# MORE ON THE MINIMUM SIZE OF GRAPHS WITH GIVEN RAINBOW INDEX 

Yan ZHaO<br>Department of Mathematics<br>Taizhou University<br>Taizhou 225300, P.R. China<br>e-mail: zhaoyan81.2008@163.com


#### Abstract

The concept of $k$-rainbow index $r x_{k}(G)$ of a connected graph $G$, introduced by Chartrand et al., is a natural generalization of the rainbow connection number of a graph. Liu introduced a parameter $t(n, k, \ell)$ to investigate the problems of the minimum size of a connected graph with given order and $k$-rainbow index at most $\ell$ and obtained some exact values and upper bounds for $t(n, k, \ell)$. In this paper, we obtain some exact values of $t(n, k, \ell)$ for large $\ell$ and better upper bounds of $t(n, k, \ell)$ for small $\ell$ and $k=3$.


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## 1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. For a graph, by size of it we mean number of its edges. Let $G$ be a nontrivial connected graph with an edge-coloring $c: E(G) \rightarrow\{1,2, \ldots, \ell\}, \ell \in \mathbb{N}$, where adjacent edges may be colored the same. A path of $G$ is a rainbow path if every two edges of the path have distinct colors. The graph $G$ is rainbow connected if for every two vertices $u$ and $v$ of $G$, there is a rainbow path connecting $u$ and $v$. The minimum number of colors for which there is an edge coloring of $G$ such that $G$ is rainbow connected is called the rainbow connection number, denoted by $\operatorname{rc}(G)$. Results on the rainbow connectivity can be found in $[2,4,5,8]$.

These concepts were introduced by Chartrand et al. in [2]. In [3], they generalized the concept of rainbow path to rainbow tree. A tree $T$ in $G$ is a
rainbow tree if no two edges of $T$ receive the same color. For $S \subseteq V(G)$, a rainbow S-tree is a rainbow tree connecting the vertices of $S$. Given a fixed integer $k$ with $2 \leq k \leq n$, the edge-coloring $c$ of $G$ is called a $k$-rainbow coloring if for every set $S$ of $k$ vertices of $G$, there exists a rainbow $S$-tree. In this case, we call $G k$-rainbow connected. The minimum number of colors that are needed in a $k$-rainbow coloring of $G$ is called the $k$-rainbow index, denoted by $r x_{k}(G)$. Clearly, when $k=2, r x_{2}(G)$ is nothing new but the rainbow connection number $r c(G)$ of $G$. For every connected graph $G$ of order $n$, it is easy to see that $r x_{2}(G) \leq r x_{3}(G) \leq \cdots \leq r x_{n}(G)$.

The Steiner distance $d_{G}(S)$ of a set $S$ of vertices in $G$ is the minimum size of a tree in $G$ containing $S$. The $k$-Steiner diameter $\operatorname{sdiam}_{k}(G)$ of $G$ is the maximum Steiner distance of $S$ among all sets $S$ with $k$ vertices in $G$. Then there is a simple upper bound and lower bound for $r x_{k}(G)$.
Observation 1 [3]. For every connected graph $G$ of order $n \geq 3$ and each integer $k$ with $3 \leq k \leq n, k-1 \leq \operatorname{sdiam}_{k}(G) \leq r x_{k}(G) \leq n-1$.

It is [3] shown that the tree is a class of graphs whose $k$-rainbow index attains the upper bound.
Proposition 2 [3]. Let $T$ be a tree of order $n \geq 3$. For each integer $k$ with $3 \leq k \leq n, r x_{k}(T)=n-1$.

Chartrand et al. also showed that the $k$-rainbow index of the unicyclic graph is $n-1$ or $n-2$.
Theorem 3 [3]. If $G$ is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then

$$
r x_{k}(G)= \begin{cases}n-2, & \text { if } k=3 \text { and } g \geq 4  \tag{1}\\ n-1, & \text { if } g=3 \text { or } 4 \leq k \leq n\end{cases}
$$

Schiermeyer [11] introduced a parameter $t(n, d)$ to investigate the rainbow connection. For integers $n$ and $d$ let $t(n, d)$ denote the minimum size (number of edges) in $d$-rainbow connected graphs of order $n$. Since a network which satisfies our certain requirements and has as few links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is significant. Later, this parameter was investigated [7, 10] and was solved completely by Lo [10]. Motivated by the parameter $t(n, d)$, Liu [9] introduced a new parameter to study the minimum size of a graph $G$ such that $G$ has a $k$-rainbow coloring using a fixed number of colors. Let $t(n, k, \ell)$ be the minimum size of a connected graph $G$ of order $n$ with $r x_{k}(G) \leq \ell$, where $2 \leq \ell \leq n-1$ and $2 \leq k \leq n$. Clearly, $t(n, k, 1) \geq t(n, k, 2) \geq \cdots \geq t(n, k, n-1)$. Liu [9] got some exact values and some upper bounds for $t(n, k, \ell)$ when $k$ and $\ell$ take specific values. In this paper, we obtain some exact values of $t(n, k, \ell)$ for large $\ell$ and better upper bounds of $t(n, k, \ell)$ for small $\ell$ and $k=3$.

## 2. Preliminaries

Definition. A rose graph $R_{p}$ with $p$ petals (or $p$-rose graph) is a graph obtained by taking $p$ cycles with just a vertex in common. The common vertex is called the center of $R_{p}$. The rose graph with $p$ petals is denoted by $R_{p}\left(\ell_{1}, \ell_{2}, \ldots, \ell_{p}\right)$ if the length of each cycle needs to be specified.

Definition. An edge-colored graph is rainbow if no two edges in the graph share the same color.

Definition. To subdivide an edge $e$ is to delete $e$, add a new vertex $x$, and join $x$ to the ends of $e$. Any graph derived from a graph $G$ by a sequence of edge subdivisions is called a subdivision of $G$.

Lemma 4 [6]. Let $H$ be a connected subgraph of a connected graph $G$. Then $r x_{k}(G) \leq r x_{k}(H)$ for $2 \leq k \leq n-1$.

Lemma 5 [6]. Let $G$ be a connected graph, and $H$ be a subdivision of $G$. Then $r x_{k}(H) \leq r x_{k}(G)+|H|-|G|$.

Theorem 6 [6]. For each integer $k$ with $k \geq 3, r x_{3}\left(K_{k, k}\right)=3$.
Lemma 7. For each integer $k$ with $k \geq 3, r x_{k}\left(K_{2, k-1}\right)=k-1$.
Proof. Let $G=K_{2, k-1}=G[X, Y]$, where $X=\{u, w\}$ and $Y=\left\{v_{1}, v_{2}, \ldots, v_{k-1}\right\}$. Define an edge-coloring $c$ as follows: $c\left(u v_{i}\right)=i$ for $1 \leq i \leq k-1, c\left(w v_{i}\right)=i+1$ for $1 \leq i \leq k-1$. It is easily checked that there exists a rainbow $S$-tree for any $S \subseteq V(G)$ and $|S|=k$. Thus $r x_{k}\left(K_{2, k-1}\right) \leq k-1$. Conversely, by Observation 1 , we have $r x_{k}\left(K_{2, k-1}\right) \geq k-1$. Therefore, $r x_{k}\left(K_{2, k-1}\right)=k-1$.
3. Some Results for $t(n, k, n-1)$ and $t(n, k, n-2)$

In this section, we first consider the case when $\ell$ is large. By Proposition 2, we get the exact value of $t(n, k, \ell)$ for $\ell=n-1$ and every $k$ with $3 \leq k \leq n$.

Theorem 8. Let $n \geq 3$ be an integer. For each integer $k$ with $3 \leq k \leq n$, $t(n, k, n-1)=n-1$.

For $\ell=n-2$, Liu [9] got $t(n, k, n-2)=n$ when $k=3$, we get results for $k=4$ and $k=n-1$.

Theorem 9. Let $n \geq 4$ be an integer. Then $t(n, 4, n-2)=n+1$.

Proof. Let $H$ be a graph obtained from $K_{2,3}$ by subdividing $n-5$ edges. Then $H$ has $n$ vertices and $n+1$ edges. Since $r x_{4}\left(K_{2,3}\right)=3$, it follows that, by Lemma 5 , $r x_{4}(H) \leq n-2$. Thus $t(n, 4, n-2) \leq n+1$. Conversely, if $G$ is a tree or unicyclic, then by Proposition 2 and Theorem $3, r x_{4}(G)=n-1$. Thus $t(n, 4, n-2) \geq n+1$. Therefore, $t(n, 4, n-2)=n+1$.

Theorem 10. Let $n \geq 4$ be an integer. Then $t(n, n-1, n-2)=2 n-4$.
Proof. since $t(n, n-1, n-2) \leq 2 n-4$ has been proved in [9], we need to prove that $t(n, n-1, n-2) \geq 2 n-4$. To the contrary, suppose $t(n, n-1, n-2) \leq 2 n-5$. Assume that $G$ is a connected graph with $2 n-5$ edges and $n-2$ colors. By the drawer principle, at least a color appears exactly once in $G$. Suppose a $c_{1}$-edge is incident to the vertex $x$. Delete the vertex $x$ from $G$, and the remaining graph $G-x$ has $n-1$ vertices but at most $n-3$ colors, it follows that $G-x$ has no rainbow tree, a contradiction.

From $t(n, 3, n-2)=n, t(n, 4, n-2)=n+1$ and $t(n, n-1, n-2)=2 n-4$, we believe that $t(n, k, n-2)=n+k-3$ for general $k$. In fact, this is true for general $k$.

Theorem 11. Let $n \geq 4$ be an integer. For each integer $k$ with $3 \leq k \leq n-1$, $t(n, k, n-2)=n+k-3$.

Proof. Let $H$ be a graph obtained from $K_{2, k-1}$ by subdividing $n-k-1$ edges. Since $r x_{k}\left(K_{2, k-1}\right)=k-1$, it follows that, by Lemma $5, r x_{k}(H) \leq n-2$. As $H$ has $n$ vertices and $n+k-3$ edges, we have $t(n, k, n-2) \leq n+k-3$. Conversely, we need to prove that $t(n, k, n-2) \geq n+k-3$. To the contrary, suppose $t(n, k, n-2) \leq n+k-4$. Let $G$ be a connected graph with $n+k-4$ edges and $n-2$ colors. Then at least $n-k$ colors appears exactly once in $G$; otherwise, at most $n-k-1$ colors appear exactly once and at least $k-1$ colors appear at least twice in $G$, it follows that $e(G) \geq n-k-1+2(k-1)=n+k-3$, a contradiction. Delete $n-k$ vertices incident to the edges colored with the $n-k$ colors which appear exactly once, then the remaining graph has $k$ vertices but at most $n-2-(n-k)=k-2$ colors. Thus the remaining graph has no rainbow spanning tree, a contradiction.

## 4. Some Results for $t(n, 3, \ell)$

In this section, we first focus on the case when $\ell$ is large.
Theorem 12. Let $n \geq 8$ be an integer. Then $t(n, 3, n-4)=n+1$.


Figure 1. Graphs for Theorem 12 and Corollary 13.
Proof. Consider the graph $G_{a}$ in Figure 1. Clearly, $G_{a}$ has 8 vertices, 9 edges and 4 colors. It is easily checked that $G_{a}$ is 3 -rainbow connection, thus $r x_{3}\left(G_{a}\right) \leq 4$. Let $H$ be a graph obtained from $G_{a}$ by subdividing $n-8$ edges. Then by Lemma $5, r x_{3}(H) \leq 4+(n-8)=n-4$. Since $H$ has $n$ vertices and $n+1$ edges, it follows that $t(n, 3, n-4) \leq n+1$. Conversely, if $G$ is a tree or a unicyclic graph, then by Proposition 2 and Theorem 3, $r x_{3}(G)$ is $n-1$ or $n-2$. So if $G$ is a graph with $r x_{3}(G)=n-4$, then $G$ has at least $n+1$ edges and then $t(n, 3, n-4) \geq n+1$. Therefore, $t(n, 3, n-4)=n+1$.

Corollary 13. Let $n \geq 7$ be an integer. Then $t(n, 3, n-3)=n+1$.
Proof. Consider the graph $G_{b}$ in Figure 1. Clearly, $G_{b}$ has 7 vertices, 8 edges and 4 colors. It is easily checked that $G_{b}$ is 3 -rainbow connection, thus $r x_{3}\left(G_{b}\right) \leq 4$ and $t(7,3,4) \leq 8$. Let $n \geq 8$. Since $t(n, 3, n-4) \geq(n, 3, n-3)$, it follows that $t(n, 3, n-3) \leq n+1$ by Theorem 12. Conversely, if $G$ is a tree or a unicyclic graph, then by Proposition 2 and Theorem $3, r x_{3}(G)$ is $n-1$ or $n-2$. So if $G$ is a graph with $r x_{3}(G)=n-3$, then $G$ has at least $n+1$ edges. Thus $t(n, 3, n-3) \geq n+1$. Therefore, $t(n, 3, n-3)=n+1$.

Remark 14. For $\ell=n-1$ and $\ell=n-2$, Liu [9] got $t(n, 3, n-1)=n-1$ and $t(n, 3, n-2)=n$. For $\frac{n}{2} \leq \ell \leq n-3$, Liu [9] (see Theorem 2.11) got $t(n, 3, \ell) \leq 2 n-\ell-1$, which implies $t(n, 3, n-3) \leq n+2, t(n, 3, n-4) \leq n+3$ and $t(n, 3, n-5) \leq n+4$. In fact, we get the exact values of $t(n, 3, n-3)$ and $t(n, 3, n-4)$ in Theorem 12 and Corollary 13, respectively.

Theorem 15. Let $n \geq 11$ be an integer. For each integer $\ell$ with $\left\lceil\frac{n+1}{2}\right\rceil \leq \ell \leq$ $n-5, t(n, 3, \ell) \leq\left\lfloor\frac{3 n-\ell-1}{2}\right\rfloor$.

Proof. We consider two cases according to the parity of $n$ and $\ell$.
Case 1. $n$ and $\ell$ have the same parity. Let $w_{0}$ be the center of $R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)$ and let $C_{i}=w_{0} u_{i} x_{i} y_{i} v_{i} w_{0}$ for $1 \leq i \leq \frac{n-\ell}{2}$. To show that $r x_{3}\left(R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)\right) \leq$
$n-\ell+1$, we provide an edge-coloring $c_{1}: E\left(R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)\right) \rightarrow\{1,2, \ldots, n-$ $\ell+1\}$ defined by

$$
c_{1}(e)= \begin{cases}2 i-1, & e=w_{0} u_{i} \text { or } y_{i} v_{i}\left(1 \leq i \leq \frac{n-\ell}{2}\right) \\ 2 i, & e=w_{0} v_{i} \text { or } u_{i} x_{i}\left(1 \leq i \leq \frac{n-\ell}{2}\right) \\ n-\ell+1, & e=x_{i} y_{i}\left(1 \leq i \leq \frac{n-\ell}{2}\right)\end{cases}
$$

For any $S \subseteq V\left(R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)\right)$ and $|S|=3$, it is easily checked that there exists a rainbow $S$-tree. Thus, $r x_{3}\left(R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)\right) \leq n-\ell+1$. Let $G$ be a graph obtained from $R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)$ by subdividing $2 \ell-n-1$ edges arbitrarily. Then $|V(G)|=4 \cdot \frac{n-\ell}{2}+1+(2 \ell-n-1)=n$ and $|E(G)|=5 \cdot \frac{n-\ell}{2}+(2 \ell-n-1)=\frac{3 n-\ell-2}{2}$. By Lemma $5, r x_{3}(G) \leq r x_{3}\left(R_{\frac{n-\ell}{2}}(5,5, \ldots, 5)\right)+(2 \ell-n-1) \leq \ell$. Since $G$ has $n$ vertices and $\frac{3 n-\ell-2}{2}$ edges, it follows that $t(n, 3, \ell) \leq \frac{3 n-\ell-2}{2}$. See $G_{c}$ in Figure 2 for an example with $n=13, \ell=7$.

Case 2. $n$ and $\ell$ have different parities. Let $w_{0}$ be the center of $R_{\frac{n-\ell+1}{2}}(3,5, \ldots$, 5), $C_{1}=w_{0} x_{1} y_{1} w_{0}$ and let $C_{i}=w_{0} u_{i} x_{i} y_{i} v_{i} w_{0}$, where $2 \leq i \leq \frac{n-\ell+1}{2}$. To show that $r x_{3}\left(R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)\right) \leq n-\ell+1$, we provide an edge-coloring $c_{2}: E\left(R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)\right) \rightarrow\{1,2, \ldots, n-\ell+1\}$ defined by

$$
c_{2}(e)= \begin{cases}1, & e=w_{0} x_{1} \text { or } w_{0} y_{1} \\ 2 i-2, & e=w_{0} u_{i} \text { or } y_{i} v_{i}\left(2 \leq i \leq \frac{n-\ell+1}{2}\right) \\ 2 i-1, & e=w_{0} v_{i} \text { or } u_{i} x_{i}\left(2 \leq i \leq \frac{n-\ell+1}{2}\right) \\ n-\ell+1, & e=x_{i} y_{i}\left(1 \leq i \leq \frac{n-\ell+1}{2}\right)\end{cases}
$$

For any $S \subseteq V\left(R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)\right)$ and $|S|=3$, it is easily checked that there exists a rainbow $S$-tree. Thus, $r x_{3}\left(R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)\right) \leq n-\ell+1$. Let $G$ be a graph obtained from $R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)$ by subdividing $2 \ell-n-1$ edges. Then $|V(G)|=4 \cdot\left(\frac{n-\ell+1}{2}-1\right)+3+(2 \ell-n-1)=n$ and $|E(G)|=5 \cdot\left(\frac{n-\ell+1}{2}-1\right)+$ $3+(2 \ell-n-1)=\frac{3 n-\ell-1}{2}$. By Lemma $5, r x_{3}(G) \leq r x_{3}\left(R_{\frac{n-\ell+1}{2}}(3,5, \ldots, 5)\right)+$ $(2 \ell-n-1) \leq \ell$. Since $G$ has $n$ vertices and $\frac{3 n-\ell-1}{2}$ edges, it follows that $t(n, 3, \ell) \leq \frac{3 n-\ell-1}{2}$. See $G_{d}$ in Figure 2 for an example with $n=11, \ell=6$.

Combining the above two cases, we have that $t(n, 3, \ell) \leq\left\lfloor\frac{3 n-\ell-1}{2}\right\rfloor$.
Remark 16. For $\frac{n}{2} \leq \ell \leq n-3$, Liu [9] got $t(n, 3, \ell) \leq 2 n-\ell-1$ (see Theorem 2.11). Since $\ell \leq n-5$ in Theorem 15, it follows that $2 n-\ell-1-\left\lfloor\frac{3 n-\ell-1}{2}\right\rfloor \geq$ $2 n-\ell-1-\frac{3 n-\ell-1}{2}=\frac{n-\ell-1}{2} \geq 2$. Thus the upper bound in Theorem 15 is better than the one in [9].
Theorem 17. Let $n \geq 17$ and $\ell$ be integers with $9 \leq \ell \leq\left\lfloor\frac{n+1}{2}\right\rfloor$. Then $t(n, 3, \ell) \leq$ $\ell\left\lceil\frac{3 n-3}{2 \ell-3}\right\rceil$.


Figure 2. Graphs for Theorem 15.

Proof. We consider three cases according to $\ell \equiv \ell^{\prime}(\bmod 3)$.
Case 1. $\quad \ell^{\prime}=0$. Set $\ell=3$ t. Let $H^{*}$ be a connected rainbow graph with $2 t$ vertices and $3 t$ edges, where $V\left(H^{*}\right)=\left\{v_{1}, v_{2}, \ldots, v_{2 t}\right\}$ and $E\left(H^{*}\right)=$ $\left\{v_{i} v_{i+1}, v_{1} v_{2 t}, v_{j} v_{2 t+2-j}, v_{1} v_{t+1}\right\}$ for $1 \leq i \leq 2 t-1$ and $2 \leq j \leq t$. Note that, $v_{1}$ and $v_{i}(2 \leq i \leq 2 t, i \neq t+1)$ have three internally-disjoint rainbow paths $P_{1}=v_{1} v_{2} \cdots v_{i-1} v_{i}, P_{2}=v_{1} v_{t+1} v_{t} \cdots v_{i+1} v_{i}, P_{3}=v_{1} v_{2 t} v_{2 t-1} \cdots v_{2 t+3-i} v_{2 t+2-i} v_{i}$; $v_{1}$ and $v_{t+1}$ have three internally-disjoint rainbow paths $P_{1}=v_{1} v_{2} \cdots v_{t} v_{t+1}, P_{2}=$ $v_{1} v_{t+1}, P_{3}=v_{1} v_{2 t} v_{2 t-1} \cdots v_{t+2} v_{t+1}$ in $H^{*}$. Also note that vertices $v_{1}$ and $v_{t+1}$ divide the cycle $C_{2 t}:=v_{1} v_{2} \cdots v_{2 t-1} v_{2 t} v_{1}$ into two segments $C_{*}=v_{1} v_{2} \cdots v_{t} v_{t+1}$ and $C_{* *}=v_{t+1} v_{t+2} \cdots v_{2 t} v_{1}$ in $H^{*}$.

Take $\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$ copies of $H^{*}$ and denote them by $H^{1}, H^{2}, \ldots, H^{\left\lfloor\frac{n-1}{2 t-1}\right\rfloor}$ with $V\left(H^{p}\right)=\left\{v_{1}^{p}, v_{2}^{p}, \ldots, v_{2 t}^{p}\right\}$ and $E\left(H^{p}\right)=\left\{v_{i}^{p} v_{i+1}^{p}, v_{1}^{p} v_{2 t}^{p}, v_{j}^{p} v_{2 t+2-j}^{p}, v_{1}^{p} v_{t+1}^{p}\right\}$ for $1 \leq$ $i \leq 2 t-1$ and $2 \leq j \leq t, 1 \leq p \leq\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$, and take a subgraph graph of $H^{*}$, denoted by $H^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$, with $n-(2 t-1)\left[\frac{n-1}{2 t-1}\right\rfloor$ vertices and corresponding edges of $H^{*}$. Let $G$ be a graph with $n$ vertices by identifying the vertices $v_{1}^{p}\left(1 \leq p \leq\left\lfloor\frac{n-1}{2 t-1}\right\rfloor\right)$ and $v_{1}^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$ if $H^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$ exists. Clearly, $e(G) \leq \ell\left\lceil\frac{n-1}{2 t-1}\right\rceil=\ell\left\lceil\frac{3 n-3}{2 \ell-3}\right\rceil$. See $G_{e}$ in Figure 3 for an example with $n=15, \ell=9$.

Let $v_{i}^{x}, v_{j}^{y}$ and $v_{k}^{z}$ be three vertices in $G$. Denote the corresponding vertex of $v_{j}^{y}\left(v_{k}^{z}\right)$ in $H^{x}$ by $v_{j}^{x}\left(v_{k}^{x}\right)$. By the drawer principle, we need to consider two subcases according to the positions of $v_{i}^{x}, v_{j}^{x}, v_{k}^{x}$.

Subcase 1.1. $v_{i}^{x}, v_{j}^{x}$ and $v_{k}^{x}$ are in the same segment of $H^{x}$ divided by $v_{1}^{x}$ and $v_{t+1}^{x}$. If $v_{i}^{x}, v_{j}^{x}, v_{k}^{x} \in C_{*}^{x}$ for $i \leq j \leq k$, then $v_{1}^{x} v_{2}^{x} \cdots v_{i-1}^{x} v_{i}^{x} \cup v_{j}^{y} v_{2 t+2-j}^{y} v_{2 t+3-j}^{y} \cdots v_{2 t}^{y}$ $v_{1}^{y} \cup v_{k}^{z} v_{k+1}^{z} \cdots v_{t}^{z} v_{t+1}^{z} v_{1}^{z}$ is a $\left\{v_{i}^{x}, v_{j}^{y}, v_{k}^{z}\right\}$-rainbow tree. Since other cases are similar, we omit them here.

Subcase 1.2. Two of $v_{i}^{x}, v_{j}^{x}$ and $v_{k}^{x}$ are in one segment of $H^{x}$, and the third one is in the other segment of $H^{x}$. Suppose $v_{i}^{x}, v_{j}^{x} \in C_{*}^{x}$ for $i \leq j$ and $v_{k}^{x} \in C_{* *}^{x}$. Then
$v_{1}^{x} v_{2}^{x} \cdots v_{i-1}^{x} v_{i}^{x} \cup v_{j}^{y} v_{j+1}^{y} \cdots v_{t}^{y} v_{t+1}^{y} v_{1}^{y} \cup v_{k}^{z} v_{k+1}^{z} \cdots v_{2 t}^{z} v_{1}^{z}$ is a $\left\{v_{i}^{x}, v_{j}^{y}, v_{k}^{z}\right\}$-rainbow tree.

Case 2. $\ell^{\prime}=1$. Set $\ell=3 t+1$. Let $H^{* *}$ be a connected rainbow graph with $2 t$ vertices and $3 t+1$ edges, which is obtained from $H^{*}$ by adding an edge $v_{1} v_{3}$ where $c\left(v_{1} v_{3}\right)$ receives a new color. Take $\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$ copies of $H^{* *}$ and denote them by $H^{1}, H^{2}, \ldots, H^{\left\lfloor\frac{n-1}{2 t-1}\right\rfloor}$ with $V\left(H^{p}\right)=\left\{v_{1}^{p}, v_{2}^{p}, \ldots, v_{2 t}^{p}\right\}$ and $E\left(H^{p}\right)=$ $\left\{v_{i}^{p} v_{i+1}^{p}, v_{1}^{p} v_{2 t}^{p}, v_{j}^{p} v_{2 t+2-j}^{p}, v_{1}^{p} v_{t+1}^{p}, v_{1}^{p} v_{3}^{p}\right\}$ for $1 \leq i \leq 2 t-1$ and $2 \leq j \leq t$, $1 \leq p \leq\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$, and take a subgraph graph of $H$, denoted by $H^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$, with $n-(2 t-1)\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$ vertices and corresponding edges of $H$. Let $G$ be a graph of order $n$ by identifying the vertex $v_{1}^{p}$ and $v_{1}^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$ if $H^{\left\lceil\frac{n-1}{2 t-1}\right\rceil}$ exists. Clearly, $e(G) \leq \ell\left\lceil\frac{n-1}{2 t-1}\right\rceil=\ell\left\lceil\frac{3(n-1)}{2 \ell-5}\right\rceil$. See $G_{f}$ in Figure 3 for an example with $n=$ $15, \ell=10$. Since the graph constructed in Case 1 is a spanning subgraph of the corresponding graph in Case 2, it follows that, by Lemma 4, every three vertices $v_{i}^{x}, v_{j}^{y}$ and $v_{k}^{z}$ in Case 2 have a rainbow tree connecting them.

Case 3. $\quad \ell^{\prime}=2$. Set $\ell=3 t+2$. Let $H^{* * *}$ be a connected rainbow graph with $2 t$ vertices and $3 t+2$ edges, which is obtained from $H^{* *}$ by adding an edge $v_{1} v_{2 t-1}$ colored with a new color. Take $\left\lfloor\frac{n-1}{2 t-1}\right\rfloor$ copies of $H$ and denote them by $H^{1}, H^{2}, \ldots, H^{\left\lfloor\frac{n-1}{2 t-1}\right\rfloor}$ with $V\left(H^{p}\right)=\left\{v_{1}^{p}, v_{2}^{p}, \ldots, v_{2 t}^{p}\right\}$ and $E(H)=$ $\left\{v_{i}^{p} v_{i+1}^{p}, v_{j}^{p} v_{2 t+2-j}^{p}, v_{1}^{p} v_{t}^{p}, v_{1}^{p} v_{3}^{p}, v_{1}^{p} v_{t-1}^{p}\right\}$ for $1 \leq i \leq 2 t-1$ and $2 \leq j \leq t$, and take
 corresponding edges of $H$. Construct a graph $G$ with $n$ vertices similar to Case 1 and Case 2, then $e(G) \leq \ell\left\lceil\frac{n-1}{2 t-1}\right\rceil=\ell\left\lceil\frac{3(n-1)}{2 \ell-7}\right\rceil$. Similarly, since the graph constructed in Case 2 is a subgraph of the corresponding graph in Case 3, it follows that, by Lemma 4, every three vertices in Case 3 have a rainbow tree connecting them.

Combining the above three cases, we get $t(n, 3, \ell) \leq \ell\left\lceil\frac{3 n-3}{2 \ell-3}\right\rceil$.

Remark 18. For $9 \leq \ell \leq\left\lfloor\frac{n+1}{2}\right\rfloor$, the upper bound in Theorem 17 is better than the one in [9].

For small $\ell$, Liu [9] just got exact values of $t(n, 3,3)$ for $n=3,4,5$. Here we get upper bounds for $t(n, 3,3)$ when $n \geq 6$.
Theorem 19. For an integer $n$ with $n \geq 6, t(n, 3,3) \leq \frac{n^{2}+2 n-3}{4}$.
Proof. We consider two cases according to whether $n$ is even or $n$ is odd.


Figure 3. Graphs for Theorem 17.

Case 1. $n$ is even. Let $n=2 k$ for some integer $k \geq 3$. Let $G$ be a regular complete bipartite graph $K_{k, k}$. Then $e(G)=\frac{n^{2}}{4}$. By Theorem $6, r x_{3}(G)=3$.

Case 2. $n$ is odd. Let $n=2 k+1$ for some integer $k \geq 3$. Let $G$ be a graph with $V(G)=\left\{u_{1}, u_{2}, \ldots, u_{k}, w_{1}, w_{2}, \ldots, w_{k}, x\right\}$ and $E(G)=\left\{u_{i} w_{j}, u_{i} x, w_{i} x\right\}$ for $1 \leq i, j \leq k$. It is easy to get $e(G)=\frac{n^{2}+2 n-3}{4}$. Define an edge-coloring $c$ : $E(G) \rightarrow\{1,2,3\}$ as follows

$$
c(e)= \begin{cases}1, & \text { if } e=u_{i} w_{j} \text { or } e=u_{i} x, 1 \leq i=j \leq k \\ 2, & \text { if } e=u_{i} w_{j} \text { or } e=w_{i} x, 1 \leq i<j \leq k \\ 3, & \text { if } e=u_{i} w_{j}, 1 \leq j<i \leq k\end{cases}
$$

Now we show that $c$ is a 3 -rainbow coloring of $G$. Let $S$ be a set of three vertices of $G$. By Case 1, we need to consider three possibilities when $S$ contain $x$. If $S=\left\{x, u_{i}, u_{j}\right\}$, where $i<j$, then $T=\left\{u_{i} w_{i}, u_{j} w_{i}, w_{i} x\right\}$ is a rainbow $S$-tree; if $S=\left\{x, w_{i}, w_{j}\right\}$, where $i<j$, then $T=\left\{u_{j} w_{i}, u_{j} w_{j}, w_{j} x\right\}$ is a rainbow $S$-tree; if $S=\left\{x, u_{i}, w_{j}\right\}$, then $T=\left\{u_{i} x, w_{j} x\right\}$ is a rainbow $S$-tree. Therefore, $r x_{3}(G) \leq 3$.

Combining the above two cases, we have that $t(n, 3,3) \leq \frac{n^{2}+2 n-3}{4}$.

Theorem 20. For an integer $n \geq 8, t(n, 3,4) \leq \frac{n^{2}+22 n+11}{8}$.

Proof. We consider four cases, according to $n \equiv n^{\prime}(\bmod 4)$.
Case 1. $n^{\prime}=0$. Let $n=4 k$ for some integer $k \geq 2$. Let $G_{1}$ be a graph with $V\left(G_{1}\right)=U_{1} \cup U_{2} \cup U_{3} \cup U_{4}$ (where $U_{i}=\left\{u_{i}^{1}, u_{i}^{2}, \ldots, u_{i}^{k}\right\}$ ) and $E\left(G_{1}\right)=$ $\left\{u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}\right\}$ for $1 \leq i, j \leq k$. It is easy to get $e\left(G_{1}\right)=2 k^{2}+2 k=$ $\frac{n^{2}+4 n}{8}$. See $G_{g}$ in Figure 4 for an example with $n=8$. We define an edge-coloring
$c_{1}: E\left(G_{1}\right) \rightarrow\{1,2,3,4\}$ as follows

$$
c_{1}(e)= \begin{cases}1, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, 1 \leq i=j \leq k, \\ 2, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, 1 \leq i<j \leq k, \\ 3, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, 1 \leq j<i \leq k, \\ 4, & \text { if } e=u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, 1 \leq i \leq k\end{cases}
$$

Now we show that $c_{1}$ is a 4 -rainbow coloring of $G_{1}$. Let $S=\{x, y, z\}$ be a set of three vertices of $G_{1}$. Since the case when $S \in U_{1} \cup U_{2}$ or $U_{3} \cup U_{4}$ have been proved in Theorem 6, by symmetry, we need to consider the following three possibilities by the positions of $x, y, z$. If $x, y \in U_{1}, z \in U_{3}$, say $x=u_{1}^{i}, y=$ $u_{1}^{j}, z=u_{3}^{k}$, then there is a rainbow $\left\{x, y, u_{1}^{k}\right\}$-tree $T^{\prime}$ in $U_{1} \cup U_{2}$ and $T=T^{\prime} \cup u_{1}^{k} z$ is a rainbow $S$-tree. If $x, y \in U_{1}, z \in U_{4}$, say $x=u_{1}^{i}, y=u_{1}^{j}, z=u_{4}^{k}$, then there is a rainbow $\left\{x, y, u_{2}^{k}\right\}$-tree $T^{\prime}$ in $U_{1} \cup U_{2}$ and $T=T^{\prime} \cup u_{2}^{k} z$ is a rainbow $S$-tree. If $x \in U_{1}, y \in U_{2}, z \in U_{3}$, say $x=u_{1}^{i}, y=u_{2}^{j}, z=u_{3}^{k}$, then there is a rainbow $\left\{x, y, u_{1}^{k}\right\}$-tree $T^{\prime}$ in $U_{1} \cup U_{2}$ and $T=T^{\prime} \cup u_{1}^{k} z$ is a rainbow $S$-tree. Therefore, $r x_{3}\left(G_{1}\right) \leq 4$.

Case 2. $n^{\prime}=1$. Let $n=4 k+1$ for some integer $k \geq 2$. Let $G_{2}$ be a graph with $V\left(G_{2}\right)=U_{1} \cup U_{2} \cup U_{3} \cup U_{4} \cup\{s\}$ (where $U_{i}=\left\{u_{i}^{1}, u_{i}^{2}, \ldots, u_{i}^{k}\right\}$ ) and $E\left(G_{2}\right)=\left\{u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{1}^{i}, s u_{2}^{i}, s u_{3}^{i}, s u_{4}^{i}\right\}$ for $1 \leq i, j \leq k$. It is easy to get $e\left(G_{2}\right)=2 k^{2}+6 k=\frac{n^{2}+10 n-11}{8}$. See $G_{h}$ in Figure 4 for an example with $n=9$. Based on the coloring $c_{1}$ in Case 1 , we define an edge-coloring $c_{2}$ : $E\left(G_{2}\right) \rightarrow\{1,2,3,4\}$ as follows

$$
c_{2}(e)= \begin{cases}1, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{1}^{i}, 1 \leq i=j \leq k, \\ 2, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{2}^{i}, 1 \leq i<j \leq k, \\ 3, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{3}^{i}, 1 \leq j<i \leq k, \\ 4, & \text { if } e=u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{4}^{i}, 1 \leq i \leq k\end{cases}
$$

Now we show that $c_{2}$ is a 4 -rainbow coloring of $G_{2}$. Let $S=\{x, y, z\}$ be a set of three vertices of $G$. Now we need to consider the subcases when $S$ contain $s$ since other subcases have been discussed in Case 1. Set $s=z$. We consider the following five possibilities by the positions of $x, y$. If $x, y \in U_{1}$, say $y=u_{1}^{i}$, then $T=\left\{x s, y u_{3}^{i}, u_{3}^{i} s\right\}$ is a rainbow $S$-tree; if $x, y \in U_{2}$, say $y=u_{2}^{i}$, then $T=\left\{x u_{1}^{i}, u_{1}^{i} s, s y\right\}$ is a rainbow $S$-tree; if $x, y \in U_{3}$, say $x=u_{3}^{i}$, then $T=\left\{x u_{1}^{i}, u_{1}^{i} s, s y\right\}$ is a rainbow $S$-tree; if $x, y \in U_{4}$, say $x=u_{4}^{i}$, then $T=\left\{x u_{3}^{i}, u_{3}^{i} s, s y\right\}$ is a rainbow $S$-tree; if $x \in U_{i}$ and $y \in U_{j}(i \neq j)$, then $T=\{x s, s y\}$ is a rainbow $S$-tree. Therefore, $r x_{3}\left(G_{2}\right) \leq 4$.

Case 3. $n^{\prime}=2$. Let $n=4 k+2$ for some integer $k \geq 2$. Let $G_{3}$ be a graph with $V\left(G_{3}\right)=U_{1} \cup U_{2} \cup U_{3} \cup U_{4} \cup\{s, t\}$ (where $U_{i}=\left\{u_{i}^{1}, u_{i}^{2}, \ldots, u_{i}^{k}\right\}$ ) and $E\left(G_{3}\right)=$


Figure 4. Graphs for Theorem 20.
$\left\{u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{1}^{i}, s u_{2}^{i}, s u_{3}^{i}, s u_{4}^{i}, t u_{1}^{i}, t u_{2}^{i}, t u_{3}^{i}, t u_{4}^{i}\right\}$ for $1 \leq i, j \leq k$. It is easy to get $e\left(G_{3}\right)=2 k^{2}+10 k=\frac{n^{2}+16 n-36}{8}$. Based on the coloring $c_{2}$ in Case 2, we define an edge-coloring $c_{3}: E\left(G_{3}\right) \rightarrow\{1,2,3,4\}$ as follows

$$
c_{3}(e)= \begin{cases}1, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{1}^{i}, t u_{1}^{i}, 1 \leq i=j \leq k, \\ 2, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{2}^{i}, t u_{1}^{i}, 1 \leq i<j \leq k, \\ 3, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{3}^{i}, t u_{1}^{i}, 1 \leq j<i \leq k, \\ 4, & \text { if } e=u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{4}^{i}, t u_{1}^{i}, 1 \leq i \leq k .\end{cases}
$$

Now we show that $c_{3}$ is a 4 -rainbow coloring of $G_{3}$. Let $S=\{x, y, z\}$ be a set of three vertices of $G$. Based on the discussion in Case 2, we need to consider the subcase when $S$ contains $s$ and $t$. Set $s=y$ and $t=z$. We consider the following four possibilities by the positions of $x$. If $x \in U_{1}$, say $x=u_{1}^{i}$, then $T=\left\{s u_{3}^{i}, u_{3}^{i} u_{1}^{i}, u_{1}^{i} t\right\}$ is a rainbow $S$-tree; if $x \in U_{2}$, say $x=u_{2}^{i}$, then $T=\left\{s u_{3}^{i}, u_{3}^{i} u_{4}^{i}, u_{4}^{i} x, x t\right\}$ is a rainbow $S$-tree; if $x \in U_{3}$, say $x=u_{3}^{i}$, then $T=$ $\left\{s x, x u_{4}^{i}, u_{4}^{i} t\right\}$ is a rainbow $S$-tree; if $x \in U_{4}$, say $x=u_{4}^{i}$, then $T=\left\{s u_{3}^{i}, u_{3}^{i} x, x t\right\}$ is a rainbow $S$-tree. Therefore, $r x_{3}\left(G_{3}\right) \leq 4$.

Case 4. $n^{\prime}=3$. Let $n=4 k+3$ for some integer $k \geq 2$. Let $G_{4}$ be a graph with $V\left(G_{4}\right)=U_{1} \cup U_{2} \cup U_{3} \cup U_{4} \cup\{s, t, p\}$ (where $U_{i}=\left\{u_{i}^{1}, u_{i}^{2}, \ldots, u_{i}^{k}\right\}$ ) and $E\left(G_{4}\right)=$ $\left\{u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{1}^{i}, s u_{2}^{i}, s u_{3}^{i}, s u_{4}^{i}, t u_{1}^{i}, t u_{2}^{i}, t u_{3}^{i}, t u_{4}^{i}, p u_{1}^{i}, p u_{2}^{i}, p u_{3}^{i}, p u_{4}^{i}, t p\right\}$ for $1 \leq i, j \leq k$. It is easy to get $e\left(G_{4}\right)=2 k^{2}+14 k+1=\frac{n^{2}+22 n+11}{8}$. Based on the edge-coloring $c_{3}$ in Case 3, we define an edge-coloring $c_{4}: E\left(G_{3}\right) \rightarrow\{1,2,3,4\}$
as follows

$$
c_{4}(e)= \begin{cases}1, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{1}^{i}, t u_{1}^{i}, p u_{1}^{i}, 1 \leq i=j \leq k \\ 2, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{2}^{i}, t u_{1}^{i}, p u_{1}^{i} \text { or } e=t p, 1 \leq i<j \leq k \\ 3, & \text { if } e=u_{1}^{i} u_{2}^{j}, u_{3}^{i} u_{4}^{j}, s u_{3}^{i}, t u_{1}^{i}, p u_{1}^{i}, 1 \leq j<i \leq k \\ 4, & \text { if } e=u_{1}^{i} u_{3}^{i}, u_{2}^{i} u_{4}^{i}, s u_{4}^{i}, t u_{1}^{i}, p u_{1}^{i}, 1 \leq i \leq k\end{cases}
$$

Now we show that $c_{4}$ is a 4-rainbow coloring of $G_{4}$. Let $S=\{x, y, z\}$ be a set of three vertices of $G$. Now we need to consider the subcase when $S=\{s, t, p\}$. Here $T=\left\{s u_{1}^{1}, u_{1}^{1} u_{3}^{1}, u_{3}^{1} t, t p\right\}$ is a rainbow $S$-tree. Therefore, $r x_{3}\left(G_{4}\right) \leq 4$.

Combining the above four cases, we have $t(n, 3,4) \leq \frac{n^{2}+22 n+11}{8}$.
Remark 21. The upper bound in Theorem 20 is better than $t(n, 3,4) \leq\binom{ n}{2}-$ $n+1$, which is got in [9].

Theorem 22. For an integer $n \geq 6, t(n, 3,5) \leq 2 n-3$.
Proof. Let $G$ be a graph with $V(G)=\left\{v, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ and $E(G)=\left\{v v_{i}\right.$, $\left.v_{j} v_{j+1}\right\}$ for $1 \leq i \leq n-1,1 \leq j \leq n-2$. It is easy to get $e(G)=2 n-3$. Define an edge-coloring $c: E(G) \rightarrow\{1,2,3,4,5\}$ as follows

$$
c(e)= \begin{cases}i, & \text { if } e=v v_{j}, 1 \leq i \leq 5,1 \leq j \leq n-1, j=i(\bmod 5) \\ i, & \text { if } e=v_{j} v_{j+1}, 1 \leq i \leq 5,1 \leq j \leq n-2, j+3=i(\bmod 5)\end{cases}
$$

It is easy to show that $c$ is a 5 -rainbow coloring of $G$. Thus $r x_{3}(G) \leq 5$, it follows that $t(n, 3,5) \leq 2 n-3$.

Remark 23. The result in Theorem 22 is better than $t(n, 3,5) \leq 2 n-2$, which is got in [9].

Theorem 24. For an integer $n \geq 7, t(n, 3,6) \leq 2 n-6$.
Proof. We consider three cases.
Case 1. $n=3 t$. Let $G_{1}$ be a graph by taking $t-2$ vertex-disjoint cliques of order 4 and 5 vertex-disjoint $K_{2}$, and identifying a vertex from each of them. That is, $G_{1}$ is a graph with $V\left(G_{1}\right)=\left\{v, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ and $E\left(G_{1}\right)=\left\{v v_{i}, v_{j} v_{j+1}\right.$, $\left.v_{j} v_{j+2}, v_{k} v_{k+1}\right\}$ for $1 \leq i \leq n-1, j=1(\bmod 3), k=2(\bmod 3), 1 \leq j, k \leq$ $3(t-2)$. It is easy to get $e\left(G_{1}\right)=2 n-7$. Define an edge-coloring $c_{1}: E\left(G_{1}\right) \rightarrow$ $\{1,2,3,4,5,6\}$ as follows

$$
c_{1}(e)= \begin{cases}1, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-5}, 1 \leq i \leq n-6, i=1(\bmod 3), \\ 2, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-4}, 1 \leq i \leq n-6, i=2(\bmod 3), \\ 3, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-3}, 1 \leq i \leq n-6, i=3(\bmod 3), \\ 4, & \text { if } e=v_{i} v_{i+1} \text { or } e=v v_{3 t-2}, i=1(\bmod 3), \\ 5, & \text { if } e=v_{i} v_{i+2} \text { or } e=v v_{3 t-1}, i=1(\bmod 3), \\ 6, & \text { if } e=v_{i} v_{i+1}, i=2(\bmod 3) .\end{cases}
$$

It is easy to show that $c_{1}$ is a 6-rainbow coloring of $G_{1}$, thus $r x_{3}\left(G_{1}\right) \leq 6$.
Case 2. $n=3 t+1$. Let $G_{2}$ be a graph with $V\left(G_{2}\right)=\left\{v, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ and $E\left(G_{2}\right)=\left\{v v_{i}, v_{j} v_{j+1}, v_{j} v_{j+2}, v_{k} v_{k+1}\right\}$ for $1 \leq i \leq n-1, j=1(\bmod 3)$, $k=2(\bmod 3), 1 \leq j, k \leq 3(t-2)$. It is easy to get $e\left(G_{2}\right)=2 n-8$. Define an edge-coloring $c_{2}: E\left(G_{2}\right) \rightarrow\{1,2,3,4,5,6\}$ as follows:

$$
c_{2}(e)= \begin{cases}1, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-5}, 1 \leq i \leq n-7, i=1(\bmod 3), \\ 2, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-4}, 1 \leq i \leq n-7, i=2(\bmod 3), \\ 3, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-3}, 1 \leq i \leq n-7, i=3(\bmod 3), \\ 4, & \text { if } e=v_{i} v_{i+1} \text { or } e=v v_{3 t-2}, i=1(\bmod 3) \\ 5, & \text { if } e=v_{i} v_{i+2} \text { or } e=v v_{3 t-1}, i=1(\bmod 3) \\ 6, & \text { if } e=v_{i} v_{i+1} \text { or } e=v v_{3 t}, i=2(\bmod 3)\end{cases}
$$

It is easy to show that $c_{2}$ is a 6 -rainbow coloring of $G_{2}$, thus $r x_{3}\left(G_{2}\right) \leq 6$.
Case 3. $n=3 t+2$. Let $G_{3}$ be a graph with $V\left(G_{3}\right)=\left\{v, v_{1}, v_{2}, \ldots, v_{n-1}\right\}$ and $E\left(G_{3}\right)=\left\{v v_{i}, v_{j} v_{j+1}, v_{j} v_{j+2}, v_{k} v_{k+1}\right\}$ for $1 \leq i \leq n-1, j=1(\bmod 3)$, $k=2(\bmod 3), 1 \leq j, k \leq 3(t-1)$. It is easy to get $e\left(G_{3}\right)=2 n-6$. Define an edge-coloring $c_{3}: E\left(G_{3}\right) \rightarrow\{1,2,3,4,5,6\}$ as follows

$$
c_{3}(e)= \begin{cases}1, & \text { if } e=v v_{i}, 1 \leq i \leq n-5, i=1(\bmod 3) \\ 2, & \text { if } e=v v_{i}, 1 \leq i \leq n-5, i=2(\bmod 3), \\ 3, & \text { if } e=v v_{i} \text { or } e=v v_{3 t-2}, 1 \leq i \leq n-5, i=3(\bmod 3), \\ 4, & \text { if } e=v_{i} v_{i+1} \text { or } e=v v_{3 t-1}, i=1(\bmod 3) \\ 5, & \text { if } e=v_{i} v_{i+2} \text { or } e=v v_{3 t}, i=1(\bmod 3), \\ 6, & \text { if } e=v_{i} v_{i+1} \text { or } e=v v_{3 t+1}, i=2(\bmod 3)\end{cases}
$$

It is easy to show that $c_{3}$ is a 6 -rainbow coloring of $G_{3}$, thus $r x_{3}\left(G_{3}\right) \leq 6$. Combining the above three cases, we have $t(n, 3,6) \leq 2 n-6$.

Remark 25. The upper bound in Theorem 24 is better than $2 n-3$, which is got in [9].

Theorem 26. For an integer $n \geq 8, t(n, 3,7) \leq 2 n-7$.
Proof. We consider three cases.
Case 1. $n=3 t+1$. Set $n^{\prime}=n-1=3 t$. Construct a graph $G_{1}$ as in Theorem 24 and let $G$ be a graph obtained from $G_{1}$ by adding an edge colored by 7 . It is easy to see that $G$ is 3 -rainbow connected and $e(G)=\left(2 n^{\prime}-7\right)+1=2 n-8$.

Case 2. $n=3 t+2$. Set $n^{\prime}=n-1=3 t+1$. Construct a graph $G_{2}$ as in Theorem 24 and let $G$ be a graph obtained from $G_{2}$ by adding an edge colored by 7. It is easy to see that $G$ is 3 -rainbow connected and $e(G)=\left(2 n^{\prime}-8\right)+1=2 n-9$.

Case 3. $n=3 t+3$. Set $n^{\prime}=n-1=3 t+2$. Construct a graph $G_{3}$ as in Theorem 24 and let $G$ be a graph obtained from $G_{3}$ by adding an edge colored by 7. It is easy to see that $G$ is 3 -rainbow connected and $e(G)=\left(2 n^{\prime}-6\right)+1=2 n-7$.

Combining the above three cases, we have $t(n, 3,7) \leq 2 n-7$.
Theorem 27. For an integer $n \geq 9, t(n, 3,8) \leq 2 n-2$.
Proof. Let $H^{*}$ be a connected rainbow graph with 5 vertices and 8 edges, where $V\left(H^{*}\right)=\left\{v_{1}, v_{2}, \ldots, v_{5}\right\}$ and $E\left(H^{*}\right)=\left\{v_{i} v_{i+1}, v_{1} v_{3}, v_{1} v_{4}, v_{1} v_{5}, v_{2} v_{5}\right\}$ for $1 \leq$ $i \leq 4$. Take $\left\lfloor\frac{n-1}{4}\right\rfloor$ copies of $H^{*}$ and denote them by $H^{1}, H^{2}, \ldots, H^{\left\lfloor\frac{n-1}{4}\right\rfloor}$ with $V\left(H^{p}\right)=\left\{v_{1}^{p}, v_{2}^{p}, \ldots, v_{5}^{p}\right\}$ and $E\left(H^{p}\right)=\left\{v_{i}^{p} v_{i+1}^{p}, v_{1}^{p} v_{3}^{p}, v_{1}^{p} v_{4}^{p}, v_{1}^{p} v_{5}^{p}, v_{2}^{p} v_{5}^{p}\right\}$ for $1 \leq$ $i \leq 4,1 \leq p \leq\left\lfloor\frac{n-1}{4}\right\rfloor$, and take a subgraph graph of $H^{*}$, denoted by $H^{\left\lceil\frac{n-1}{4}\right\rceil}$, with $n-4\left\lfloor\frac{n-1}{4}\right\rfloor$ vertices and corresponding edges of $H^{*}$. Let $G$ be a graph with $n$ vertices by identifying the vertex $v_{1}^{p}\left(1 \leq p \leq\left\lfloor\frac{n-1}{4}\right\rfloor\right)$ and $v_{1}^{\left\lceil\frac{n-1}{4}\right\rceil}$ if $H^{\left\lceil\frac{n-1}{4}\right\rceil}$ exists. Clearly, $e(G) \leq 2 n-2$. Similar to the discussion in Theorem 17, it is shown that $G$ is 3 -rainbow connected. Thus $t(n, 3,8) \leq 2 n-2$.

## 5. Summary

In Section 3, we get the exact values of $t(n, k, n-1)$ and $t(n, k, n-2)$ for $3 \leq$ $k \leq n-1$. In Section 4, the exact values of $t(n, 3, n-3)$ and $t(n, 3, n-4)$ are obtained. In other cases for $k=3$, the upper bounds we got are better than the ones in [9], but they are not tight. In fact, it is challenging to get the exact values of $t(n, k, \ell)$ for all cases. We will continue to focus on this problem in the future.

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