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ON GRAPHS REPRESENTABLE BY PATTERN-AVOIDING WORDS

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Abstract

In this paper we study graphs defined by pattern-avoiding words. Word-representable graphs have been studied extensively following their introduction in 2004 and are the subject of a book published by Kitaev and Lozin in 2015. Recently there has been interest in studying graphs represented by pattern-avoiding words. In particular, in 2016, Gao, Kitaev, and Zhang investigated 132-representable graphs, that is, word-representable graphs that can be represented by a word which avoids the pattern 132. They proved that all 132-representable graphs are circle graphs and provided examples and properties of 132-representable graphs. They posed several questions, some of which we answer in this paper.

One of our main results is that not all circle graphs are 132-representable, thus proving that 132-representable graphs are a proper subset of circle graphs, a question that was left open in the paper by Gao *et al.* We show that 123-representable graphs are also a proper subset of circle graphs, and are different from 132-representable graphs. We also study graphs represented by pattern-avoiding 2-uniform words, that is, words in which every letter appears exactly twice.

Keywords: pattern-avoidance, word-representability, circle graphs.

2010 Mathematics Subject Classification: 05C99.

1. Introduction

In this paper we study graphs defined by pattern-avoiding words. Word-representable graphs have been investigated extensively following their introduction by Kitaev in 2004 and their first systematic study in [3]. They are also the subject of the book [6]. There is general interest in study various patterns in words as is witnessed by [4, 5]. Recently there has been interest (see [1] and page 183 in [6]) in studying graphs represented by pattern-avoiding words. In particular, Gao, Kitaev, and Zhang studied 132-representable graphs, that is, word-representable graphs which can be represented by a word which avoids the pattern 132 [1]. They proved that all 132-representable graphs are circle graphs and provided examples and properties of 132-representable graphs. They also posed several questions, some of which we answer in this paper.

One of the main results in this paper is that not all circle graphs are 132-representable, thus proving that 132-representable graphs are a strict subset of circle graphs, a question that was left open in [1]. We show that 123-representable graphs are also a proper subset of circle graphs, and are different from 132-representable graphs. We also study graphs represented by pattern-avoiding 2-uniform words, that is, words in which every letter appears exactly twice.

This paper is organized as follows. In Section 2 we introduce important definitions, notation, and past results that will be used in the paper. In Section 3 we prove several results about 123-representable graphs. In Section 4 we discuss and prove some properties of graphs which can be represented by 2-uniform pattern-avoiding words, and provide some examples of such graphs. In Section 5 we prove that not all circle graphs are 132-representable, answering one of the questions posed in [1]. Finally in Section 6 we give some research directions.

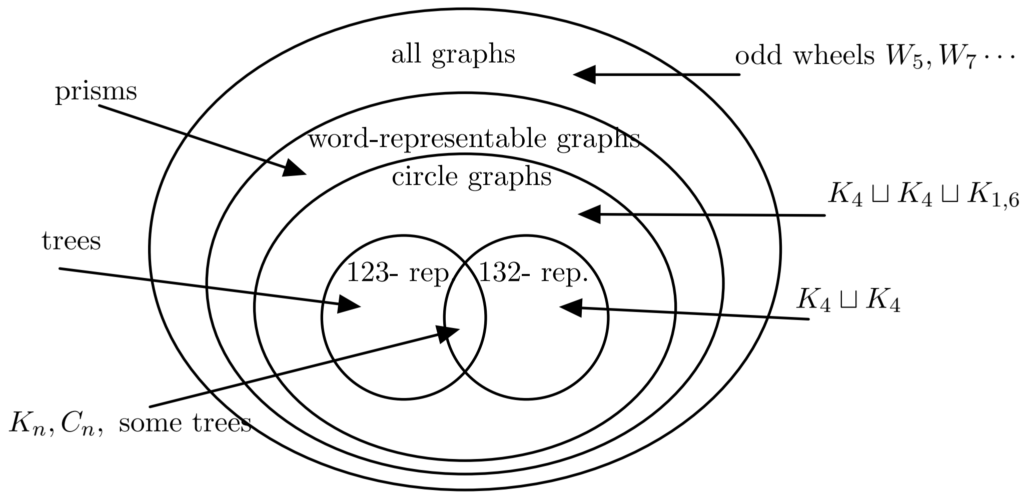

Figure 1. The place of 123 and 132-representable graphs in the hierarchy of graph classes.

Figure 1 shows the hierarchy of graph classes, as established in this paper, with some examples of graphs fitting into each category.

2. Preliminaries and Definitions

We will now introduce notation and definitions.

2.1. Words and patterns

Throughout this paper, w refers to a word $w_1w_2\cdots w_n$ over a totally ordered alphabet. For any letter x in a word w, x_i denotes the i-th instance of the letter x in w from left to right.

Definition 2.1. A word v is a *factor* of the word w if $w = xvy$ for (possibly empty) words x and y.

Definition 2.2. A word w is *k-uniform* if there are exactly k copies of each letter in w.

For example, the word 12432143 is 2-uniform, whereas 1232342 is not.

Definition 2.3. Two letters x and y *alternate* in a word w if there is an instance of x between any two instances of y (if there is more than one y), and an instance of y between any two instances of x (if there is more than one instance of x).

For example, in the word *abcbd*, the following pairs of letters are alternating: $(a, c), (a, d), (b, c), (c, d)$, while the pairs (a, b) and (b, d) are not.

Definition 2.4. A word w *contains* the pattern $\tau = \tau_1\cdots\tau_k$ if there are indices $1 \le i_1 < \cdots < i_k \le n$ such that $w_{i_1}\cdots w_{i_k}$ is order-isomorphic to τ.

In particular, w contains the pattern 123 if there is a strictly increasing substring of length 3 in w. For example, 31247 contains the pattern 123.

Definition 2.5. A word w *avoids* a pattern if it does not contain it.

Thus, the word 7546231 is 123-avoiding, whereas 7534621 is not.

2.2. Word-representable graphs

In this paper all graphs are simple. The degree of a vertex is denoted by $d(v)$.

Definition 2.6. A *circle graph* $G = (V, E)$ is a graph whose vertices can be associated with chords of a circle such that two chords a and b intersect if and only if $ab \in E$.

Definition 2.7. A graph $G = (V, E)$ is *word-representable* if there exists a word w over V such that x and y alternate in w if and only if $xy \in E$. Any such w is said to represent G.

Definition 2.8. Given a pattern τ, a graph is τ-representable if, possibly after relabeling the vertices of the graph, it can be represented by a word which avoids τ.

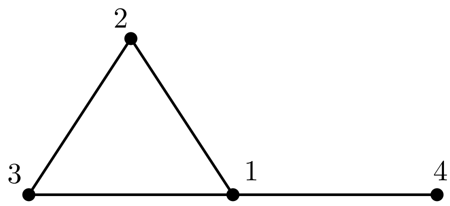

Figure 2. An example of a word-representable graph. A 123-avoiding word that represents this graph is 32414.

In particular, a graph is 123-representable if it can be represented by a 123-avoiding word. See Figure 2 for an example of a 123-representable graph.

Note that labeling is important when dealing with τ-representable graphs, for a pattern τ as opposed to simply word-representable graphs. Figure 3 shows the importance of a correct labeling. The graphs on the right and left are the same graph, but while the one on the left is 132-representable with its current labeling (by the word 4321234), the one on the right is not. In the graph on the right, at least two of the letters 1, 2, and 3 must appear twice, as otherwise there would be at least two appearing once, and thus they would be alternating. Thus if a and b (with $a \neq b$, $a, b < 4$) each appear twice and are not alternating with each other, but are both alternating with 4, then there must be a subsequence in the word $ab4ba$, and then either $a4b$ or $b4a$ forms a 132-pattern.

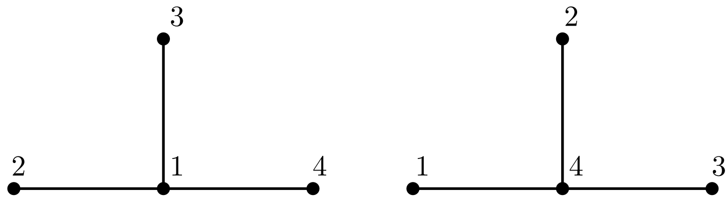

Figure 3. An example showing the importance of labeling correctly.

2.3. Preliminaries

We need the following results.

Theorem 2.9 [1]**.** *Any* 132*-representable graph is a circle graph.*

Theorem 2.10 [2]**.** *A graph G is word-representable and has a representant with at most two copies of each letter if and only if G is a circle graph.*

Lemma 2.11 [1]**.** *If $G_1, G_2, \dots, G_k$ are the connected components of a graph G that can be* 132*-represented by* 2*-uniform words then G is* 132*-representable by a* 2*-uniform word.*

3. 123-Representable Graphs

In this section we discuss properties of words representing 123-representable graphs and properties of the graphs themselves.

We first present a simple, but useful generalization of Theorem 3.1 in [1].

Theorem 3.1. *Let G be a word-representable graph, which can be represented by a word avoiding a pattern τ of length $k+1$. Let x be a vertex in G such that $d(x) \geq k$. Then, any word w representing G that avoids τ must contain no more than k instances of x.*

Proof. Since $d(x) \geq k$, there must be vertices $a_1, a_2, \ldots, a_k$ that are adjacent to x, and each of $a_1, \ldots, a_k$ must be alternating with x in w.

Now suppose there are at least $k+1$ copies of x in w. Then there exists a subsequence $x_1 w_1 x_2 \cdots w_k x_{k+1}$ in w, where each $w_1, \ldots, w_k$ must have a copy of each $a_1, \ldots, a_k$. However, now every single possible permutation of $x, a_1, a_2, \ldots, a_k$ can be found in the word, since we can get any permutation of the a_i's by simply taking the first element from w_1, the second from w_2, etc. Then we can find an x between any two elements of the permutation of the a_is, so we have all possible permutations of $x, a_1, a_2, \ldots, a_k$. Therefore, all possible patterns of length $k+1$ can be found in w. It follows that x can appear in w no more than k times for w to avoid a pattern of length $k+1$. ∎

Note that in the case of 123-representability, Theorem 3.1 implies that in a graph G, if x has degree at least 2, then x can appear at most two times in any 123-avoiding word representant for G.

Corollary 3.2. *Let w be a word-representant for a graph which avoids a pattern of length $k+1$. If some vertex a adjacent to x has degree at least k, then x occurs at most $k+1$ times in w.*

Proof. By Theorem 3.1, a occurs at most k times in w. Since x must alternate with a, there can be no more than $k+1$ instances of x, for otherwise, x and a would not be alternating. ∎

Note that if x is a vertex of degree 1 in a graph G and a vertex a adjacent to x has degree at least 2, then x occurs at most three times in any word that is a 123-representant for G.

The following lemma will be very useful in the proof of Theorem 3.4.

Lemma 3.3. *If w is a 123-representant for $G=(V,E)$ and has a factor ab, with $a<b$, then w', formed by switching a and b, is also 123-avoiding. Furthermore, w' represents G if $ab \notin E$, and a and b do not alternate in w'.*

Proof. Assume for the sake of contradiction that w' is not 123-avoiding, and thus contains an increasing subsequence of length 3. Since the only difference between w and w' is the order of a and b, this means that they both must be in this subsequence. However, they are in decreasing order in w', so this is impossible. Thus w' is 123-avoiding.

The only possible edges (or lack thereof) that could have been affected by the switch are edges incident with a or b. However, the relative order of a and b is irrelevant when considering an edge only involving one of a or b. Thus if $ab \notin E$, and a and b do not alternate in w', then w' represents G. ∎

Note that Lemma 3.3 says that if we have a word $w = w_1 a_1 a_2 \cdots a_k x w_2$ that is 123-avoiding, with all $a_i < x$ for $1 \leq i \leq k$, then the word $w_1 x a_1 a_2 \cdots a_k w_2$ is also 123-avoiding and represents almost the same graph as w, except with possible changes in the connections between x and all of the a_i's.

The following theorem will be the main ingredient in proving that all 123-representable graphs are circle graphs.

Theorem 3.4. *If a graph G is 123-representable, then there exists a 123-avoiding word w representing G such that any letter in w appears at most twice.*

Proof. Let w be a 123-representant for $G = (V, E)$. We will show that if there is a letter x which appears more than two times, we can create a new word that is still a 123-representant of G but has x appearing only twice and does not change the frequency of the other letters. Then this process can be repeated until every letter appears at most twice.

We begin by noting that if a vertex in V has degree at least two, then by Theorem 3.1, the corresponding letter appears at most twice. Hence we need only consider vertices which have degree 1 or 0.

Case 1. $d(x) = 0$. Let w_1 be the word obtained by deleting all instances of x from w, and then re-label the graph so that the vertex previously labeled x now has the largest value of any of the vertices, while preserving the relative order of the other vertices. We denote this value x'. The desired word is $x'x'w_1$. This word is still 123-avoiding since x' cannot participate in a 123-pattern as it is at the beginning of the word. Furthermore, the word still represents G because x' cannot alternate with anything, and all other alternating pairs have not been affected.

Case 2. $d(x) = 1$ and the vertex a connected to x also has degree 1. This case means that the edge xa is disconnected from the rest of the graph. Let w_1 be the word obtained from w after deleting all instances of a and x. Now we form the final word $x'a'x'a'w_1$, where x' and a' are greater than all the values of the letters in w_1. The new word is still 123-avoiding and it represents G because x' alternates with a' and neither of them alternates with any other letter.

Case 3. $d(x) = 1$ and the vertex a adjacent to x has degree at least 2. By Corollary 3.2, x cannot appear more than 3 times in w, and by Theorem 3.1, a cannot appear more than twice. Then we have the word $w = w_1x_1w_2a_1w_3x_2w_4a_2w_5x_3w_6$ where w_i are words such that $x, a \notin w_i$.

Now consider the word w_3. If w_3 is nonempty, let c be a letter in w_3. Then 5 of the 6 possible permutations of axc appear in w, namely, x_1a_1c, x_1ca_2, cx_2a_2, ca_2x_3, and a_1cx_2. The only one that does not appear is axc. Thus, for w to be 123-avoiding, $a < x < c$ must hold for any $c \in w_3$.

Similarly, if w_4 is nonempty, and it contains some c, then the only possible ordering of a, x, and c is $c < x < a$, as w contains x_1a_1c, x_1ca_2, ca_2x_3, a_1x_2c, and a_1cx_3.

Because both inequalities cannot hold simultaneously, w_3 and w_4 cannot both be nonempty.

Let b be another vertex adjacent to a. Then we know b must be either in w_3 or w_4 since it alternates with a. There are six possible relative orders of a, b, and x.

1. $x < a < b$: This is impossible since x_1a_1b is a 123-pattern.

2. $x < b < a$: This is impossible since x_1ba_2 is a 123-pattern.

3. $b < a < x$: This is impossible since ba_2x_3 is a 123-pattern.

4. $a < b < x$: This is impossible since a_1bx_3 is a 123-pattern.

5. $a < x < b$: This is only possible if $b \in w_3$. Otherwise, if $b \in w_4$, then a_1x_2b is a 123-pattern.

6. $b < x < a$: This is only possible if $b \in w_4$. Otherwise, if $b \in w_3$, then bx_2a_2 is a 123-pattern.

This gives us two subcases.

Subcase 3.1. $a < x < b$ and $b \in w_3$. This implies that w_4 is empty since w_3 is nonempty. This gives us $w = w_1x_1w_2a_1w_3'bx_2a_2w_5x_3w_6$ where $w_3 = w_3'b$

If w_5 is nonempty, consider some $c \in w_5$. Note that w contains the patterns a_1cx_3 and a_1x_2c. Then, since w is 123-avoiding, $c < a$. Thus every letter in w_5 is less than a (vacuously true if w_5 is empty). Now let $w' = w_1x_1w_2a_1w_3'bx_2a_2x_3w_5w_6$ be a subword of w with x_3 and w_5 swapped, and $w'' = w_1w_2a_1w_3'bx_2a_2x_3w_5w_6$ be w' with x_1 removed. By Lemma 3.3, w' is 123-avoiding, and thus it is clear that w'' is as well. To show that w'' represents G, it only remains to make sure that x shares an edge with a and does not share an edge with any vertex in w_5 (since $d(x) = 1$). Since a_2 is the only letter between x_2 and x_3, x can only alternate with a. In fact, it clearly does alternate with a, so we have successfully created a word w'' which only contains x twice.

Subcase 3.2. $b < x < a$ and $b \in w_4$. This implies that w_3 is empty since w_4 is nonempty. This gives us $w = w_1 x_1 w_2 a_1 x_2 b w_4' a_2 w_5 x_3 w_6$ where $w_4 = b w_4'$

If w_2 is nonempty, consider some $c \in w_2$. Note that w contains the subword $c x_2 a_2$. Thus, since w is 123-avoiding and $x < a$, we know $x < c$. Therefore every letter in w_2 is greater than x (vacuously true if w_2 is empty). Now let $w' = w_1 w_2 x_1 a_1 x_2 b w_4 a_2 w_5 w_6$.

By Lemma 3.3, w' is 123-avoiding and it only remains to make sure that x is only alternating with a. It is clear that x alternates with a, and since the only letter between x_1 and x_2 is a_1, x does not alternate with any other letters. Thus we have successfully created a word w' which is a 123-representant of G and only contains x twice.

In both cases we were able to create a new word which represents the same graph but only contains two copies of the letter x.

We have gone through all possible cases, and have shown that it is possible to create a 123-representant for a graph G which has no more than two copies of each letter, since we can repeat the process outlined above for every letter which appears more than two times. Therefore there exists a word for any 123-representable graph which contains no more than two copies of each letter. ■

The next corollary follows easily.

Corollary 3.5. *Any* 123*-representable graph is a circle graph.*

Proof. By Theorem 3.4, any 123-representable graph can be represented by a 123-avoiding word with at most two copies of each letter. From Theorem 2.10, this implies that any 123-representable graph is a circle graph. ■

Finally, we prove that not all circle graphs are 123-representable. We begin with a lemma.

Lemma 3.6. *Let x be a vertex in a graph $G = (V, E)$ with $xa, xb \in E$. If $ab \notin E$, and a and b appear on both sides of x in some word w representing G with at most two copies of each letter, then a and b appear in opposite orders on both sides of x.*

Proof. Assume a and b appear in the same order (without loss of generality, a followed by b) on each side of x. Then a alternates with b, but since $ab \notin E$ this is impossible. Thus they must appear in opposite order. ■

Theorem 3.7. *The star $K_{1,6}$ is not* 123*-representable.*

Proof. Consider the star $K_{1,6}$ (Figure 4), and suppose w is some 123-representant for it. At most one of the vertices labeled a through f can appear only once in the word, since if two appear only once then they alternate with each other,

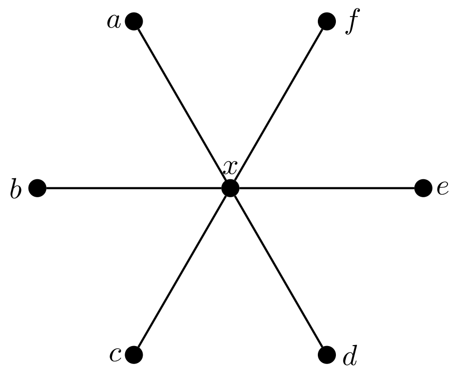

Figure 4. The star $K_{1,6}$.

a contradiction. Thus, without loss of generality, $a, b, \ldots, e$ appear twice in w. There are two cases.

Case 1. x appears once. In this case, it is sufficient to consider only a, b and c. We have one copy of each of these letters on either side of x. Without loss of generality, let $a < b < c$. Note that b must appear after c on the right of x, for otherwise $a_1b_2c_2$ forms a 123-pattern. Furthermore, a must come after b on the left of x because otherwise $a_1b_1c_2$ would form a 123-pattern. Now, by Lemma 3.6 the only two possibilities for the order in which the letters appear in w are $b_1a_1c_1xc_2a_2b_2$ and $b_1c_1a_1xa_2c_2b_2$.

We have $x > c$ because otherwise a_1xc_2 would form a 123-pattern. However, if $c < x$, a_1c_1x and b_1c_1x form 123-patterns in the first and second options, respectively. Thus the star cannot be 123-represented with only one instance of x.

Case 2. x appears twice. We must have each a, b, c, d, e appearing between x_1 and x_2. The second copies of a, b, c, d, e can be either to the right of x_2 or to the left of x_1. By the pigeonhole principle, there must be at least three letters to one side of the x's. Without loss of generality, let a, b, and c appear on both sides of x_1. Then we have reduced this argument to the same one examined in Case 1, so we are done.

Therefore, the star $K_{1,6}$ is not 123-representable. ■

Corollary 3.8. *Not all circle graphs are* 123*-representable.*

Proof. In the previous theorem we showed that the 7-star is not 123-representable. However, since it is a tree, it is a circle graph [6]. Thus not all circle graphs are 123-representable. ■

3.1. Examples of 123-representable graphs

In this section we give three families of 123-representble graphs.

Theorem 3.9. *Complete graphs are* 123*-representable.*

Proof. The complete graph on n vertices can be represented by the word $n(n-1)\cdots 21$. Clearly this is 123-avoiding and represents the complete graph since each letter appears once, thereby alternating with every other letter. ∎

Theorem 3.10. *Paths are* 123*-representable.*

Proof. A path on n vertices, with the natural labeling (consecutively labeled with 1 at one endpoint and n at the other) can be represented by the 123-avoiding word $n(n-1)n(n-2)(n-1)(n-3)(n-2)\cdots 23121$. This is clearly 123-avoiding and represents a path since every vertex alternates only with the one before it and after it, with the exception of 1 and n, which alternate only with 2 and $n-1$, respectively. ∎

Theorem 3.11. *Cycles are* 123*-representable.*

Proof. A cycle on n vertices with the natural labeling (same as the path in Theorem 3.10 but with 1 and n adjacent) can be represented by the 123-avoiding word $(n-1)n(n-2)(n-1)(n-3)(n-2)\cdots 2312$. This is the same word as the one which represents a path, except with the first n and last 1 deleted to make it alternate with 1 as well as with $n-1$. ∎

4. Graphs Representable by Pattern Avoiding 2-Uniform Words

In this section we provide some properties of graphs which are representable by either 123-avoiding or 132-avoiding 2-uniform words. As stated in Lemma 2.11, the disjoint union of graphs which can be represented by 132-avoiding 2-uniform words is a 132-representable graph as well. Here, we prove a more general result.

Theorem 4.1. *Let $k \geq 1$ and $G_1, G_2, \ldots, G_k$ be connected* 132(123)*-representable components of a graph G. Then G is* 132(123)*-representable if and only if at most one of the connected components cannot be* 132(123)*-represented by a* 2*-uniform word.*

Proof. First we show that if at least 2 of the components are not 132(123)-representable by 2-uniform words, then G is not 132(123)-representable. We first prove this for 132-representable graphs.

Assume for the sake of contradiction that G is 132-representable and can be represented by some w. Let components G_i and G_j be graphs not representable by a 132-avoiding 2-uniform word. By Theorems 2.9 and 2.10 we can assume that w has at most 2 copies of each letter.

It is clear that w must have subwords w_i and w_j, formed by removing all letters from w that do not appear in G_i and G_j, respectively, which represent both G_i and G_j (otherwise it would be impossible for them to be components of G). It is impossible for either w_i or w_j not to be 132-avoiding, thus G_i and G_j must both be 132-representable. Now, if w_i is not 2-uniform, then some letter a must appear only once (as no letter appears more than two times). Similarly, some b must appear only once in w_j. However, this would mean that a and b alternate, and then G_i and G_j are not disconnected components of G. We have reached a contradiction, thus we are done.

Next we show that G is 132-representable if it has at most one component that is not representable by a 132-avoiding 2-uniform word.

First we relabel the components the following way: G_1 has vertices $1, 2, \ldots, t_1$; G_2 has vertices $t_1 + 1, t_1 + 2, \ldots, t_2$; $\ldots$; G_k has vertices $t_{k-1} + 1, \ldots, t_k$. Let $w_1, w_2, \ldots, w_k$ be 132-representants for $G_1, G_2, \ldots, G_k$, respectively. Now we show that the word $w = w_k w_{k-1} \cdots w_1$ is a 132-representant for G. It is easy to see that w is 132-representable, as no w_i contains a 132-pattern, and all letters appearing after w_i are smaller than all letters appearing in w_i. Furthermore, w represents G since it is impossible for any letter appearing twice in w to alternate with anything that is not in its component's word. Since at most one of the w_i's can have letters appearing fewer than two times, they will not alternate with any letters in any other w_j. Thus G is 132-representable.

The proof for 123-representable graphs is exactly the same, except we use Theorem 3.4 instead of Theorems 2.9 and 2.10. ∎

Note that this gives us a new class of 123- and 132-representable graphs, namely those formed by the disjoint union of several graphs representable by 2-uniform words and one that possibly is not 2-uniform.

4.1. Graphs representable by 123-avoiding 2-uniform words

Here we prove that linear forests and P_3-free graphs can be 123-represented by 2-uniform words. We were not able to find examples of 123-representable graphs that require non-2-uniform words but we suspect that cycle graphs could be such examples. See Question 2.

Theorem 4.2. *Any linear forest is 123-representable by a 2-uniform word.*

Proof. We can see that paths are 123-representable by a 2-uniform word, which follows directly from the construction given in Theorem 3.10. Furthermore, note that the proof of Theorem 4.1 gives a construction for a 123-representant for the disjoint union G of graphs $G_1, \ldots, G_k$ representable by 2-uniform 123-avoiding words $w_1, \dots, w_k$ which maintains the number of copies of each letter. Thus the word representing G given in this construction is also 2 uniform, thus telling us that linear forests, i.e., disjoint unions of paths, are also 123-representable by 2-uniform words. ■

Theorem 4.3. *Any P_3-free graph is 123-representable by a 2-uniform word.*

Proof. A complete graph on n vertices can be represented by the 2-uniform word $n(n-1)\cdots 1n(n-1)\cdots 1$. Then, by the same reasoning as in Theorem 4.2 and by Theorem 4.1, any disjoint union of complete graphs (i.e., a P_3-free graph) is also 123-representable by a 2-uniform word. ■

4.2. Graphs representable by 132-avoiding 2-uniform words

Theorem 4.4. *Any forest is 132-representable by a 2-uniform word.*

Proof. The recursive algorithm provided in [1] for 132-representants for trees gives a word which has one copy of the root vertex and two copies of every other vertex. The recursively generated word is $w(T_r)w(T_{r-1})\cdots w(T_1)1n_1n_2\cdots n_r$, where 1 is the root vertex, $n_1, n_2, \dots, n_r$ are the children of the root vertex, and $w(T_m)$ denotes the word generated in the same way, but representing the subtree with n_m's as the root vertices. We claim that the word $w(T_r)w(T_{r-1})\cdots w(T_1)1n_1 n_2\cdots n_r1$ is also 132-avoiding and represents the same tree. It is easy to see that it is still 132-avoiding since adding 1 at the end cannot possibly form a 132-pattern. Furthermore, 1 is still alternating with all of n_i, and not with any other vertices. Thus any tree can be represented by a 2-uniform 132-avoiding word. Then, by Theorem 4.1 and the same reasoning as in Theorem 4.2, we conclude that any forest is 132-representable by a 2-uniform word. ■

Theorem 4.5. *The complete graph K_n, with $n > 3$, is not 132-representable by a 2-uniform word.*

Proof. We prove this by contradiction. Assume there is a 132-avoiding 2-uniform word w that represents a complete graph on at least 4 vertices. Since 1 must appear twice in w, we have $w = w_11w_21w_3$, with 2, 3, and 4 appearing in w_2. It is easy to see that they must appear in increasing order, since otherwise there will be a 132-pattern. Each of 2, 3, and 4 must also appear either in w_1 or in w_3. Neither 2 nor 3 cannot appear in w_3 since otherwise they form a 132-pattern with the first 1 and the 4 in w_2. Thus they appear in w_1 in increasing order because otherwise the 2 and 3 would not be alternating. Finally, a 4 cannot appear in w_3 because then it would not be alternating with 2 or with 3. If the 4 is in w_1 it must come before the 2, since otherwise it would form a 132-pattern with the 2 in w_1 and the 3 in w_2. However, this makes it impossible for the 4 to alternate with 2 and 3, a contradiction. This completes the proof. ■

5. Circle Graphs

Our final result in this paper answers a question posed in [1].

Theorem 5.1. *Not all circle graphs are* 132*-representable.*

Proof. As shown in Theorem 4.5, K_4 is not representable by a 2-uniform 132-avoiding word. Then, by Theorem 4.1, the disjoint union of two complete graphs of size 4 (Figure 5) is not 132-representable. However, Figure 6 demonstrates that it is a circle graph. Thus not all circle graphs are 132-representable. ∎

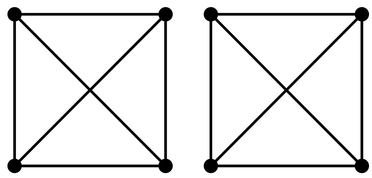

Figure 5. The disjoint union of two complete graphs of size 4.

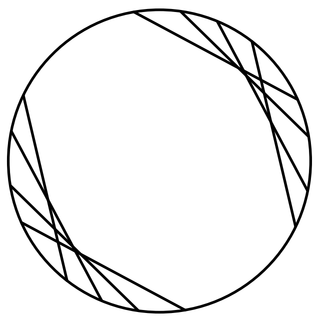

Figure 6. The circle representation of the disjoint union of two complete graphs of size 4, which demonstrates that this is indeed a circle graph.

One might wonder if all circle graphs are either 123- or 132-representable. This is not true, as can be seen in the simple counterexample in Figure 7. It is not 123-representable for the same reason that a star on 7 vertices is not, and it is not 132-representable for the same reason that the disjoint union of two complete graphs greater than K_3 is not 132-representable.

6. Open Research Directions

A natural next step in the study of pattern-representable graphs would be to investigate longer patterns, or to find more examples of 132/123-representable

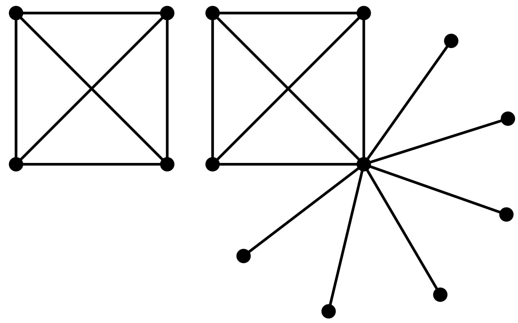

Figure 7. An example of a circle graph which is neither 132-representable nor 123-representable.

and non-representable graphs. In particular, the following questions may be of interest.

Question 1. *Is the disjoint union of two complete graphs of size* 4 *the smallest non*-132-*representable circle graph*?

Question 2. *Can all* 123-*representable graphs be* 2-*uniformly representable*?

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