}^x$. Then

$v_1^x v_2^x \cdots v_{i-1}^x v_i^x \cup v_j^y v_{j+1}^y \cdots v_t^y v_{t+1}^y v_1^y \cup v_k^z v_{k+1}^z \cdots v_{2t}^z v_1^z$ is a $\{v_i^x, v_j^y, v_k^z\}$-rainbow tree.

Case 2. $\ell' = 1$. Set $\ell = 3t+1$. Let H^{**} be a connected rainbow graph with $2t$ vertices and $3t+1$ edges, which is obtained from H^* by adding an edge v_1v_3 where $c(v_1v_3)$ receives a new color. Take $\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ copies of H^{**} and denote them by $H^1, H^2, \ldots, H^{\lfloor \frac{n-1}{2t-1} \rfloor}$ with $V(H^p) = \{v_1^p, v_2^p, \ldots, v_{2t}^p\}$ and $E(H^p) = \left\{v_i^p v_{i+1}^p, v_1^p v_{2t}^p, v_j^p v_{2t+2-j}^p, v_1^p v_{t+1}^p, v_1^p v_3^p\right\}$ for $1 \leq i \leq 2t-1$ and $2 \leq j \leq t$, $1 \leq p \leq \left\lfloor \frac{n-1}{2t-1} \right\rfloor$, and take a subgraph graph of H, denoted by $H^{\lceil \frac{n-1}{2t-1} \rceil}$, with $n - (2t-1)\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ vertices and corresponding edges of H. Let G be a graph of order n by identifying the vertex v_1^p and $v_1^{\lceil \frac{n-1}{2t-1} \rceil}$ if $H^{\lceil \frac{n-1}{2t-1} \rceil}$ exists. Clearly, $e(G) \leq \ell \left\lceil \frac{n-1}{2t-1} \right\rceil = \ell \left\lceil \frac{3(n-1)}{2\ell-5} \right\rceil$. See G_f in Figure 3 for an example with $n = 15, \ell = 10$. Since the graph constructed in Case 1 is a spanning subgraph of the corresponding graph in Case 2, it follows that, by Lemma 4, every three vertices v_i^x, v_j^y and v_k^z in Case 2 have a rainbow tree connecting them.

Case 3. $\ell' = 2$. Set $\ell = 3t+2$. Let H^{***} be a connected rainbow graph with $2t$ vertices and $3t+2$ edges, which is obtained from H^{**} by adding an edge v_1v_{2t-1} colored with a new color. Take $\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ copies of H and denote them by $H^1, H^2, \ldots, H^{\lfloor \frac{n-1}{2t-1} \rfloor}$ with $V(H^p) = \{v_1^p, v_2^p, \ldots, v_{2t}^p\}$ and $E(H) = \left\{v_i^p v_{i+1}^p, v_j^p v_{2t+2-j}^p, v_1^p v_t^p, v_1^p v_3^p, v_1^p v_{t-1}^p\right\}$ for $1 \leq i \leq 2t-1$ and $2 \leq j \leq t$, and take a subgraph graph of H, denoted by $H^{\lceil \frac{n-1}{2t-1} \rceil}$, with $n-(2t-1)\left\lfloor \frac{n-1}{2t-1} \right\rfloor$ vertices and corresponding edges of H. Construct a graph G with n vertices similar to Case 1 and Case 2, then $e(G) \leq \ell \left\lceil \frac{n-1}{2t-1} \right\rceil = \ell \left\lceil \frac{3(n-1)}{2\ell-7} \right\rceil$. Similarly, since the graph constructed in Case 2 is a subgraph of the corresponding graph in Case 3, it follows that, by Lemma 4, every three vertices in Case 3 have a rainbow tree connecting them.

Combining the above three cases, we get $t(n,3,\ell) \leq \ell \left\lceil \frac{3n-3}{2\ell-3} \right\rceil$. ■

Remark 18. For $9 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$, the upper bound in Theorem 17 is better than the one in [9].

For small ℓ, Liu [9] just got exact values of $t(n,3,3)$ for $n = 3,4,5$. Here we get upper bounds for $t(n,3,3)$ when $n \geq 6$.

Theorem 19. *For an integer n with $n \geq 6$, $t(n,3,3) \leq \frac{n^2+2n-3}{4}$.*

Proof. We consider two cases according to whether n is even or n is odd.

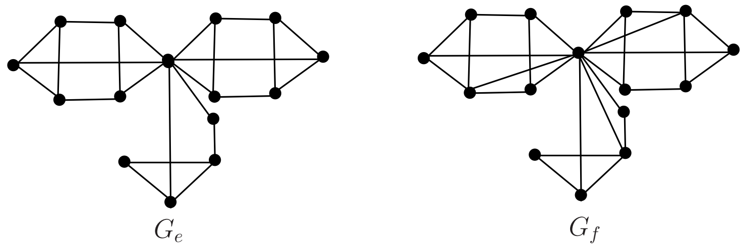

Figure 3. Graphs for Theorem 17.

Case 1. n is even. Let $n = 2k$ for some integer $k \geq 3$. Let G be a regular complete bipartite graph $K_{k,k}$. Then $e(G) = \frac{n^2}{4}$. By Theorem 6, $rx_3(G) = 3$.

Case 2. n is odd. Let $n = 2k+1$ for some integer $k \geq 3$. Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_k, x\}$ and $E(G) = \{u_iw_j, u_ix, w_ix\}$ for $1 \leq i, j \leq k$. It is easy to get $e(G) = \frac{n^2+2n-3}{4}$. Define an edge-coloring $c : E(G) \to \{1, 2, 3\}$ as follows

$$c(e) = \begin{cases} 1, & \text{if } e = u_iw_j \text{ or } e = u_ix,\ 1 \leq i = j \leq k, \\ 2, & \text{if } e = u_iw_j \text{ or } e = w_ix,\ 1 \leq i < j \leq k, \\ 3, & \text{if } e = u_iw_j,\ 1 \leq j < i \leq k. \end{cases}$$

Now we show that c is a 3-rainbow coloring of G. Let S be a set of three vertices of G. By Case 1, we need to consider three possibilities when S contain x. If $S = \{x, u_i, u_j\}$, where $i < j$, then $T = \{u_iw_i, u_jw_i, w_ix\}$ is a rainbow S-tree; if $S = \{x, w_i, w_j\}$, where $i < j$, then $T = \{u_jw_i, u_jw_j, w_jx\}$ is a rainbow S-tree; if $S = \{x, u_i, w_j\}$, then $T = \{u_ix, w_jx\}$ is a rainbow S-tree. Therefore, $rx_3(G) \leq 3$.

Combining the above two cases, we have that $t(n, 3, 3) \leq \frac{n^2+2n-3}{4}$. ■

Theorem 20. *For an integer $n \geq 8$, $t(n, 3, 4) \leq \frac{n^2+22n+11}{8}$.*

Proof. We consider four cases, according to $n \equiv n' \pmod 4$.

Case 1. $n' = 0$. Let $n = 4k$ for some integer $k \geq 2$. Let G_1 be a graph with $V(G_1) = U_1 \cup U_2 \cup U_3 \cup U_4$ (where $U_i = \{u_i^1, u_i^2, \ldots, u_i^k\}$) and $E(G_1) = \{u_1^iu_2^j, u_3^iu_4^j, u_1^iu_3^i, u_2^iu_4^i\}$ for $1 \leq i, j \leq k$. It is easy to get $e(G_1) = 2k^2 + 2k = \frac{n^2+4n}{8}$. See G_g in Figure 4 for an example with $n = 8$. We define an edge-coloring

c_1: $E(G_1) \to \{1,2,3,4\}$ as follows

$$c_1(e) = \begin{cases} 1, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j,\ 1 \le i = j \le k, \\ 2, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j,\ 1 \le i < j \le k, \\ 3, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j,\ 1 \le j < i \le k, \\ 4, & \text{if } e = u_1^i u_3^i, u_2^i u_4^i,\ 1 \le i \le k. \end{cases}$$

Now we show that c_1 is a 4-rainbow coloring of G_1. Let $S = \{x, y, z\}$ be a set of three vertices of G_1. Since the case when $S \in U_1 \cup U_2$ or $U_3 \cup U_4$ have been proved in Theorem 6, by symmetry, we need to consider the following three possibilities by the positions of x, y, z. If $x, y \in U_1$, $z \in U_3$, say $x = u_1^i, y = u_1^j, z = u_3^k$, then there is a rainbow $\{x, y, u_1^k\}$-tree T' in $U_1 \cup U_2$ and $T = T' \cup u_1^k z$ is a rainbow S-tree. If $x, y \in U_1$, $z \in U_4$, say $x = u_1^i, y = u_1^j, z = u_4^k$, then there is a rainbow $\{x, y, u_2^k\}$-tree T' in $U_1 \cup U_2$ and $T = T' \cup u_2^k z$ is a rainbow S-tree. If $x \in U_1$, $y \in U_2$, $z \in U_3$, say $x = u_1^i, y = u_2^j, z = u_3^k$, then there is a rainbow $\{x, y, u_1^k\}$-tree T' in $U_1 \cup U_2$ and $T = T' \cup u_1^k z$ is a rainbow S-tree. Therefore, $rx_3(G_1) \le 4$.

Case 2. $n' = 1$. Let $n = 4k + 1$ for some integer $k \ge 2$. Let G_2 be a graph with $V(G_2) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{s\}$ (where $U_i = \{u_i^1, u_i^2, \dots, u_i^k\}$) and $E(G_2) = \{u_1^i u_2^j, u_3^i u_4^j, u_1^i u_3^i, u_2^i u_4^i, su_1^i, su_2^i, su_3^i, su_4^i\}$ for $1 \le i, j \le k$. It is easy to get $e(G_2) = 2k^2 + 6k = \frac{n^2+10n-11}{8}$. See G_h in Figure 4 for an example with $n = 9$. Based on the coloring c_1 in Case 1, we define an edge-coloring c_2: $E(G_2) \to \{1,2,3,4\}$ as follows

$$c_2(e) = \begin{cases} 1, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j, su_1^i,\ 1 \le i = j \le k, \\ 2, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j, su_2^i,\ 1 \le i < j \le k, \\ 3, & \text{if } e = u_1^i u_2^j, u_3^i u_4^j, su_3^i,\ 1 \le j < i \le k, \\ 4, & \text{if } e = u_1^i u_3^i, u_2^i u_4^i, su_4^i,\ 1 \le i \le k. \end{cases}$$

Now we show that c_2 is a 4-rainbow coloring of G_2. Let $S = \{x, y, z\}$ be a set of three vertices of G. Now we need to consider the subcases when S contain s since other subcases have been discussed in Case 1. Set $s = z$. We consider the following five possibilities by the positions of x, y. If $x, y \in U_1$, say $y = u_1^i$, then $T = \{xs, yu_3^i, u_3^i s\}$ is a rainbow S-tree; if $x, y \in U_2$, say $y = u_2^i$, then $T = \{xu_1^i, u_1^i s, sy\}$ is a rainbow S-tree; if $x, y \in U_3$, say $x = u_3^i$, then $T = \{xu_1^i, u_1^i s, sy\}$ is a rainbow S-tree; if $x, y \in U_4$, say $x = u_4^i$, then $T = \{xu_3^i, u_3^i s, sy\}$ is a rainbow S-tree; if $x \in U_i$ and $y \in U_j$ $(i \ne j)$, then $T = \{xs, sy\}$ is a rainbow S-tree. Therefore, $rx_3(G_2) \le 4$.

Case 3. $n' = 2$. Let $n = 4k+2$ for some integer $k \ge 2$. Let G_3 be a graph with $V(G_3) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{s, t\}$ (where $U_i = \{u_i^1, u_i^2, \dots, u_i^k\}$) and $E(G_3) =$

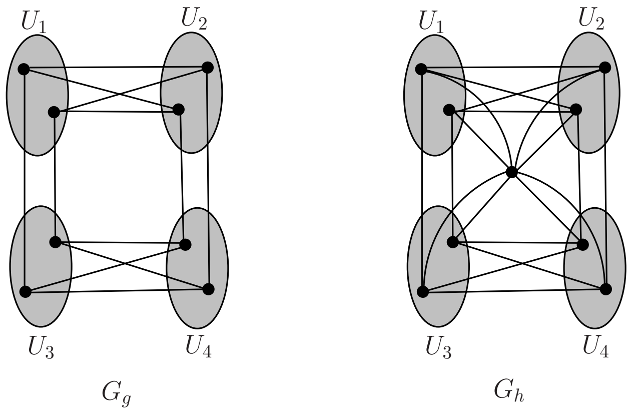

Figure 4. Graphs for Theorem 20.

$\big\{u_1^iu_2^j, u_3^iu_4^j, u_1^iu_3^i, u_2^iu_4^i, su_1^i, su_2^i, su_3^i, su_4^i, tu_1^i, tu_2^i, tu_3^i, tu_4^i\big\}$ for $1 \le i, j \le k$. It is easy to get $e\big(G_3\big) = 2k^2 + 10k = \frac{n^2+16n-36}{8}$. Based on the coloring c_2 in Case 2, we define an edge-coloring $c_3 : E(G_3) \to \{1, 2, 3, 4\}$ as follows

$$c_3(e) = \begin{cases} 1, & \text{if } e = u_1^iu_2^j, u_3^iu_4^j, su_1^i, tu_1^i,\ 1 \le i = j \le k, \\ 2, & \text{if } e = u_1^iu_2^j, u_3^iu_4^j, su_2^i, tu_1^i,\ 1 \le i < j \le k, \\ 3, & \text{if } e = u_1^iu_2^j, u_3^iu_4^j, su_3^i, tu_1^i,\ 1 \le j < i \le k, \\ 4, & \text{if } e = u_1^iu_3^i, u_2^iu_4^i, su_4^i, tu_1^i,\ 1 \le i \le k. \end{cases}$$

Now we show that c_3 is a 4-rainbow coloring of G_3. Let $S = \{x, y, z\}$ be a set of three vertices of G. Based on the discussion in Case 2, we need to consider the subcase when S contains s and t. Set $s = y$ and $t = z$. We consider the following four possibilities by the positions of x. If $x \in U_1$, say $x = u_1^i$, then $T = \big\{su_3^i, u_3^iu_1^i, u_1^it\big\}$ is a rainbow S-tree; if $x \in U_2$, say $x = u_2^i$, then $T = \big\{su_3^i, u_3^iu_4^i, u_4^ix, xt\big\}$ is a rainbow S-tree; if $x \in U_3$, say $x = u_3^i$, then $T = \big\{sx, xu_4^i, u_4^it\big\}$ is a rainbow S-tree; if $x \in U_4$, say $x = u_4^i$, then $T = \big\{su_3^i, u_3^ix, xt\big\}$ is a rainbow S-tree. Therefore, $rx_3(G_3) \le 4$.

Case 4. $n' = 3$. Let $n = 4k+3$ for some integer $k \ge 2$. Let G_4 be a graph with $V(G_4) = U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{s, t, p\}$ (where $U_i = \big\{u_i^1, u_i^2, \ldots, u_i^k\big\}$) and $E(G_4) = \big\{u_1^iu_2^j, u_3^iu_4^j, u_1^iu_3^i, u_2^iu_4^i, su_1^i, su_2^i, su_3^i, su_4^i, tu_1^i, tu_2^i, tu_3^i, tu_4^i, pu_1^i, pu_2^i, pu_3^i, pu_4^i, tp\big\}$ for $1 \le i, j \le k$. It is easy to get $e(G_4) = 2k^2 + 14k + 1 = \frac{n^2+22n+11}{8}$. Based on the edge-coloring c_3 in Case 3, we define an edge-coloring c_4: $E(G_3) \to \{1, 2, 3, 4\}$

as follows

$$c_4(e)=\begin{cases}1, & \text{if } e=u_1^iu_2^j, u_3^iu_4^j, su_1^i, tu_1^i, pu_1^i,\ 1\le i=j\le k,\\ 2, & \text{if } e=u_1^iu_2^j, u_3^iu_4^j, su_2^i, tu_1^i, pu_1^i \text{ or } e=tp,\ 1\le i<j\le k,\\ 3, & \text{if } e=u_1^iu_2^j, u_3^iu_4^j, su_3^i, tu_1^i, pu_1^i,\ 1\le j<i\le k,\\ 4, & \text{if } e=u_1^iu_3^i, u_2^iu_4^i, su_4^i, tu_1^i, pu_1^i,\ 1\le i\le k.\end{cases}$$

Now we show that c_4 is a 4-rainbow coloring of G_4. Let $S=\{x,y,z\}$ be a set of three vertices of G. Now we need to consider the subcase when $S=\{s,t,p\}$. Here $T=\left\{su_1^1, u_1^1u_3^1, u_3^1t, tp\right\}$ is a rainbow S-tree. Therefore, $rx_3(G_4)\le 4$.

Combining the above four cases, we have $t(n,3,4)\le \frac{n^2+22n+11}{8}$. ■

Remark 21. The upper bound in Theorem 20 is better than $t(n,3,4)\le \binom{n}{2}-n+1$, which is got in [9].

Theorem 22. *For an integer $n\ge 6$, $t(n,3,5)\le 2n-3$.*

Proof. Let G be a graph with $V(G)=\{v, v_1, v_2, \ldots, v_{n-1}\}$ and $E(G)=\{vv_i, v_jv_{j+1}\}$ for $1\le i\le n-1$, $1\le j\le n-2$. It is easy to get $e(G)=2n-3$. Define an edge-coloring c: $E(G)\to\{1,2,3,4,5\}$ as follows

$$c(e)=\begin{cases} i, & \text{if } e=vv_j,\ 1\le i\le 5,\ 1\le j\le n-1,\ j=i \pmod 5,\\ i, & \text{if } e=v_jv_{j+1},\ 1\le i\le 5,\ 1\le j\le n-2,\ j+3=i \pmod 5).\end{cases}$$

It is easy to show that c is a 5-rainbow coloring of G. Thus $rx_3(G)\le 5$, it follows that $t(n,3,5)\le 2n-3$. ■

Remark 23. The result in Theorem 22 is better than $t(n,3,5)\le 2n-2$, which is got in [9].

Theorem 24. *For an integer $n\ge 7$, $t(n,3,6)\le 2n-6$.*

Proof. We consider three cases.

Case 1. $n=3t$. Let G_1 be a graph by taking $t-2$ vertex-disjoint cliques of order 4 and 5 vertex-disjoint K_2, and identifying a vertex from each of them. That is, G_1 is a graph with $V(G_1)=\{v, v_1, v_2, \ldots, v_{n-1}\}$ and $E(G_1)=\{vv_i,\ v_jv_{j+1}, v_jv_{j+2}, v_kv_{k+1}\}$ for $1\le i\le n-1$, $j=1 \pmod 3$, $k=2 \pmod 3$, $1\le j,k\le 3(t-2)$. It is easy to get $e(G_1)=2n-7$. Define an edge-coloring c_1: $E(G_1)\to\{1,2,3,4,5,6\}$ as follows

$$c_1(e)=\begin{cases}1, & \text{if } e=vv_i \text{ or } e=vv_{3t-5},\ 1\le i\le n-6,\ i=1 \pmod 3,\\ 2, & \text{if } e=vv_i \text{ or } e=vv_{3t-4},\ 1\le i\le n-6,\ i=2 \pmod 3,\\ 3, & \text{if } e=vv_i \text{ or } e=vv_{3t-3},\ 1\le i\le n-6,\ i=3 \pmod 3,\\ 4, & \text{if } e=v_iv_{i+1} \text{ or } e=vv_{3t-2},\ i=1 \pmod 3,\\ 5, & \text{if } e=v_iv_{i+2} \text{ or } e=vv_{3t-1},\ i=1 \pmod 3,\\ 6, & \text{if } e=v_iv_{i+1},\ i=2 \pmod 3.\end{cases}$$

It is easy to show that c_1 is a 6-rainbow coloring of G_1, thus $rx_3(G_1) \leq 6$.

Case 2. $n = 3t+1$. Let G_2 be a graph with $V(G_2) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ and $E(G_2) = \{vv_i, v_jv_{j+1}, v_jv_{j+2}, v_kv_{k+1}\}$ for $1 \leq i \leq n-1$, $j = 1 \,(\mathrm{mod}\ 3)$, $k = 2 \,(\mathrm{mod}\ 3)$, $1 \leq j, k \leq 3(t-2)$. It is easy to get $e(G_2) = 2n-8$. Define an edge-coloring c_2: $E(G_2) \to \{1,2,3,4,5,6\}$ as follows:

$$c_2(e) = \begin{cases} 1, & \text{if } e = vv_i \text{ or } e = vv_{3t-5},\ 1 \leq i \leq n-7,\ i = 1 \ (\mathrm{mod}\ 3), \\ 2, & \text{if } e = vv_i \text{ or } e = vv_{3t-4},\ 1 \leq i \leq n-7,\ i = 2 \ (\mathrm{mod}\ 3), \\ 3, & \text{if } e = vv_i \text{ or } e = vv_{3t-3},\ 1 \leq i \leq n-7,\ i = 3 \ (\mathrm{mod}\ 3), \\ 4, & \text{if } e = v_iv_{i+1} \text{ or } e = vv_{3t-2},\ i = 1 \ (\mathrm{mod}\ 3), \\ 5, & \text{if } e = v_iv_{i+2} \text{ or } e = vv_{3t-1},\ i = 1 \ (\mathrm{mod}\ 3), \\ 6, & \text{if } e = v_iv_{i+1} \text{ or } e = vv_{3t},\ i = 2 \ (\mathrm{mod}\ 3). \end{cases}$$

It is easy to show that c_2 is a 6-rainbow coloring of G_2, thus $rx_3(G_2) \leq 6$.

Case 3. $n = 3t+2$. Let G_3 be a graph with $V(G_3) = \{v, v_1, v_2, \ldots, v_{n-1}\}$ and $E(G_3) = \{vv_i, v_jv_{j+1}, v_jv_{j+2}, v_kv_{k+1}\}$ for $1 \leq i \leq n-1$, $j = 1 \,(\mathrm{mod}\ 3)$, $k = 2 \,(\mathrm{mod}\ 3)$, $1 \leq j, k \leq 3(t-1)$. It is easy to get $e(G_3) = 2n-6$. Define an edge-coloring c_3: $E(G_3) \to \{1,2,3,4,5,6\}$ as follows

$$c_3(e) = \begin{cases} 1, & \text{if } e = vv_i,\ 1 \leq i \leq n-5,\ i = 1 \ (\mathrm{mod}\ 3), \\ 2, & \text{if } e = vv_i,\ 1 \leq i \leq n-5,\ i = 2 \ (\mathrm{mod}\ 3), \\ 3, & \text{if } e = vv_i \text{ or } e = vv_{3t-2},\ 1 \leq i \leq n-5,\ i = 3 \ (\mathrm{mod}\ 3), \\ 4, & \text{if } e = v_iv_{i+1} \text{ or } e = vv_{3t-1},\ i = 1 \ (\mathrm{mod}\ 3), \\ 5, & \text{if } e = v_iv_{i+2} \text{ or } e = vv_{3t},\ i = 1 \ (\mathrm{mod}\ 3), \\ 6, & \text{if } e = v_iv_{i+1} \text{ or } e = vv_{3t+1},\ i = 2 \ (\mathrm{mod}\ 3). \end{cases}$$

It is easy to show that c_3 is a 6-rainbow coloring of G_3, thus $rx_3(G_3) \leq 6$.
Combining the above three cases, we have $t(n,3,6) \leq 2n-6$. ■

Remark 25. The upper bound in Theorem 24 is better than $2n-3$, which is got in [9].

Theorem 26. *For an integer $n \geq 8$, $t(n,3,7) \leq 2n-7$.*

Proof. We consider three cases.

Case 1. $n = 3t+1$. Set $n' = n-1 = 3t$. Construct a graph G_1 as in Theorem 24 and let G be a graph obtained from G_1 by adding an edge colored by 7. It is easy to see that G is 3-rainbow connected and $e(G) = (2n'-7)+1 = 2n-8$.

Case 2. $n = 3t+2$. Set $n' = n-1 = 3t+1$. Construct a graph G_2 as in Theorem 24 and let G be a graph obtained from G_2 by adding an edge colored by 7. It is easy to see that G is 3-rainbow connected and $e(G) = (2n'-8)+1 = 2n-9$.

Case 3. $n = 3t+3$. Set $n' = n-1 = 3t+2$. Construct a graph G_3 as in Theorem 24 and let G be a graph obtained from G_3 by adding an edge colored by 7. It is easy to see that G is 3-rainbow connected and $e(G) = (2n'-6)+1 = 2n-7$.

Combining the above three cases, we have $t(n,3,7) \leq 2n-7$. ∎

Theorem 27. *For an integer* $n \geq 9$, $t(n,3,8) \leq 2n-2$.

Proof. Let H^* be a connected rainbow graph with 5 vertices and 8 edges, where $V\big(H^*\big) = \big\{v_1, v_2, \ldots, v_5\big\}$ and $E\big(H^*\big) = \big\{v_iv_{i+1}, v_1v_3, v_1v_4, v_1v_5, v_2v_5\big\}$ for $1 \leq i \leq 4$. Take $\left\lfloor \frac{n-1}{4} \right\rfloor$ copies of H^* and denote them by $H^1, H^2, \ldots, H^{\left\lfloor \frac{n-1}{4} \right\rfloor}$ with $V\big(H^p\big) = \{v_1^p, v_2^p, \ldots, v_5^p\}$ and $E\big(H^p\big) = \big\{v_i^pv_{i+1}^p, v_1^pv_3^p, v_1^pv_4^p, v_1^pv_5^p, v_2^pv_5^p\big\}$ for $1 \leq i \leq 4$, $1 \leq p \leq \left\lfloor \frac{n-1}{4} \right\rfloor$, and take a subgraph graph of H^*, denoted by $H^{\left\lceil \frac{n-1}{4} \right\rceil}$, with $n - 4\left\lfloor \frac{n-1}{4} \right\rfloor$ vertices and corresponding edges of H^*. Let G be a graph with n vertices by identifying the vertex $v_1^p(1 \leq p \leq \left\lfloor \frac{n-1}{4} \right\rfloor)$ and $v_1^{\left\lceil \frac{n-1}{4} \right\rceil}$ if $H^{\left\lceil \frac{n-1}{4} \right\rceil}$ exists. Clearly, $e(G) \leq 2n-2$. Similar to the discussion in Theorem 17, it is shown that G is 3-rainbow connected. Thus $t(n,3,8) \leq 2n-2$. ∎

5. Summary

In Section 3, we get the exact values of $t(n,k,n-1)$ and $t(n,k,n-2)$ for $3 \leq k \leq n-1$. In Section 4, the exact values of $t(n,3,n-3)$ and $t(n,3,n-4)$ are obtained. In other cases for $k = 3$, the upper bounds we got are better than the ones in [9], but they are not tight. In fact, it is challenging to get the exact values of $t(n,k,\ell)$ for all cases. We will continue to focus on this problem in the future.

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