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# THE COMPARED COSTS OF DOMINATION LOCATION-DOMINATION AND IDENTIFICATION

Olivier Hudry

*LTCI, Télécom ParisTech*
*Université Paris-Saclay*
*46 rue Barrault, 75634 Paris Cedex 13 – France*

**e-mail:** olivier.hudry@telecom-paristech.fr

and

Antoine Lobstein

*Centre National de la Recherche Scientifique*
*Laboratoire de Recherche en Informatique, UMR 8623*
*Université Paris-Sud, Université Paris-Saclay*
*Bâtiment 650 Ada Lovelace, 91405 Orsay Cedex – France*

**e-mail:** antoine.lobstein@lri.fr

**Abstract**

Let $G = (V, E)$ be a finite graph and $r \geq 1$ be an integer. For $v \in V$, let $B_r(v) = \{x \in V : d(v, x) \leq r\}$ be the ball of radius $r$ centered at $v$. A set $C \subseteq V$ is an $r$-dominating code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$; it is an $r$-locating-dominating code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$, and for any two distinct non-codewords $x \in V \setminus C$, $y \in V \setminus C$, we have $B_r(x) \cap C \neq B_r(y) \cap C$; it is an $r$-identifying code if for all $v \in V$, we have $B_r(v) \cap C \neq \emptyset$, and for any two distinct vertices $x \in V$, $y \in V$, we have $B_r(x) \cap C \neq B_r(y) \cap C$. We denote by $\gamma_r(G)$ (respectively, $ld_r(G)$ and $id_r(G)$) the smallest possible cardinality of an $r$-dominating code (respectively, an $r$-locating-dominating code and an $r$-identifying code). We study how small and how large the three differences $id_r(G) - ld_r(G)$, $id_r(G) - \gamma_r(G)$ and $ld_r(G) - \gamma_r(G)$ can be.

**Keywords:** graph theory, dominating set, locating-dominating code, identifying code, twin-free graph.

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## 1. Introduction

### 1.1. Definitions and notation

For graph theory, we refer to, e.g., [1, 2] or [8]; for the vast topic of domination in graphs, see [13]. For locating-dominating codes, see the first papers [7] and [18], for identifying codes, see the seminal paper [14]; for both, see also the large bibliography at [15].

We shall denote by $G = (V, E)$ a finite, simple, undirected graph with vertex set $V$ and edge set $E$, where an *edge* between $x \in V$ and $y \in V$ is indifferently denoted by $xy$ or $yx$. The *order* of the graph is its number of vertices, $|V|$. Our graphs will generally be connected. The *distance* between two vertices $x \in V$, $y \in V$, will be denoted by $d_G(x, y)$, or $d(x, y)$ when there is no ambiguity.

For an integer $k \geq 2$, the *k-th transitive closure*, or *k-th power*, of $G = (V, E)$ is the graph $G^k = (V, E^k)$ defined by $E^k = \{uv : u \in V, v \in V, 0 < d_G(u, v) \leq k\}$. For a given graph $G^*$, any graph $G$ such that $G^k = G^*$ is called a *k-th root* of $G^*$; such roots do not always exist.

For any integer $r \geq 1$, and for every vertex $x \in V$, we denote by $B_{G,r}(x)$ (and $B_r(x)$ when there is no ambiguity) the *ball of radius $r$ centered at $x$*, i.e., the set of vertices at distance at most $r$ from $x$:

$$B_r(x) = \{y \in V : d(x, y) \leq r\}.$$

Whenever $x \in B_r(y)$ (which is equivalent to $y \in B_r(x)$), we say that $x$ and $y$ *$r$-dominate* or *$r$-cover* each other. A vertex $x \in V$ is said to be *$r$-universal* if it $r$-dominates all the vertices, i.e., if $B_r(x) = V$. When three vertices $x, y, z$ are such that $z \in B_r(x)$ and $z \notin B_r(y)$, we say that $z$ *$r$-sepatares* $x$ and $y$ in $G$ (note that $z = x$ is possible). A set of vertices is said to $r$-separate $x$ and $y$ if at least one of its element does.

Let $C \subseteq V$ be a set of vertices; the set $C$ is called a *code*, and its elements *codewords*.

A code $C$ is said to be an *$r$-dominating set* or an *$r$-dominating code* ($r$-D code for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$. One can also find the terminology *dominating set at distance $r$*, or *distance $r$ dominating set*.

A code $C$ is said to be *$r$-locating-dominating* ($r$-LD for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$, and for any two distinct non-codewords $x \in V \setminus C$, $y \in V \setminus C$, we have $B_r(x) \cap C \neq B_r(y) \cap C$.

A code $C$ is said to be *$r$-identifying* ($r$-ID for short) if for all $x \in V$, we have $B_r(x) \cap C \neq \emptyset$, and for any two distinct vertices $x \in V$, $y \in V$, we have $B_r(x) \cap C \neq B_r(y) \cap C$.

In other words: every vertex must be $r$-dominated by at least one codeword for the three definitions; in addition, every pair of distinct non-codewords (respectively, vertices) must be $r$-separated by an $r$-LD (respectively, $r$-ID) code.

Two vertices $x \in V$, $y \in V$, $x \neq y$, are said to be *$r$-twins* if $B_r(x) = B_r(y)$. Dominating and locating-dominating codes exist for all graphs. On the other hand, it is easy to see that a graph $G$ admits an $r$-identifying code if and only if

$$\forall x \in V,\ \forall y \in V,\ x \neq y : \ B_r(x) \neq B_r(y). \tag{1}$$

A graph satisfying (1) is called *$r$-identifiable* or *$r$-twin-free*.

### 1.2. Aim of the paper

For all three concepts, we are often interested in finding the minimum sized codes. We denote by $\gamma_r(G)$ (respectively, $ld_r(G)$ and $id_r(G)$) the smallest possible cardinality of an $r$-dominating code (respectively, an $r$-locating-dominating code and an $r$-identifying code when $G$ is $r$-twin-free). We call $\gamma_r(G)$ the *$r$-domination number* of $G$. Since obviously an $r$-ID code (when it exists) is an $r$-LD code which in turn is an $r$-D code, the following inequalities hold:

$$\gamma_r(G) \leq ld_r(G) \leq id_r(G).$$

In other words, location-domination is more "expensive" than domination, and identification is more expensive than location-domination. In this paper, we compare the respective "costs" for these three definitions.

More precisely, denoting

$$\mathcal{G}_{r,n} = \{G : G \text{ is } r\text{-twin-free, connected, with order } n \geq 2\},$$

$$\text{and } \mathcal{G}^{tw}_{r,n} = \{G : G \text{ has } r\text{-twins and is connected, with order } n \geq 2\},$$

we study the following maximum and minimum differences:

- $F_{id,ld}(r,n) = \max\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{id,ld}(r,n) = \min\{id_r(G) - ld_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $F_{id,\gamma}(r,n) = \max\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{id,\gamma}(r,n) = \min\{id_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$.

For D- and LD-codes, we have two cases, (a) and (b), which study graphs which are without or with twins, respectively:

(a)
- $F_{ld,\gamma}(r,n) = \max\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$,
- $f_{ld,\gamma}(r,n) = \min\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}_{r,n}\}$;

these two functions are considered on the same set of graphs (the twin-free graphs) as the four functions involving identification, unlike the two functions below:

(b)
- $F^{tw}_{ld,\gamma}(r,n) = \max\left\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}^{tw}_{r,n}\right\}$,
- $f^{tw}_{ld,\gamma}(r,n) = \min\left\{ld_r(G) - \gamma_r(G) : G \in \mathcal{G}^{tw}_{r,n}\right\}$.

Finally, if we want to consider all the connected graphs of order $n$, twin-free or not, the result is obviously obtained by taking $\max\left\{F_{ld,\gamma}(r,n), F^{tw}_{ld,\gamma}(r,n)\right\}$ and $\min\left\{f_{ld,\gamma}(r,n), f^{tw}_{ld,\gamma}(r,n)\right\}$.

## 2. Some Earlier Results

The following easy four lemmas are as old as the definitions of dominating, locating-dominating or identifying codes. We give the proofs only for the first two.

**Lemma 1.** (a) *For any graph $G=(V,E)$ of order $n$ and any integer $r \ge 1$, we have*

$$ld_r(G) \ge \lceil \log_2(n - ld_r(G) + 1) \rceil . \tag{2}$$

(b) *For any integer $r \ge 1$ and any $r$-twin-free graph $G=(V,E)$ of order $n$, we have*

$$id_r(G) \ge \lceil \log_2(n+1) \rceil . \tag{3}$$

***Proof.*** (a) Let $C$ be any $r$-LD code in $G$. All the $n-|C|$ non-codewords $v \in V \setminus C$ must be given nonempty and distinct sets $B_r(v) \cap C$ constructed with the $|C|$ codewords, so $2^{|C|}-1 \ge n-|C|$, from which (2) follows when $C$ is optimal;

(b) the argument is the same, but we have to consider all the $n$ vertices $v \in V$, so $2^{|C|}-1 \ge n$. ■

**Lemma 2.** *Let $r \ge 2$ be any integer and $G=(V,E)$ be a graph.*

(a) *A code $C$ is 1-locating-dominating in $G^r$, the $r$-th power of $G$, if and only if it is $r$-locating-dominating in $G$.*
(b) *A code $C$ is 1-identifying in $G^r$ if and only if it is $r$-identifying in $G$.*
(c) *A code $C$ is 1-dominating in $G^r$ if and only if it is $r$-dominating in $G$.*

***Proof.*** (a) For every vertex $v \in V$, we have

$$\{c \in C : d_G(v,c) \le r\} = \{c \in C : d_{G^r}(v,c) \le 1\},$$

so if for all $v \in V \setminus C$, the sets on the left-hand side of the equality are nonempty and distinct, then the sets on the right side also are, and *vice-versa*; (b) the same proof, for all $v \in V$; (c) the same proof, for all $v \in V$, with only nonemptiness to be checked. ■

**Lemma 3.** (a) *For any integer $r \ge 1$, if $G$ is a connected graph of order $n$, then*

$$ld_r(G) \le n-1. \tag{4}$$

(b) *If $G$ is an $r$-twin-free graph of order $n$, then $n \geq 2r+1$, and the only $r$-twin-free graph of order $2r+1$ is the path.*

(c) *If $G$ is an $r$-twin-free cycle of order $n$, then $n \geq 2r+2$.*

The following obvious lemma is often used implicitly.

**Lemma 4.** *Let $r \geq 1$ be any integer and $G=(V,E)$ be a graph.*

(a) *If $C$ is $r$-dominating in $G$, so is any set $S \supset C$.*

(b) *If $C$ is $r$-locating-dominating in $G$, so is any set $S \supset C$.*

(c) *If $C$ is $r$-identifying in $G$, so is any set $S \supset C$.*

**Proposition 5.** (a) [16], [13, p. 41] *If $G$ has no isolated vertices (in particular, if $G$ is connected) and has order $n$, then $\gamma_1(G) \leq \frac{n}{2}$.*

(b) [11] *If $G$ is a 1-twin-free graph, then $id_1(G) \leq 2ld_1(G)$.*

The following result is from [3], but a shorter proof can be found in [12].

**Proposition 6.** *If $G$ is a connected 1-twin-free graph of order $n$, then $id_1(G) \leq n-1$.*

**Corollary 7.** *Let $r \geq 1$ be any integer.*

(i) *If $G$ is a connected graph of order $n$, then $\gamma_r(G) \leq \frac{n}{2}$.*

(ii) *If $G$ is an $r$-twin-free graph, then $id_r(G) \leq 2ld_r(G)$.*

(iii) *If $G$ is a connected $r$-twin-graph of order $n$, then*

$$id_r(G) \leq n-1. \tag{5}$$

***Proof.*** Use the $r$-th power of $G$, together with the previous two propositions. ∎

Both lower bounds (2), (3) and upper bounds (4), (5) for $r$-LD and $r$-ID codes can be reached [6], as well as all intermediate values [4], [5].

The graphs $G$ of order $n$ such that $id_1(G)=n-1$ have been characterized in [10], but the case $r \geq 2$ remains open.

## 3. Some Important Graphs

The following three lemmas describe three useful graphs, which have been used in previous papers. The first graph is the "star".

**Lemma 8.** *For $n \geq 3$, let $G_n$ be the tree consisting of $n$ vertices $v_0, v_1, \ldots, v_{n-1}$, and $n-1$ edges $v_0v_i$, $1 \leq i \leq n-1$. Then*

$$\gamma_1(G_n)=1,\ ld_1(G_n)=n-1 \text{ and } id_1(G_n)=n-1.$$

***Proof.*** (a) Since $v_0$ is a 1-universal vertex, we have $\gamma_1(G_n) = 1$.

(b) It is quite straightforward to check that taking for codewords any set of $n-1$ vertices is necessary and sufficient to obtain a 1-LD or 1-ID code, except for $n = 3$, when only $\{v_1, v_2\}$ is a 1-ID code. ■

The second graph, denoted $G^*_{2p}$, has even order and is the complete graph (or clique) minus a perfect matching; see Figure 1.

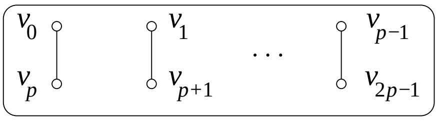


Figure 1. The complement of $G^*_{2p}$: only the missing edges $v_0v_p, \ldots, v_{p-1}v_{2p-1}$ are represented.

**Lemma 9.** *Let $p \geq 2$ and $G^*_{2p} = \left(V^*_{2p}, E^*_{2p}\right)$, with $V^*_{2p} = \{v_0, v_1, \ldots, v_{2p-1}\}$, $E^*_{2p} = \left\{v_iv_j : v_i \in V^*_{2p}, v_j \in V^*_{2p},\ i \neq j,\ i \neq j + p \bmod 2p\right\}$. Then*

$$\gamma_1\left(G^*_{2p}\right) = 2,\ ld_1\left(G^*_{2p}\right) = p \text{ and } id_1\left(G^*_{2p}\right) = 2p - 1.$$

***Proof.*** For every $v_i \in V^*_{2p}$, we have $B_1(v_i) = V^*_{2p} \setminus \left\{v_{i+p \bmod 2p}\right\}$, and for every pair of distinct vertices $v_i \in V^*_{2p}$, $v_j \in V^*_{2p}$, we have $B_1(v_i)\Delta B_1(v_j) = \left\{v_{i+p \bmod 2p}, v_{j+p \bmod 2p}\right\}$, where $\Delta$ stands for the symmetric difference.

(a) The fact that $\gamma_1(G^*_{2p}) = 2$ is easy to check.

(b) Obviously, $C = \{v_0, \ldots, v_{p-1}\}$ is a 1-LD code, of size $p$. Assume that there is a minimum 1-LD code $C$ with fewer than $p$ elements. Then there is at least one $j$ such that $v_j \notin C$ and $v_{j+p \bmod 2p} \notin C$. Without loss of generality, we may assume that $v_0 \notin C$, $v_p \notin C$. Then $B_1(v_0)\Delta B_1(v_p) = \{v_0, v_p\}$ leads to $C \cap (B_1(v_0)\Delta B_1(v_p)) = \emptyset$, contradicting the definition of a 1-LD code: $v_0$ and $v_p$ are non-codewords not 1-separated by any codeword.

(c) We know that at most $2p-1$ codewords are necessary in any minimum 1-ID code $C$; therefore, assume, without loss of generality, that $v_0 \notin C$. Then for all $j \neq p$, $B_1(v_p)\Delta B_1(v_j) = \left\{v_0, v_{j+p \bmod 2p}\right\}$, and, since $v_p$ and $v_j$ are 1-separated by at least one codeword, we have $\emptyset \neq (B_1(v_p)\Delta B_1(v_j)) \cap C \subseteq \left\{v_{j+p \bmod 2p}\right\}$. So for all values of $j$ but one, the $2p-1$ distinct vertices $v_{j+p \mathrm{mod} 2p}$ are codewords, and $|C| \geq 2p-1$, i.e., $|C| = 2p-1$. ■

The third graph is obtained from the previous one by adding one 1-universal vertex, and its order is odd.

**Lemma 10.** *Let $p \geq 2$ and $G^*_{2p+1} = \left(V^*_{2p+1}, E^*_{2p+1}\right)$, with $V^*_{2p+1} = \{v_0, v_1, \ldots, v_{2p}\}$, $E^*_{2p+1} = \left\{v_iv_j : v_i \in V^*_{2p+1} \setminus \{v_{2p}\}, v_j \in V^*_{2p+1} \setminus \{v_{2p}\},\ i \neq j,\ i \neq j + p \bmod 2p\right\} \cup \left\{v_{2p}v_j : v_j \in V^*_{2p+1} \setminus \{v_{2p}\}\right\}$. Then*

$$\gamma_1\left(G^*_{2p+1}\right) = 1,\ ld_1\left(G^*_{2p+1}\right) = p \text{ and } id_1\left(G^*_{2p+1}\right) = 2p.$$

***Proof.*** (a) The fact that $v_{2p}$ is 1-universal shows that $\gamma_1(G^*_{2p+1}) = 1$.

(b) For 1-LD codes, the argument of the Case (b) of the previous proof can be applied *mutatis mutandis*, because the 1-universal vertex does not change anything when considering symmetric differences of balls of radius one.

(c) For $i \in \{0, \ldots, 2p-1\}$, we have $B_1(v_{2p})\Delta B_1(v_i) = \left\{v_{i+p \bmod 2p}\right\}$, therefore all vertices but $v_{2p}$ must be codewords. ■

Now, what is more difficult and interesting is that the two graphs $G^*_{2p}$ and $G^*_{2p+1}$ just described in Lemmas 9 and 10 admit $r$-th roots for any $r$, if $p$ is sufficiently large [6]. More precisely:

**Proposition 11.** *Let $r \geq 2$ and $p \geq 2$ be integers.*

(a) [6, Theorem 5] *If $2p \geq 3r^2$, then there exists a graph $G_{2p}$ of order $2p$ such that $(G_{2p})^r = G^*_{2p}$.*

(b) [6, Theorem 6] *If $2p \geq 3r^2$, then there exists a graph $G_{2p+1}$ of order $2p+1$ such that $(G_{2p+1})^r = G^*_{2p+1}$.*

(c) *For $n \geq 3r^2$, there exists a graph $G_n$ of even order $n$ such that $\gamma_r(G_n) = 2$, $ld_r(G_n) = \frac{n}{2}$ and $id_r(G_n) = n-1$.*

(d) *For $n \geq 3r^2+1$, there exists a graph $G_n$ of odd order $n$ such that $\gamma_r(G_n) = 1$, $ld_r(G_n) = \frac{n-1}{2}$ and $id_r(G_n) = n-1$.*

***Proof.*** (c)–(d). Use the properties of $r$-th powers of graphs (Lemma 2). ■

See also the constructions presented and discussed immediately after Proposition 23 in Section 7.1.

## 4. The Very Small Cases: $n \leq 4$

Here, we denote by $T_r(G)$ the triple $(\gamma_r(G), ld_r(G), id_r(G))$, with the convention that $id_r(G) = ?$ if $G$ is not $r$-twin-free. Figure 2 gives all the nonisomorphic unlabeled connected graphs with two, three or four vertices, together with their triples for $r = 1$.

For $r = 2$, the triples are, for the nine graphs of Figure 2, respectively: $(1,1,?)$; $(1,2,?)$ and $(1,2,?)$; $(1,2,?)$, $(1,3,?)$, $(1,3,?)$, $(1,3,?)$, $(1,3,?)$, and $(1,3,?)$. For $r \geq 3$, the triples are $(1, n-1, ?)$ for all nine graphs. From this, we have the following result.

**Proposition 12.** *We have*

(a) $r = 1$

$n = 2: F^{tw}_{ld,\gamma}(1,2) = f^{tw}_{ld,\gamma}(1,2) = 0;$

$n=3: F_{id,ld}(1,3)=f_{id,ld}(1,3)=0;\ F_{id,\gamma}(1,3)=f_{id,\gamma}(1,3)=1;$
$\quad F_{ld,\gamma}(1,3)=f_{ld,\gamma}(1,3)=1;\ F^{tw}_{ld,\gamma}(1,3)=f^{tw}_{ld,\gamma}(1,3)=1;$

$n=4: F_{id,ld}(1,4)=1,\ f_{id,ld}(1,4)=0;\ F_{id,\gamma}(1,4)=2,\ f_{id,\gamma}(1,4)=1;$
$\quad F_{ld,\gamma}(1,4)=2,\ f_{ld,\gamma}(1,4)=0;\ F^{tw}_{ld,\gamma}(1,4)=2,\ f^{tw}_{ld,\gamma}(1,4)=1;$

(b) $r=2$

$n=2: F^{tw}_{ld,\gamma}(2,2)=f^{tw}_{ld,\gamma}(2,2)=0;$

$n=3: F^{tw}_{ld,\gamma}(2,3)=f^{tw}_{ld,\gamma}(2,3)=1;$

$n=4: F^{tw}_{ld,\gamma}(2,4)=2,\ f^{tw}_{ld,\gamma}(2,4)=1;$

(c) $r\geq 3$

$n\in\{2,3,4\}: F^{tw}_{ld,\gamma}(r,n)=f^{tw}_{ld,\gamma}(r,n)=n-2.$

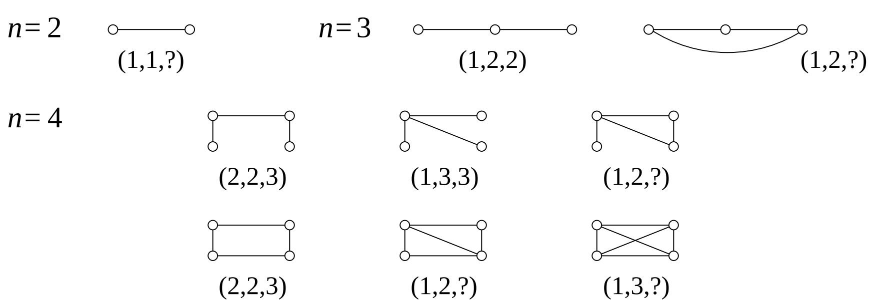


Figure 2. Small graphs, $r=1$.

## 5. Identification vs Domination

First, we construct an infinite family of graphs $G^*_n$ such that $G^*_n$ has order $n$ and satisfies $id_r(G^*_n)=\gamma_r(G^*_n)$.

These graphs will have order $n=k(r+1)$ and consist of one cycle of order $k$ and $k$ strings with $r$ vertices each: $G^*_n=(V^*_n,E^*_n)$, with $V^*_n=V_0\cup\big(\bigcup_{1\leq i\leq k}V_i\big)$ and $E^*_n=E_0\cup\big(\bigcup_{1\leq i\leq k}E_i\big)$, where $V_0=\{v_{1,0},v_{2,0},\dots,v_{k,0}\}$, $V_i=\{v_{i,j}:1\leq j\leq r\}$ for $i\in\{1,2,\dots,k\}$, $E_0=\{v_{i,0}v_{i+1,0}:1\leq i\leq k-1\}\cup\{v_{k,0}v_{1,0}\}$ and $E_i=\big\{v_{i,j}v_{i,j+1}:0\leq j\leq r-1\big\}$ for $i\in\{1,2,\dots,k\}$ (see Figure 3(a)).

**Proposition 13.** *For all $r\geq 1$ and $k\geq 2r+2$, the graph $G^*_n$ is such that*

$$\gamma_r(G^*_n)=id_r(G^*_n).$$

***Proof.*** The $k$ leaves $v_{i,r}$ must be $r$-dominated by at least one codeword, and no vertex can $r$-dominate two leaves, so $\gamma_r(G^*_n)\geq k$. On the other hand, the code $C=V_0$ represented by the black vertices in Figure 3(a) has cardinality $k$, and it is

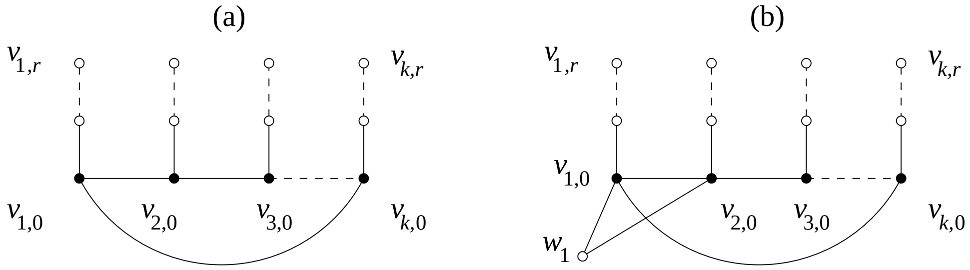


Figure 3. (a) The graph $G_n^*$. (b) The graph $G_{n+1}$. The $k$ black vertices represent codewords.

straightforward to check that it is $r$-identifying. Note in particular that vertices in $V_0$ are $r$-dominated by exactly $2r+1$ codewords (this is where the assumption $k \geq 2r+2$ is crucial, cf. Lemma 3(c)), and vertices $v_{i,j} \in V_i$ are $r$-dominated by exactly $2r-2j+1$ codewords. See also the proof of Proposition 28 for $r$-LD codes, which is analogous but more intricate.

So $k \leq \gamma_r(G_n^*) \leq id_r(G_n^*) \leq k$. ■

Second, if we want $n$ to reach all intermediate values between $k(r+1)$ and $(k+1)(r+1)-1$, we can do so by adding $p \in \{0,\dots,r\}$ vertices to $G_n^*$ in the following way: since $p < \frac{k}{2}$, we can add the set of $p$ vertices $W_p = \{w_1,\dots,w_p\}$ together with the set of edges $X_p = \{w_1v_{1,0}, w_1v_{2,0}, w_2v_{3,0}, w_2v_{4,0},\dots,w_pv_{2p-1,0}, w_pv_{2p,0}\}$, see Figure 3(b) for $p=1$. Setting $G_{n+p} = (V_n^* \cup W_p, E_n^* \cup X_p)$, we obtain a graph of order $n+p$, for which, due to the assumption $k \geq 2r+2$ and the remark in the proof of Proposition 13 stating that all vertices in $G_n^*$ are $r$-dominated by an odd number of codewords, it is again straightforward to check that $C = V_0$ is still a (minimum) $r$-ID code. Therefore we have the following.

**Proposition 14.** *For all $r \geq 1$, $k \geq 2r+2$ and $p \in \{0,\dots,r\}$, the graph $G_{k(r+1)+p}$ is such that $\gamma_r(G_{k(r+1)+p}) = id_r(G_{k(r+1)+p})$. As a consequence, for all $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have*

$$f_{id,\gamma}(r,n) = 0.$$

In advance of the next sections, we have the following obvious consequence.

**Corollary 15.** *For all $r \geq 1$ and $n \geq (2r+2)(r+1)$, we have*

$$f_{id,ld}(r,n) = f_{ld,\gamma}(r,n) = 0.$$

For $r=1$, the construction for Propositions 13 and 14 works for $n \geq 8$; however, we have the exact value of $f_{id,\gamma}(1,n)$ for all $n$, due to an alternative construction.

Proposition 12(a) has already settled the cases $n=3$, $n=4$. Lemma 1(b) and Proposition 5(a) establish that any (1-twin-free) graph $G$ with five vertices is such that $id_1(G) \geq 3$ and $\gamma_1(G) \leq 2$; on the other hand it is easy to find graphs $G$ of order five with $id_1(G) = 3$ and $\gamma_1(G) = 2$, e.g., the path, so that $f_{id,\gamma}(1, 5) = 1$. For even $n$, $n \geq 6$, and odd $n$, $n \geq 7$, it is easy to see that Figure 4 gives graphs $G$ such that $id_1(G) = \gamma_1(G) = k$.

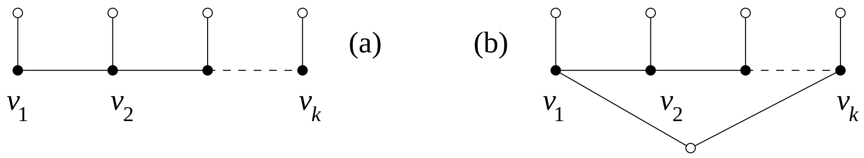


Figure 4. (a) $n$ even. (b) $n$ odd. The $k$ black vertices represent codewords constituting both a minimum 1-identifying and 1-dominating code.

**Proposition 16.** (a) *For all* $n \geq 6$, *we have* $f_{id,\gamma}(1, n) = 0$; *consequently,* $f_{id,ld}(1, n) = f_{ld,\gamma}(1, n) = 0$.
(b) *For* $n \in \{3, 4, 5\}$, *we have* $f_{id,\gamma}(1, n) = 1$.

Now how large can the difference $id_r(G) - \gamma_r(G)$ be? By Corollary 7(iii), it is at most $n - 2$, obtained by graphs $G$ with $id_r(G) = n - 1$ and $\gamma_r(G) = 1$.

We first treat the case $r = 1$, which is easy: the star on $n$ vertices (Lemma 8) is an example of a graph $G$ with $id_1(G) = n - 1$ and $\gamma_1(G) = 1$.

**Proposition 17.** *For all* $n \geq 3$, *we have* $F_{id,\gamma}(1, n) = n - 2$.

We now turn to the case $r \geq 2$. When $n$ is odd, the answer is given by Proposition 11(d). Again, we can reach $n - 2$ for the difference $id_r(G) - \gamma_r(G)$. When $n$ is even, the study of all the graphs $G$ of even order $n$ such that $id_1(G) = n - 1$ [10] shows that none of them contains a 1-universal vertex, i.e., none of them is such that $\gamma_1(G) = 1$, except the star; but the star cannot be the power of any graph. Therefore, for $r \geq 2$, there can exist no graph $G$ with even order $n$ such that $id_r(G) = n-1$ and $\gamma_r(G) = 1$, since the $r$-th power of this graph would contradict the characterization from [10]; consequently the difference $id_r(G) - \gamma_r(G)$ is at most $n - 3$. On the other hand, Proposition 11(c) gives an example achieving $n - 3$, and we have proved the following.

**Proposition 18.** (a) *For all* $r \geq 2$ *and even* $n \geq 3r^2$, *we have* $F_{id,\gamma}(r, n) = n-3$.
(b) *For all* $r \geq 2$ *and odd* $n \geq 3r^2 + 1$, *we have* $F_{id,\gamma}(r, n) = n - 2$.

## 6. Identification vs Location-Domination

We have already seen in Corollary 15 that, for $r \geq 1$ and $n \geq (2r + 2)(r + 1)$, we have $f_{id,ld}(r, n) = 0$.

For $r = 1$, and for all values of $n$, Propositions 12(a) and 16(a) completely settle this case except when $n = 5$, where $f_{id,ld}(1, 5) = 0$ thanks to the graph $G_5$ with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_5\}$, $id_1(G_5) = ld_1(G_5) = 3$. Therefore we have the following.

**Proposition 19.** *For all $n \geq 3$, we have $f_{id,ld}(1, n) = 0$.*

What about $F_{id,ld}(r, n)$? We can use Corollary 7(ii) and obtain that any (connected) $r$-twin-free graph $G$ is such that $id_r(G) \leq 2ld_r(G)$. Therefore, $-ld_r(G) \leq -\frac{id_r(G)}{2}$, and $id_r(G)-ld_r(G) \leq id_r(G)-\frac{id_r(G)}{2} \leq \frac{n-1}{2}$, leading to $id_r(G)-ld_r(G) \leq \lceil\frac{n}{2}\rceil-1$. On the other hand, Proposition 11(c)–(d) gives examples of graphs reaching $\lceil\frac{n}{2}\rceil - 1$.

**Proposition 20.** *For all $r \geq 1$ and $n \geq 3r^2 + 1$, we have $F_{id,ld}(r, n) = \lceil\frac{n}{2}\rceil - 1$.*

**Proposition 21.** (a) *For all $n \geq 4$, we have $F_{id,ld}(1, n) = \lceil\frac{n}{2}\rceil - 1$.*
(b) $F_{id,ld}(1, 3) = 0$.

***Proof.*** Proposition 12(a) settles the case $n = 3$. ∎

## 7. Location-Domination vs Domination

### 7.1. Twin-free graphs

We have already seen in Corollary 15 that, for $r \geq 1$ and $n \geq (2r + 2)(r + 1)$, we have $f_{ld,\gamma}(r, n) = 0$. Moreover, for $r = 1$, Propositions 12(a) and 16(a) treat all values of $n$ but $n = 5$, for which the path shows that $f_{ld,\gamma}(1, 5) = 0$.

**Proposition 22.** (a) *For all $n \geq 4$, we have $f_{ld,\gamma}(1, n) = 0$.*
(b) $f_{ld,\gamma}(1, 3) = 1$.
(c) *For all $r \geq 1$ and $n \geq (2r + 2)(r + 1)$, we have $f_{ld,\gamma}(r, n) = 0$.*

We know, using the example of the star (Lemma 8), that $F_{ld,\gamma}(1, n) = n - 2$. What about $F_{ld,\gamma}(r, n)$ for general $r$?

On the one hand, Proposition 11(c)–(d) immediately gives examples proving that $F_{ld,\gamma}(r, n) \geq \lceil\frac{n}{2}\rceil - 2$, for all $r \geq 2$ and $n \geq 3r^2 + 1$. On the other hand, the characterization [10] of the graphs $G$ of order $n$ such that $id_1(G) = n - 1$ gives graphs which, apart from the star which is not the power of any graph, are such that $ld_1(G) \leq n - 2$. This allows to conclude that $F_{ld,\gamma}(r, n) \leq n - 3$. Indeed, $F_{ld,\gamma}(r, n) = n - 2$ is possible only if a graph $G$ of order $n$ satisfies $\gamma_r(G) = 1$ and $ld_r(G) = n - 1$, which implies $\gamma_1(G^r) = 1$ and $ld_1(G^r) = n - 1 = id_1(G^r)$, contradicting the previous sentence.

**Proposition 23.** (a) *For all $n \geq 3$, we have $F_{ld,\gamma}(1, n) = n - 2$.*

(b) *For all* $r \geq 2$ *and* $n \geq 3r^2 + 1$, *we have* $F_{ld,\gamma}(r, n) \geq \lceil \frac{n}{2} \rceil - 2$.
(c) *For all* $r \geq 2$ *and* $n \geq 2r + 1$, *we have* $F_{ld,\gamma}(r, n) \leq n - 3$.

We now present a general framework using Theorem 5 in [6], and, to a lesser extent, Theorem 6 in [6], cf. Section 3, Proposition 11(a)–(b). We shall use it in the case $r = 2$, when this gives a lower bound for $F_{ld,\gamma}(2, n)$ which is better than $\lceil \frac{n}{2} \rceil - 2$, for all $n \geq 24$; for $r = 3$, $n = 30$, this gives no improvement, and we shall informally explain why for $r = 3$ and larger $n$, or for larger $r$, this method is doomed to fail.

Let $m = 2p \geq 3r^2 + 1$. We consider the Euclidean division of $p$ by $r$: $p = rQ + R$, $0 \leq R \leq r - 1$, and set $k = Q + 1$, $A = r - R$, so that $p = rk - A$ with $A \in \{1, 2, \ldots, r\}$. We build $G_m = (V_m, E_m)$ in the following way:

$$V_m = \{v_i : 0 \leq i \leq m - 1\}, \tag{6}$$

$$E_m = \big\{v_i v_{i+j \bmod m} : 0 \leq i \leq m - 1, j \in J = \{1, 2, \ldots, k - A - 1, k\}\big\}. \tag{7}$$

The graph $G_m$ can be seen as a cycle with chords added according to the set $J$, where every vertex plays the same role, see Figure 5(a). Theorem 5 from [6] states that the $r$-th power of $G_m$ is the graph $G_m^*$ of Lemma 9, with $m = 2p$. We now need to be more specific with respect to $r$.

In the case $r = 2$, we can improve on Proposition 23(b) and build, for $n$ large enough, graphs of order $n$ proving that $\frac{F_{ld,\gamma}(2,n)}{n}$ tends to $\frac{5}{8}$ when $n$ goes to infinity. The first step is the study of graphs with order a multiple of eight.

**Proposition 24.** *For* $n = 8t \geq 24$, *there exists a* 2*-twin-free graph* $G_n$ *of order* $n$, *with* $\gamma_2(G_n) = 2$ *and* $ld_2(G_n) = 5t - 1$.

***Proof.*** Let $m = 6t = 2p \geq 18$ and $p = 2k - A$ with $A \in \{1, 2\}$. Because $A \in \{1, 2\}$ and $p \geq 9$, we have $p \geq 3A + 3 \Rightarrow p - A - 1 \geq \frac{2p}{3} = 2t \Rightarrow 2k - 2A - 1 \geq 2t$, and finally

$$k - A \geq 2t - (k - A - 1). \tag{8}$$

When $p = 9$, then $A = 1$ and $\frac{p}{3} \geq A + 2$, which also holds whenever $p \geq 12$. Therefore, $\frac{2k-A}{3} \geq A + 2 \Rightarrow k \geq 2A + 3 \Rightarrow 3k - 3A - 3 \geq 2k - A \Rightarrow k - A - 1 \geq \frac{p}{3}$, and finally

$$2t + k - A - 1 = \frac{2p}{3} + k - A - 1 \geq p. \tag{9}$$

These two inequalities, (8) and (9), will be used later on. We have already seen that the graph $G_m = (V_m, E_m)$ defined by (6) and (7) is such that the square power of $G_m$ is the graph $G_m^*$ of Lemma 9, with $m = 2p$. This means that

$B_{G_m,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$, for all $i \in \{0, \ldots, m-1\}$; in other words, from any $v_i$ we can go in two steps to any vertex in $V_m$ but $v_{i+p \bmod m}$. Next, we diverge from Theorem 6 in [6]: we add $\frac{m}{3} = 2t$ vertices $Z_j$, $0 \le j \le 2t-1$; every $Z_j$ is linked to three vertices in $V_m$, namely $v_j$, $v_{j+2t}$ and $v_{j+4t}$. By setting $G_n = (V_n, E_n)$ with $V_n = V_m \cup \{Z_j : 0 \le j \le 2t-1\}$ and $E_n = E_m \cup \{Z_j v_j, Z_j v_{j+2t}, Z_j v_{j+4t} : 0 \le j \le 2t-1\}$, we obtain a graph of order $8t$, see Figure 5(b).

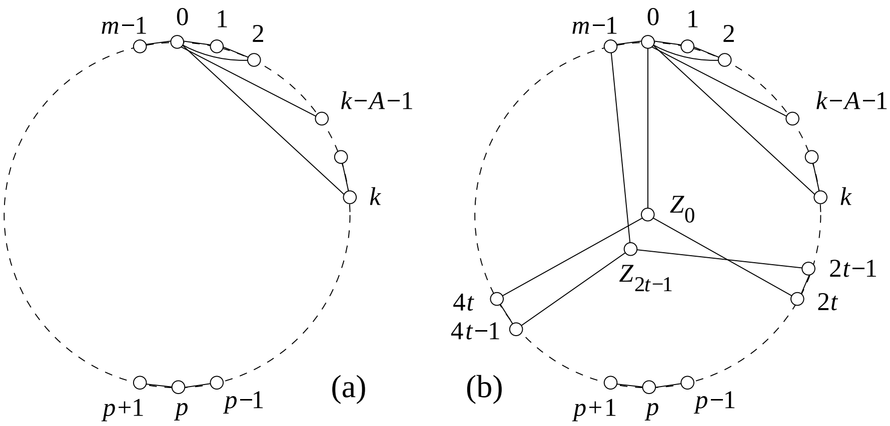


Figure 5. (a) The graph $G_m$. (b) The graph $G_n$ ($r = 2$). Not all vertices nor edges are represented. Only the indices of the vertices $v_i$ are given.

We claim that

(a) $B_{G_n,2}(Z_j) = V_m \setminus \{Z_\ell : 0 \le \ell \le 2t-1, \ell \ne j\}$, for all $j \in \{0, \ldots, 2t-1\}$;

(b) $B_{G_n,2}(v_i) = V_m \setminus \{v_{i+p \bmod m}\}$, for all $i \in \{0, \ldots, m-1\}$.

(a) That we cannot go in two steps from $Z_j$ to $Z_\ell$ is obvious since $B_{G_n,1}(Z_j) \cap B_{G_n,1}(Z_\ell) = \emptyset$. Note already that this could not be directly transposed to the case $r \ge 3$, since then the existence of paths such as $Z_j v_j, v_j v_{j+1}, v_{j+1} Z_{j+1}$ would lead to a contradiction; see the discussion below for $r = 3$.

Next, we show that we can go in two moves from any $Z_j$ to any $v_i$; because of the symmetries of the graph, we need to do it only for, say, $Z_0$, and the vertices going from $v_0$ to $v_p$. Thanks to the edge $Z_0 v_0$, $Z_0$ can reach $v_1, v_2, \ldots, v_{k-1-A}$ (and $v_k$, but we do not need it) in two moves; thanks to the edge $Z_0 v_{2t}$, $Z_0$ can also reach $v_{2t+1}, \ldots, v_{2t+k-1-A}$ (and $v_{2t+k}$), as well as $v_{2t-1}, \ldots, v_{2t-(k-1-A)}$ (and $v_{2t-k}$). Using (8), we can see that in the worst case, $v_{2t-(k-1-A)} = v_{k-A}$ and all the vertices between $v_0$ and $v_{2t}$ can be reached, including in particular $v_{k-1-A}$ and $v_{k-A}$. In other words, the areas reached in one move by going clockwise from $v_0$ or anticlockwise from $v_{2t}$ do meet. Similarly, by (9), we have in the worst case $v_{2t+k-A-1} = v_p$ and all the vertices between $v_{2t+1}$ and $v_p$ can be reached in two moves from $Z_0$. Claim (a) is proved.

(b) The proof is the same as for the graph $G_m$; we just have to check that the additional vertices $Z_j$ and their edges do not make it possible to go in two moves from $v_0$ to $v_p$ (this is sufficient for reasons of symmetry).

Claims (a) and (b) show that $G_n$ is 2-twin-free; they also show that the square power of $G_n$ is the following graph. $(G_n)^2$ has vertex set $V_n$ and edge set all the possible edges except the edges $v_i v_{i+p \bmod m}$, $0 \leq i \leq p$, and $Z_j Z_\ell$, $\{j, \ell\} \subset \{0, \ldots, 2t-1\}$, $j \neq \ell$, see Figure 6.

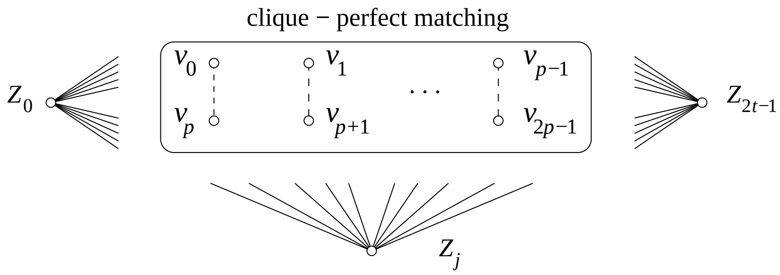


Figure 6. A partial representation of $(G_n)^2$. Dotted lines are non-edges.

Now what are $\gamma_1\big((G_n)^2\big)$ and $ld_1((G_n)^2)$ (or equivalently, $\gamma_2(G_n)$ and $ld_2(G_n)$)? Obviously, $\gamma_1\big((G_n)^2\big) = 2$. Next, the argument of Case (b) of the proofs of Lemmas 9 and 10 can be used to show that it is necessary to take half of the vertices $v_i$ in $V_m$ for a 2-LD code. Then, for $j \neq \ell$, we have $B_{(G_n)^2,1}(Z_j) \Delta B_{(G_n)^2,1}(Z_\ell) = \{Z_j, Z_\ell\}$, which implies that we have to take all vertices $Z_j$ but one as codewords, together with $p$ vertices in $V_m$, and this is sufficient, $ld_1\big((G_n)^2\big) = p + (2t-1) = 5t - 1$. ■

In order to reach the values of $n$ other than multiples of eight, we might consider $m = 6t+2$ or $6t+4$ instead of $m = 6t$, but it is more efficient to stick to $m = 6t$ and simply add a number of vertices $Z_j$ smaller (by a number between 1 and 7) than $2t$. From $m = 6t \geq 18$ we constructed a graph with $8t$ vertices; now, we start from $6(t+1)$, and, instead of building a graph with order $6(t+1)+2(t+1) = 8(t+1)$, we build a graph with $6(t+1)+[2(t+1)-q] = 8t+(8-q)$ vertices, by adding only $2(t+1)-q$ vertices $Z_j$, with $1 \leq q \leq 7$. The resulting graph has its 2-domination number equal to 1 (in the unique case when $t = 3$, $q = 7$ and we add only one vertex, $Z_0$) or 2; any minimum 2-LD code has size $3(t+1) + [2(t+1) - q - 1] = 5(t+1) - q - 1$, including when $2(t+1) - q = 1$.

So, letting $i = 8-q$, $1 \leq i \leq 7$, we obtain graphs $G_n$ with order $n = 8t+i$ and $ld_2(G_n) = 5t + i - 4$ (the borderline case $i = 0$, i.e., dropping eight vertices $Z_j$, logically leads to a worse result, namely $5t - 4$, than if we start from $6t$ to reach $8t$, in which case we have just seen that we obtain $5t-1$). Since the 2-domination number of these graphs is at most 2, we have the following result.

**Proposition 25.** (a) *Let $n = 8t \geq 24$. Then $F_{ld,\gamma}(2, n) \geq 5t - 3$.*
(b) *Let $n = 8t + i \geq 24$, with $1 \leq i \leq 7$. Then $F_{ld,\gamma}(2, n) \geq 5t + i - 6$.*

The least favorable case is when $i = 1$, which leads to

$$F_{ld,\gamma}(2, n) \geq \frac{5n - 45}{8}. \tag{10}$$

The case $m = 6t$ works best because we have a miraculously large number of $Z_j$'s, namely $2t$, which is advantageous when we look for a "large" LD-code, since we have to take all of them but one in a 1-LD code in $(G_n)^2$. If we consider $m = 6t+2$ or $m = 6t + 4$, we cannot take as many vertices $Z_j$; yet, if we can take a number of $Z_j$'s which is only a fraction $\frac{m}{\beta}$ with $\beta > 3$, then we obtain a graph $G_n$ with order $n = m + \frac{m}{\beta}$ and $ld_2(G_n) = ld_1\big((G_n)^2\big) = \frac{m}{2} + \frac{m}{\beta} - 1$, leading to the ratio $\frac{F_{ld,\gamma}(2,n)}{n}$ greater than $\frac{ld_2(G_n)-2}{n} \approx \frac{\beta+2}{2\beta+2}$, which is not as good as $\frac{5}{8}$.

For $r = 3$, we consider $m = 30 = 2p = 2(3k - A)$, leading to $p = 15$, $k = 6$, $A = 3$, $J = \{1, 2, 6\}$. To the graph $G_{30}$ defined by (6) and (7), whose third power, by [6, Theorem 5], is $G^*_{30}$, we add first the vertex $Z_0$ together with the edges $Z_0v_0$, $Z_0v_{10}$ and $Z_0v_{20}$. Then we add the vertex $Z_1$. Because we want no path of the type $Z_0v_0v_iZ_1$ for some $i$, among the vertices $\{v_1, \ldots, v_{15}\}$, $Z_1$ cannot be linked to $v_1$, $v_2$ nor $v_6$; because of $Z_0v_{10}$, this also excludes $v_4$, $v_8$, $v_9$, $v_{11}$ and $v_{12}$ as neighbours of $Z_1$. Finally we can take, e.g., $Z_1$ with the edges $Z_1v_3$, $Z_1v_{13}$, $Z_1v_{23}$, $Z_2$ with $Z_2v_7$, $Z_2v_{17}$, $Z_2v_{27}$, and no more. Exactly as before, this leads to a graph $G_{33}$ whose third power is a graph of the type given by Figure 6, with 33 vertices, $ld_3(G_{33}) = 15 + (3 - 1) = 17$, 3-domination number equal to 2, and $ld_3(G_{33}) - \gamma_3(G_{33}) = 15 = \lceil \frac{33}{2} \rceil - 2$, i.e., not better than Proposition 23(b).

It is impossible to take fewer than three neighbours for each vertex $Z_\ell$. On the other hand, as discussed above when studying the possible neighbours of $Z_1$, if $v_0$ is the neighbour of $Z_0$, the "first" neighbour of $Z_1$ will be $v_i$ with $i \geq k - A$, for $Z_2$ it will be $v_j$ with $j \geq i + (k - A) \geq 2(k - A)$, ... So, roughly speaking, the total number of possible neighbours for the vertices $Z_\ell$ is at most

$$\frac{m}{k - A} = \frac{m}{\frac{p}{3} - \frac{2A}{3}} = \frac{6p}{p - 2A}, \tag{11}$$

and therefore, the number of vertices $Z_\ell$ is at most $\frac{2p}{p-2A}$. When $p = 15$, this leads to at most three vertices $Z_\ell$, and things only worsen when $m$, hence $p$ increases. Anyway, with only three additional vertices $Z_\ell$, all we can reach is a graph $G_n$ with $n = m + 3$ vertices and

$$ld_3(G_n) - \gamma_3(G_n) = (p + (3 - 1)) - 2 = p = \frac{n - 3}{2}.$$

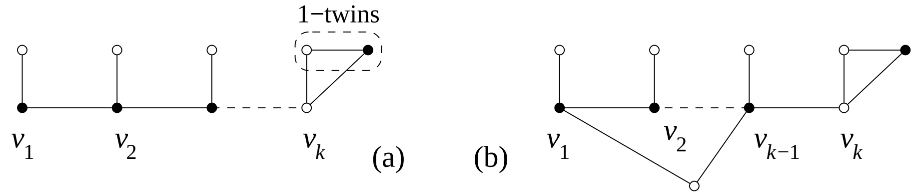


Figure 7. (a) $n$ odd: $n = 2k+1 \geq 5$. (b) $n$ even: $n = 2k+2 \geq 8$. The $k$ black vertices represent codewords constituting both a minimum 1-dominating and 1-locating-dominating code.

When we place ourselves again in the general case for $r$, we must have the "first" neighbour of $Z_1$, say $v_i$, such that $i \geq (r-2)(k-A)$, in order to avoid a path of length $r$ between $Z_0$ and $Z_1$, and equalities (11) now read

$$\frac{m}{(r-2)(k-A)} = \frac{m}{(r-2)\left(\frac{p}{r} - \frac{(r-1)A}{r}\right)} = \frac{2rp}{(r-2)(p-(r-1)A)}.$$

Even with $p = \frac{3r^2}{2}$ and $A = r$, this can lead only to approximately $\frac{3r^3}{(r-2)\frac{r^2}{2}} \approx 6$, hence at most two vertices $Z_\ell$, and again, things only worsen when $p$ increases. Therefore, other constructions should be invented—that is, if improvements do exist in Proposition 23(b).

**Open Problem.** Reduce the gap between lower and upper bounds for $F_{ld,\gamma}(r,n)$, when $r > 1$.

### 7.2. Graphs with twins

The study of $F^{tw}_{ld,\gamma}(r,n)$ is trivial, because of the clique, or complete graph on $n$ vertices, $K_n$, which obviously contains $r$-twins, and is such that $\gamma_r(K_n) = 1$ and $ld_r(K_n) = n-1$.

We are going to prove that (i) for $r = 1$ and $n \in \{2,5\}$ or $n \geq 7$ (Proposition 26) and (ii) for any $r \geq 2$ and $n$ large enough (Proposition 28), we have $f^{tw}_{ld,\gamma}(r,n) = 0$.

**Proposition 26.** (a) *For $n = 2$, $n = 5$ and all $n \geq 7$, we have $f^{tw}_{ld,\gamma}(1,n) = 0$;*
(b) *For $n \in \{3,4,6\}$, we have $f^{tw}_{ld,\gamma}(1,n) = 1$.*

***Proof.*** We already know by Proposition 12(a) that

$$f^{tw}_{ld,\gamma}(1,2) = 0;\ f^{tw}_{ld,\gamma}(1,3) = 1;\ f^{tw}_{ld,\gamma}(1,4) = 1.$$

For $n = 6$, Lemma 1(a) and Proposition 5(a) state that for any connected graph $G$ with six vertices, $ld_1(G) \geq 3$ and $\gamma_1(G) \leq 3$; but a study of the graphs with 1-twins shows that for them, $\gamma_1(G) \leq 2$ (alternatively, one can use the characterization of the graphs with even order and 1-domination number half their order [13, p. 42], [9, 17]), and eventually $f^{tw}_{ld,\gamma}(1,6) = 1$. For $n = 5$ and $n \geq 7$, we consider the graphs in Figure 7, obtained from the graphs in Figure 4 by a slight modification, intended to create one pair of 1-twins. The study of these graphs is straightforward and gives the desired result. ■

We now turn to the case $r \geq 2$ (even if the results below are also valid for $r = 1$); first, we give an analogue of Proposition 13 for $r$-LD codes. We take the graphs $G^*_n = (V^*_n, E^*_n)$ represented in Figure 3(a) and described just before Proposition 13, and transform them into graphs $G^y_{n+1}$ by applying the same type of modification just performed for $r = 1$. We simply add one vertex $y$ which is the $r$-twin of $v_{k,r}$, see Figure 8(a). The order of $G^y_{n+1}$ is $n + 1 = k(r + 1) + 1$.

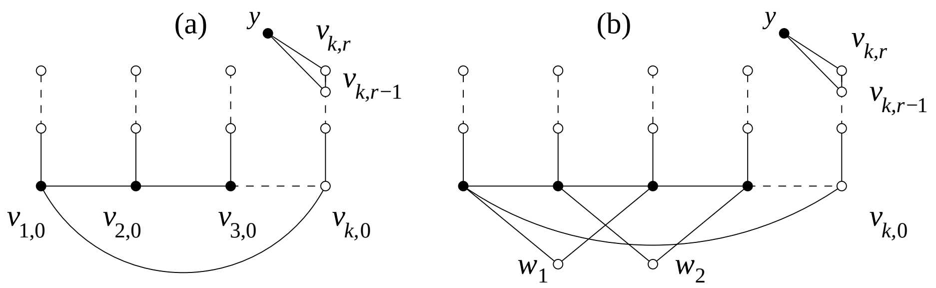


Figure 8. The $k$ black vertices represent codewords.

**Observation 27.** *Because here we deal with $r$-LD codes, not $r$-ID codes like in Proposition* 13, *the bound for $k$ could be lowered, down to $k \geq 2r$. For simplicity and because this does not represent a significant improvement, we keep the bound $k \geq 2r + 2$.*

**Proposition 28.** *For all $r \geq 2$ and $k \geq 2r + 2$, the graphs $G^y_{n+1}$ are such that $\gamma_r\big(G^y_{n+1}\big) = ld_r\big(G^y_{n+1}\big)$.*

***Proof.*** Obviously, $k \leq \gamma_r\big(G^y_{n+1}\big)$ (and $k \leq ld_r\big(G^y_{n+1}\big)$), and the code $C = \{v_{1,0}, v_{2,0}, \ldots, v_{k-1,0}, y\}$, with $k$ codewords, is an $r$-D code. We are going to prove that $C$ is also $r$-LD. In spite of the fact that all we have to check is that any two distinct non-codewords are $r$-separated by $C$, the proof is a little more intricate than the proof of Proposition 13 for $r$-ID codes, because of the "missing" codeword $v_{k,0}$, so we present it in detail.

(a) The non-codewords $v_{k,j}$, $0 \leq j \leq r$, are the only non-codewords $r$-dominated by $y$, so they all are $r$-separated by $y \in C$ from other non-codewords; each of them is $r$-dominated by a different number of codewords, because $k$ is large enough, and therefore they are pairwise $r$-separated by $C$.

(b) Consider any two non-codewords $v_{i,j}$, $v_{i,t}$ on the same string $i$, $1 \leq i \leq k-1$, $\{j,t\} \subseteq \{1,\ldots,r\}$, $j < t$; then $v_{i,j}$ is $r$-dominated by at least one codeword more than $v_{i,t}$ (it would be at least two if we had all $k$ elements of the cycle in the code), and so these two vertices are $r$-separated by $C$.

(c) Let us consider two non-codewords $v_{i,j}$ and $v_{s,t}$ belonging to two different strings, other than the $k$-th string: $\{i,s\} \subset \{1,\ldots,k-1\}$, $i \neq s$, $\{j,t\} \subseteq \{1,\ldots,r\}$; without loss of generality, we may assume that $j \leq t$.

If $j < t$, then again, $v_{i,j}$ is $r$-dominated by at least one codeword more than $v_{s,t}$; so from now on, we assume that $j = t$. The set of codewords $r$-dominating $v_{i,j}$ has cardinality $2r-2j+1$ or $2r-2j$, and consists, with computations performed modulo $k$, of $v_{i,0}$, $v_{i-1,0},\ldots,v_{i-r+j,0}$, $v_{i+1,0},\ldots,v_{i+r-j,0}$, or of the same set without $v_{k,0}$, which is not a codeword. In both cases, it cannot be the same as the set of codewords $r$-dominating $v_{s,j}$.

We have just proved that $C$ $r$-separates the non-codewords $v_{k,j}$ between themselves and from the other non-codewords; the non-codewords belonging to the same string; the non-codewords belonging to different strings. Therefore $C$ is an $r$-LD code. ∎

Now, like in Proposition 14, we want to reach all intermediate values between $k(r+1)+1$ and $(k+1)(r+1)$. We do so by adding a set $W_p = \{w_1,\ldots,w_p\}$ of $p$ vertices, $p \in \{0,\ldots,r\}$. However, if we proceed exactly as for Proposition 14 by creating the edge set $X_p = \{w_1v_{1,0}, w_1v_{2,0}, w_2v_{3,0}, w_2v_{4,0},\ldots,w_pv_{2p-1,0}, w_pv_{2p,0}\}$ but now considering the code $C = \{v_{1,0},\ldots,v_{k-1,0}\} \cup \{y\}$, we might have one or two pairs of vertices not $r$-separated by $C$. We show one such pair ($v_{4,1}$ and $w_2$) in Figure 9 when $r = 4$, $k = 11$, $p = 4$; more generally, this may occur whenever $r$ is even. Moreover, a symmetrical situation appears when $k - r$ is odd, see the same Figure with $w_4$ and $v_{7,1}$. The existence of both pairs is due to the fact that $v_{k,0} \notin C$.

So we choose another way of linking the vertices $w_i$ to the vertices $v_{s,0}$: $X_p = \{w_1v_{1,0}, w_1v_{3,0}, w_2v_{2,0}, w_2v_{4,0},\ldots\}$, see Figure 8(b) for $p = 2$. Setting $G^y_{n+1+p} = \big(V^*_n \cup \{y\} \cup W_p, E^*_n \cup \{yv_{k,r}, yv_{k,r-1}\} \cup X_p\big)$, we obtain a graph of order $n+1+p = k(r+1)+1+p$.

**Proposition 29.** *For all $r \geq 2$, $k \geq 2r+2$ and $p \in \{0,\ldots,r\}$, the graph $G^y_{k(r+1)+1+p}$ is such that $\gamma_r\left(G^y_{k(r+1)+1+p}\right) = ld_r\left(G^y_{k(r+1)+1+p}\right)$.*

***Proof.*** Again, we take $C = \{v_{i,0} : 1 \leq i \leq k-1\} \cup \{y\}$. Using anew the proof of the previous proposition, we can see that we have only to prove in addition the following two assertions about the $w_i$'s.

(a) If $p \geq 2$, any two non-codewords $w_i$ and $w_s$, $\{i,s\} \subseteq \{1,2,\ldots,p\}$, $i < s$, are $r$-separated by $C$. If $w_i$ is linked to $v_{\ell,0}$ and $v_{\ell+2,0}$, then the set of codewords $r$-dominating $w_i$ has size $3+2(r-1)$ or $2+2(r-1)$, and consists (with computations modulo $k$) of $v_{\ell,0}$, $v_{\ell+1,0}$, $v_{\ell+2,0}$, $v_{\ell-1,0}, \ldots, v_{\ell-r+1,0}, v_{\ell+3,0}, \ldots, v_{\ell+2+r-1,0}$, or of the same set without $v_{k,0}$. In both cases, it cannot be the same as the set of codewords $r$-dominating $w_s$.

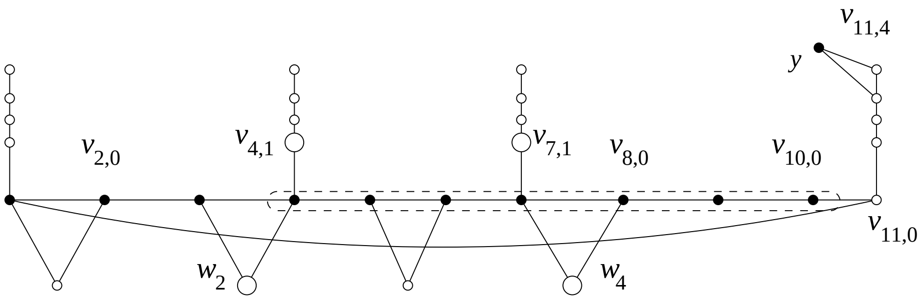


Figure 9. $r = 4$, $n = 60$. The eleven black vertices represent codewords. Not all strings are shown. The vertices $w_2$ and $v_{4,1}$ are not 4-separated by $C$; neither are $w_4$ and $v_{7,1}$, which are both 4-dominated by $v_{i,0}$, $4 \leq i \leq 10$, as indicated by the dotted-line box.

(b) Two non-codewords $w_i$, $i \in \{1, \ldots, p\}$, and $v_{s,t}$, $1 \leq s \leq k-1$, $1 \leq t \leq r$, are $r$-separated by $C$. If $w_i$ is linked to $v_{\ell,0}$ and $v_{\ell+2,0}$, the most crucial cases are when $s \in \{\ell, \ell+1, \ell+2\}$ and $t = 1$, but even here, $w_i$ is $r$-dominated by more codewords than $v_{s,1}$ (note that this "W-construction" would not have worked for $r$-ID codes, because then $w_i$ and $v_{\ell+1,0}$ would not be $r$-separated by the code). ■

**Corollary 30.** *For all $r \geq 2$ and $n \geq (2r+2)(r+1)+1$, we have $f^{tw}_{ld,\gamma}(r, n) = 0$.*

## 8. Conclusion

In the following tables, we recapitulate our results on the different minimum and maximum differences between cardinalities of minimum dominating, locating-dominating or identifying codes in connected graphs, first for $r = 1$, then for $r \geq 2$. For $r = 1$, we have exact values for all $n$ and all functions.

| $n$ | 2 | 3 | 4 | 5 | 6 | $\geq 7$ | Proposition |
|---|---|---|---|---|---|---|---|
| $f_{id,\gamma}(1, n)$ | $\times$ | 1 | 1 | 1 | 0 | 0 | 16 |
| $F_{id,\gamma}(1, n)$ | $\times$ | 1 | 2 | 3 | 4 | $n-2$ | 17 |
| $f_{id,ld}(1, n)$ | $\times$ | 0 | 0 | 0 | 0 | 0 | 19 |
| $F_{id,ld}(1, n)$ | $\times$ | 0 | 1 | 2 | 2 | $\lceil \frac{n}{2} \rceil - 1$ | 21 |
| $f_{ld,\gamma}(1, n)$ | $\times$ | 1 | 0 | 0 | 0 | 0 | 22(a)–(b) |
| $F_{ld,\gamma}(1, n)$ | $\times$ | 1 | 2 | 3 | 4 | $n-2$ | 23(a) |
| $f^{tw}_{ld,\gamma}(1, n)$ | 0 | 1 | 1 | 0 | 1 | 0 | 26 |
| $F^{tw}_{ld,\gamma}(1, n)$ | 0 | 1 | 2 | 3 | 4 | $n-2$ | (clique) |

For $r \geq 2$, most results are valid for $n$ large (typically, $n$ is in $r^2$).

| | |
|---|---|
| $id$ vs $\gamma$ | $\forall r \geq 2,\ f_{id,\gamma}(r,n) = 0$ [Proposition 14]<br>$n$ even, $\forall r \geq 2,\ F_{id,\gamma}(r,n) = n-3$ [Proposition 18(a)]<br>$n$ odd, $\forall r \geq 2,\ F_{id,\gamma}(r,n) = n-2$ [Proposition 18(b)] |
| $id$ vs $ld$ | $\forall r \geq 2,\ f_{id,ld}(r,n) = 0$ [Corollary 15]<br>$\forall r \geq 2,\ F_{id,ld}(r,n) = \lceil \frac{n}{2} \rceil - 1$ [Proposition 20] |
| $ld$ vs $\gamma$ (twin-free graphs) | $\forall r \geq 2,\ f_{ld,\gamma}(r,n) = 0$ [Corollary 15, Proposition 22(c)]<br>$F_{ld,\gamma}(2,n) \geq \frac{5n-45}{8} \approx \frac{5n}{8}$ [Prop. 25(b), case $i=1$, ineq. (10)]<br>$\forall r \geq 3,\ F_{ld,\gamma}(r,n) \geq \lceil \frac{n}{2} \rceil - 2$ [Proposition 23(b)]<br>$\forall r \geq 2,\ F_{ld,\gamma}(r,n) \leq n-3$ [Proposition 23(c)] |
| $ld$ vs $\gamma$ (with twins) | $\forall r \geq 2,\ f^{tw}_{ld,\gamma}(r,n) = 0$ [Corollary 30]<br>$\forall r \geq 2,\ F^{tw}_{ld,\gamma}(r,n) = n-2$ (clique) |

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