

AN IMPROVED UPPER BOUND ON NEIGHBOR EXPANDED SUM DISTINGUISHING INDEX

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Abstract

A total k -weighting f of a graph G is an assignment of integers from the set $\{1, \dots, k\}$ to the vertices and edges of G . We say that f is neighbor expanded sum distinguishing, or NESD for short, if $\sum_{w \in N(v)} (f(vw) + f(w))$ differs from $\sum_{w \in N(u)} (f(uw) + f(w))$ for every two adjacent vertices v and u of G . The neighbor expanded sum distinguishing index of G , denoted by $\text{egndi}_\Sigma(G)$, is the minimum positive integer k for which there exists an NESD weighting of G . An NESD weighting was introduced and investigated by Flandrin *et al.* (2017), where they conjectured that $\text{egndi}_\Sigma(G) \leq 2$ for any graph G . They examined some special classes of graphs, while proving that $\text{egndi}_\Sigma(G) \leq \chi(G) + 1$. We improve this bound and show that $\text{egndi}_\Sigma(G) \leq 3$ for any graph G . We also show that the conjecture holds for all bipartite, 3-regular and 4-regular graphs.

Keywords: general edge coloring, total coloring, neighbor sum distinguishing index.

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1. INTRODUCTION

All graphs considered in this paper are finite, simple and undirected. Let $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ and $\chi(G)$ denote the vertex set, the edge set, the maximum degree, the minimum degree, and the chromatic number of a graph G , respectively. Let $N_G(v)$ and $\deg_G(v)$ denote the set of neighbors and the degree of a vertex v in G , respectively. For all other terminology used in this paper, we refer the reader to [1].

A total k -weighting f of a graph G is an assignment of integers from the set $\{1, \dots, k\}$ to the vertices and edges of G . We say that f is neighbor expanded sum distinguishing, or NESD for short, if

$$(1) \quad \sigma(v) = \sum_{u \in N(v)} (f(uv) + f(u))$$

yields a proper vertex coloring of G . The minimum positive integer k for which an NESD weighting of a graph G exists is called the neighbor expanded sum distinguishing index of G and denoted by $\text{egndi}_{\Sigma}(G)$. An NESD weighting was introduced and investigated by Flandrin *et al.* in [2], where they proposed the following conjecture.

Conjecture 1. *For any graph G , $\text{egndi}_{\Sigma}(G) \leq 2$.*

They examined some classes of graphs, including paths, cycles, complete graphs and trees, and proved a relaxed upper bound $\text{egndi}_{\Sigma}(G) \leq \chi(G) + 1$ for any graph G . Our main result is an improvement over this bound stated in the next theorem.

Theorem 2. *For any graph G , $\text{egndi}_{\Sigma}(G) \leq 3$.*

Proofs of our theorems are deferred to the Section 2. Flandrin *et al.* [2] proved the following theorem.

Theorem 3. *Let $G = (X, Y, E)$ be a connected bipartite graph. If any of the bipartite sets X and Y has an even number of vertices, or there is a vertex of odd degree in G , then $\text{egndi}_{\Sigma}(G) \leq 2$.*

In our next theorem we extend their claim to any bipartite graph.

Theorem 4. *If G is a bipartite graph, then $\text{egndi}_{\Sigma}(G) \leq 2$*

Furthermore, we show that for any 3-regular and any 4-regular graph there exists an NESD weighting using only the weights 1 and 2.

Theorem 5. *If G is a 3-regular or a 4-regular graph, then $\text{egndi}_{\Sigma}(G) \leq 2$.*

Before we proceed with proofs of Theorems 2, 4 and 5, we will mention a closely related variation of weighting. An edge k -weighting f of a graph G is called neighbor sum distinguishing, or NSD for short, if a function $\phi(v) = \sum_{e \ni v} f(e)$ yields a proper vertex coloring of G . The minimum positive integer k for which such a weighting exists is the neighbor sum distinguishing index of G , denoted by $\text{gndi}_{\Sigma}(G)$. The following problem, also known as the 1-2-3 conjecture, was proposed by Karonski *et al.* [3], and it received a significant attention.

Conjecture 6. *For any graph G without isolated edges, $\text{gndi}_{\Sigma}(G) \leq 3$.*

Kalkowski *et al.* [4] proved that $\text{gndi}_{\Sigma}(G) \leq 5$ for any graph G without isolated edges, which is by now the best known bound for $\text{gndi}_{\Sigma}(G)$. In the proof of Theorem 2 we use an approach similar to that used in [4].

2. PROOFS OF OUR THEOREMS

Proof of Theorem 2. If the statement holds for any connected graph, an immediate consequence is that it also holds for any disconnected graph. Thus we may assume that G is a connected graph.

We prove a slightly stronger claim, that is, there exists a total 3-weighting f such that $\left\lfloor \frac{\sigma(v)}{2} \right\rfloor \neq \left\lfloor \frac{\sigma(u)}{2} \right\rfloor$ for any two adjacent vertices v and u of G . Let $V = V(G) = \{v_1, \dots, v_k\}$ be the set of vertices of G arranged in an arbitrary order. Let $E = E(G)$ be the set of edges of G and $f: E \cup V \rightarrow \{1, 2, 3\}$. For $1 \leq j \leq k$, let $S_1(V, E, f, j)$ and $S_2(V, E, f, j)$ denote the following two statements:

$$S_1(V, E, f, j): \left\lfloor \frac{\sigma(v_i)}{2} \right\rfloor \neq \left\lfloor \frac{\sigma(v_l)}{2} \right\rfloor \text{ for every } v_i v_l \in E \text{ and } 1 \leq i < l \leq j.$$

$$S_2(V, E, f, j): f(v_i) = 2, \text{ and } f(v_i u) = 2 \text{ for every } v_i \in V, j < i \leq k, \text{ and } v_i u \in E.$$

We start by assigning the weight 2 to each vertex of V and each edge of E . Hence $S_1(V, E, f, 1)$ and $S_2(V, E, f, 1)$ are trivially true. Suppose that $S_1(V, E, f, j)$ and $S_2(V, E, f, j)$ hold for some weighting f and integer j with $1 \leq j < k$. Let d denote the value of $\sigma(v_{j+1})$ for the current weighting f . Let $U = \{u_1, \dots, u_n\}$ be the subset of vertices from $\{v_1, \dots, v_j\}$ that are adjacent to v_{j+1} in G . Next, let U_o and U_e be the subsets of U , with $U_o \cup U_e = U$, such that $\sigma(v)$ is odd for every $v \in U_o$, and $\sigma(u)$ is even for every $u \in U_e$. Let $n_o = |U_o|$ and $n_e = |U_e|$, thus $n_o + n_e = n$. We now consider possible adjustments of $f(v_{j+1})$ and $f(u_l v_{j+1})$ for $u_l \in U$, where both $S_1(V, E, f, j)$ and $S_2(V, E, f, j+1)$ remain satisfied.

1. Since $f(u_l v_{j+1}) = 2$ for every $l \in \{1, \dots, n\}$, we can increase by 1 the weight of any of the n_e edges joining v_{j+1} with the vertices of U_e , while keeping $S_1(V, E, f, j)$ satisfied. Similarly, we can decrease by 1 the weight of any of the n_o edges joining v_{j+1} with the vertices of U_o . Hence we can adjust the weights of these edges to obtain that $\sigma(v_{j+1})$ equals any integer value from $[d - n_o, d + n_e]$.

2. By changing $f(v_{j+1})$ to 1, and $f(u_l v_{j+1})$ to 3, for every $l \in \{1, \dots, n\}$, the value of $\sigma(v_{j+1})$ increases by n , while $\sigma(u_l)$ remains the same for every $l \in \{1, \dots, n\}$. Now, we can decrease by 1 the weight of any of the n_o edges joining v_{j+1} with the vertices of U_o , while keeping $S_1(V, E, f, j)$ satisfied. Hence we can achieve that $\sigma(v_{j+1})$ equals any integer value from $[d + n_e, d + n]$.

3. By changing $f(v_{j+1})$ to 3 and $f(u_l v_{j+1})$ to 1, for every $l \in \{1, \dots, n\}$, the value of $\sigma(v_{j+1})$ decreases by n , while $\sigma(u_l)$ remains the same for every $l \in \{1, \dots, n\}$. Now, similarly to the previous case, we can achieve that $\sigma(v_{j+1})$ equals any integer value from $[d - n, d - n_o]$.

Therefore, we can make adjustments to the weights of v_{j+1} and $u_l v_{j+1}$, with $u_l \in U$, so that $\sigma(v_{j+1})$ equals any integer value from $[d-n, d+n]$, while keeping $S_1(V, E, f, j)$ and $S_2(V, E, f, j+1)$ satisfied. Since there are n neighboring vertices of v_{j+1} preceding it, and there are $2n+1$ reachable values for $\sigma(v_{j+1})$, we can adjust the weights of v_{j+1} and $u_l v_{j+1}$, with $1 \leq l \leq n$, so that $S_1(V, E, f, j+1)$ holds. In the above described procedure we do not change the weight of any vertex $v_l \in V$, with $l > j+1$, nor the weight of any edge incident with v_l . Thus $S_2(V, E, f, j+1)$ holds. Continuing in this manner until $j+1 = k$, we obtain a desired weighting. ■

In the proof of Theorem 4 we follow the idea used in [3], and later implemented in the proof of Theorem 3.

Proof of Theorem 4. As stated in the proof of Theorem 2, it suffices to show that the theorem holds for every connected graph. Thus we may assume that G is a connected graph. If X or Y have an even number of vertices, or there exists a vertex of odd degree in G , the statement is true by Theorem 3. So we may assume that both X and Y have an odd number of vertices, and every vertex of G has even degree.

Let $X = \{x_1, \dots, x_{2k+1}\}$. We may assume, without loss of generality, that X contains a vertex with degree equal to $\delta(G)$, and that x_{2k+1} is such a vertex. First, we assign the weight 2 to every edge of G and vertex of X , and the weight 1 to every vertex of Y . Since every vertex of G has even degree, after the described assignment $\sigma(v)$ is even for every $v \in X \cup Y$. We now change the weights of some edges of G so that $\sigma(x_i)$ becomes odd for every $i \in \{1, 2k\}$, while $\sigma(y)$ remains even for every $y \in Y$. We subsequently prove that $\sigma(x_{2k+1}) < \sigma(y)$ for every $y \in N_G(x_{2k+1})$, which implies the statement of the theorem.

Let P_i denote a path from x_i to x_{i+k} , for every integer i with $1 \leq i \leq k$. For each path P_i , with $1 \leq i \leq k$, we change the weight of every edge on the path, that is, from 1 to 2, and from 2 to 1. This way the parity of $\sigma(u)$ stays the same for every vertex u on P_i different from x_i and x_{i+k} . After this procedure the value of $\sigma(x_i)$ is odd for every integer i with $1 \leq i \leq 2k$, while $\sigma(y)$ is even for every $y \in Y$. Next, let $y \in Y$ be an arbitrary neighbor of x_{2k+1} . Let $d = \deg(x_{2k+1})$. Since $d = \delta(G)$, we have $d \leq \deg(y)$. Since $f(u) = 1$ for every $u \in N_G(x_{2k+1})$, we have $\sigma(x_{2k+1}) \leq 3(d-1) + f(x_{2k+1}y) + 1$. On the other hand, since $f(u) = 2$ for every $u \in N_G(y)$, we have $\sigma(y) \geq 3(d-1) + f(x_{2k+1}y) + 2$. Therefore, $\sigma(x_{2k+1}) < \sigma(y)$ for every $y \in N_G(x_{2k+1})$, completing the proof. ■

The proof of Theorem 5 is organized as follows. We start from some proper vertex coloring c of G . Then we define a total weighting f of G by assigning a weight from the set $\{1, 2\}$ to every vertex v of G depending on the value of $c(v)$,

and to every edge uv of G depending on the values of $c(u)$ and $c(v)$. Finally, we adjust some of these weights and prove that such a weighting is NESD.

Proof of Theorem 5. Let G be a k -regular graph with $k \in \{3, 4\}$. Flandrin *et al.* proved in [2] that $\text{egndi}_\Sigma(G) = 2$ for any complete graph G . Thus we may assume that G is not a complete graph. Then $\chi(G) \leq k$ according to Brooks' Theorem [5]. For $1 \leq i \leq k$, let V_i be the color classes of $V(G)$. We may assume that every vertex v of V_j , with $1 < j \leq k$, has at least one neighbor in every V_i , with $1 \leq i < j$. Otherwise, while there exists $v \in V_j$ that has no neighbor in V_i , $1 \leq i < j$, we move v to V_i . This way sets V_i remain independent for every $1 \leq i \leq k$, hence c remains a proper coloring with k colors.

First, we prove the case when G is a 3-regular graph. As observed above, we may assume that $\chi(G) \leq 3$. Let c be any proper vertex coloring of G with colors from $\{1, 2, 3\}$. As noted earlier, we may assume that every vertex colored j , $1 < j \leq 3$, has a neighbor colored i for every $1 \leq i < j$. We now define a total 2-weighting f of G . We assign the weight 1 to every $v \in V_i$ with $i \in \{1, 3\}$, and the weight 2 to every $v \in V_2$. For every two adjacent vertices u and v of G , if $c(u) = 3$ or $c(v) = 3$, then we assign 2 to $f(uv)$; otherwise we assign 1 to $f(uv)$. For every $v \in V_j$, the vertex v has a neighbor in every V_i with $1 \leq i < j$, so we have the following:

1. for $j = 1$, we have $\sigma(v) = 9$,
2. for $j = 2$, we have $\sigma(v) < 9$,
3. for $j = 3$, we have $\sigma(v) > 9$.

Because V_j is an independent set for every $j \in \{1, 2, 3\}$, it follows that $\sigma(v) \neq \sigma(u)$ for every two adjacent vertices v and u of G , completing the first part of the proof.

We now prove that the statement holds for every 4-regular graph G . As before, we may assume that $\chi(G) \leq 4$. Let c be an arbitrary proper vertex coloring of G with colors from $\{1, 2, 3, 4\}$. Again, we may assume that every $v \in V_j$, with $1 < j \leq 4$, has at least one neighbor in every V_i , with $1 \leq i < j$. We now define a total 2-weighting f . We assign the weight 1 to every vertex of V_1 and V_3 , and the weight 2 to every vertex of V_2 and V_4 . We assign the weight 1 to edges joining the vertices of V_1 with the vertices of V_2 , and also to edges incident with the vertices of V_4 , while to all other edges we assign the weight 2. Now, for every $v \in V_j$, the vertex v has a neighbor in every V_i with $1 \leq i < j$, so we have the following:

1. for $j = 1$, we have $\sigma(v) = 12$,
2. for $j = 2$, we have $\sigma(v) < 12$,
3. for $j = 3$, we have $\sigma(v) > 12$,

4. for $j = 4$, we have $\sigma(v) \in \{9, 10\}$.

Clearly, for any two adjacent vertices v and u of G , the values of $\sigma(v)$ and $\sigma(u)$ may be equal only when one of these two vertices is from V_2 and the other is from V_4 . We now change the weights of some of the vertices and edges of G to obtain an NESD coloring.

First, for every $v \in V_4$ with $\sigma(v) = 10$, we do the following. Since $\sigma(v) = 10$, the vertex v has exactly one neighbor in both V_1 and V_3 , and two neighbors in V_2 . Denote by $\{v_1, v'_2, v''_2, v_3\}$ the neighbors of v , where $v_1 \in V_1$, $v'_2, v''_2 \in V_2$ and $v_3 \in V_3$. We consider two cases, depending on the value of $\sigma(v_3)$.

Case 1. $\sigma(v_3) = 14$. We adjust the weights as follows: $f(v) = 1$, $f(v_1v) = 2$. Now, we have $\sigma(v_1) = 12$, $\sigma(v'_2) \leq 10$, $\sigma(v''_2) \leq 10$, $\sigma(v_3) = 13$ and $\sigma(v) = 11$.

Case 2. $\sigma(v_3) = 13$. We make the following changes: $f(v) = 1$, $f(v_1v) = 2$, $f(v'_2v) = 2$, $f(v''_2v) = 2$, $f(v_3v) = 2$. Hence the values of $\sigma(v_1)$, $\sigma(v'_2)$, $\sigma(v''_2)$ and $\sigma(v_3)$ remain the same, while now $\sigma(v) = 14$. Thus $\sigma(v)$ differs from $\sigma(w)$ for every $w \in N(v)$.

Note that since the value of $\sigma(v_3)$ for $v_3 \in V_3$ never changes to 14, there is no conflict between the two adjustments above.

Next, for every $v \in V_4$ with $\sigma(v) = 9$, we do the following. In this case v has only one neighbor u in V_2 . If $\sigma(u) \neq 9$, then $\sigma(v)$ is different from $\sigma(w)$ for every $w \in N_G(v)$, and we do not need to change any weight. Otherwise, since $f(v) = 2$ and $\sigma(u) = 9$, it follows that u does not have any neighbor in V_3 , and also every edge incident with u has the weight 1. We now change the weight of u to 1, and the weight of every edge incident with u to 2. This way $\sigma(y)$ remains the same for every $y \in N_G(u)$, while the value of $\sigma(u)$ becomes 13. The vertex u has no neighbor in V_3 , while $\sigma(y) \neq 13$ for every $y \in V_4$. Thus $\sigma(u) \neq \sigma(w)$ for every $w \in N_G(u)$.

After the procedure above is finished, we have $\sigma(v) \neq \sigma(u)$ for every two adjacent vertices v and u of G , and the proof is completed. ■

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