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# GREGARIOUS KITE FACTORIZATION OF TENSOR PRODUCT OF COMPLETE GRAPHS

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**Abstract**

A kite factorization of a multipartite graph is said to be gregarious if every kite in the factorization has all its vertices in different partite sets. In this paper, we show that there exists a gregarious kite factorization of $K_m \times K_n$ if and only if $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$, where $\times$ denotes the tensor product of graphs.

**Keywords:** tensor product, kite, decomposition, gregarious factor, factorization.

**2010 Mathematics Subject Classification:** 05C70.

## 1. Introduction

A *latin square* of order $n$ is an $n \times n$ array such that each row and each column of the array contains each of the symbols from $\{1, 2, \ldots, n\}$ exactly once. Two latin squares $L_1$ and $L_2$ of order $n$ are said to be *orthogonal* if for each $(x, y) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ there is exactly one cell $(i, j)$ in which $L_1$ contains the symbol $x$ and $L_2$ contains the symbol $y$. In other words, if $L_1$ and $L_2$ are superimposed, the resulting set of $n^2$ ordered pairs are distinct. The latin squares $L_1, L_2, \ldots, L_t$ of order $n$ are said to be *mutually orthogonal* ($MOLS(n)$) if for $1 \leq a \neq b \leq t$, $L_a$ and $L_b$ are orthogonal. $N(n)$ denotes the maximum number of $MOLS(n)$.

Partition of $G$ into subgraphs $G_1, G_2, \ldots, G_r$ such that $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$, $i, j \in \{1, 2, \ldots, r\}$ and $E(G) = \bigcup_{i=1}^{r} E(G_i)$ is called *decomposition* of $G$; in this case we write $G$ as $G = G_1 \oplus G_2 \oplus \cdots \oplus G_r$, where $\oplus$ denotes edge-disjoint sum of subgraphs. If $G_i \cong H$, $1 \leq i \leq r$, then we say that *$H$-decomposes* $G$; in notation $H \mid G$. A spanning subgraph of $G$ such that each component of it is isomorphic to some graph $H$ is called an *$H$-factor* of $G$. A partition of $G$ into edge-disjoint $H$-factors is called an *$H$-factorization* of $G$; in notation $H \parallel G$. Let $C_k$, $K_k$ and $I_k$, respectively denote a cycle, a complete graph and a null graph on $k$ vertices. A $k$-regular spanning subgraph of $G$ is called a *$k$-factor* of $G$. A *$C_k$-factor* of $G$ is a 2-factor in which each component is a $C_k$. Decomposition of $G$ into $C_k$-factors is called a *$C_k$-factorization* of $G$. A cycle containing all the vertices of $G$ is called a *Hamilton cycle*. We say that $G$ has a *Hamilton cycle decomposition* if its edge set can be partitioned into edge-disjoint Hamilton cycles. For an integer $\lambda$, $\lambda G$ denotes a graph with $\lambda$ components each isomorphic to $G$.

The *tensor product* $G \times H$ and the *wreath product* $G \otimes H$ of two simple graphs $G$ and $H$ are defined as follows: $V(G \times H) = V(G \otimes H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$. $E(G \times H) = \{(u, v)(x, y) \mid ux \in E(G)$ and $vy \in E(H)\}$ and $E(G \otimes H) = \{(u, v)(x, y) \mid u = x$ and $vy \in E(H)$, or $ux \in E(G)\}$. It is well known that tensor product is commutative and distributive over an edge-disjoint union of subgraphs, that is, if $G{=}G_1 \oplus G_2 \oplus \cdots \oplus G_r$, then $G \times H = (G_1 \times H) \oplus (G_2 \times H) \oplus \cdots \oplus (G_r \times H)$. A graph $G$ having partite sets $V_1, V_2, \ldots, V_m$ with $\mid V_i \mid = n$, $1 \leq i \leq n$, and $E(G) = \{uv \mid u \in V_i$ and $v \in V_j, i \neq j\}$ is called *complete $m$-partite graph* and is denoted by $K_m(n)$. Note that $K_m(n)$ is same as the $K_m \otimes I_n$.

A *kite* is a graph which is obtained by attaching an edge to a vertex of the triangle, see Figure 1. We denote the kite with edge set $\{ab, bc, ca, cd\}$ by $(a, b, c; cd)$.

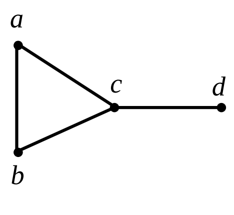


Figure 1. The kite graph.

A subgraph of a multipartite graph $G$ is said to be *gregarious* if each of its vertices lies in different partite sets of $G$. A kite factorization of a multipartite graph is said to be *gregarious* if each kite in the factorization has its vertices in four different partite sets.

The study of kite-design is not new. Bermond and Schonheim [3] proved that a kite-design of order $n$ exists if and only if $n \equiv 0, 1 \pmod 8$. Wang and Chang [18, 19] considered the existence of $(K_3+e)$ and $(K_3+e, \lambda)$-group divisible designs of type $g^t u^1$. Wang [17] has shown that the obvious necessary conditions for the existence of resolvable $(K_3+e)$-group divisible design of type $g^u$ are also sufficient. Fu *et al.* [5] have shown that there exists a gregarious kite decomposition of $K_m(n)$ if and only if $n \equiv 0, 1 \pmod 8$ for odd $m$ or $n \geq 4$ for even $m$. Gionfriddo and Milici [6] considered the existence of uniformly resolvable decompositions of $K_v$ and $K_v - I$ into paths and kites. For more results on kite designs, see [4, 7, 9, 11, 12].

In this direction, in [15] we have shown that the necessary conditions for the existence of a gregarious kite decomposition of tensor product of complete graphs are also sufficient. Further, in this paper, we show that there exists a gregarious kite factorization of $K_m \times K_n$ if and only if $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$.

We require the following to prove our main results.

## 2. Preliminary Results

**Theorem 1** [10]**.** *There exists a pair of mutually orthogonal latin squares ($MOLS(n)$) of order $n$ for every $n \neq 2, 6$.*

**Theorem 2** [1]**.** *If $n = p^d$ is a prime power, then $N(n) = n - 1$.*

**Corrolary 3** [2]**.** *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each number $p_i$ is a distinct prime number and $\alpha_i \geq 1, i = 1, 2, \ldots, t$, then $N(n) \geq \min\{p_i^{\alpha_i} \mid i = 1, 2, \ldots, t\}$.*

**Theorem 4** [8]**.** *Let $G$ be a graph with chromatic number $\chi(G)$. Then*

(i) *$G \mid G \otimes I_n$ if $\chi(G) \leq N(n) + 2$ and*

(ii) *$G \parallel G \otimes I_n$ if $\chi(G) \leq N(n) + 1$.*

**Theorem 5** [17]**.** *The necessary conditions for the existence of a kite factorization of $K_m(n)$, namely, $m \geq 3$, $n(m-1) \equiv 0 \pmod 2$, $mn \equiv 0 \pmod 4$ are also sufficient.*

**Theorem 6** [13]**.** *$C_3 \parallel K_m$ if and only if $m \equiv 3 \pmod 6$.*

**Note 7.** Let $G_1 = v_1v_2v_3v_4v_5 \cdots v_{p-1}v_pv_1$, $G_2 = v_1v_3v_5 \cdots v_pv_2v_4v_6 \cdots v_{p-3}v_{p-1}v_1$ and $G_3 = v_1v_5v_9 \cdots v_{p-1}v_3v_7v_{11} \cdots v_{p-3}v_1$ be three cycles of length $p$ ($p$ is odd). Now consider two graphs $G = G_1 \oplus G_2$ and $H = G_1 \oplus G_2 \oplus G_3$ as shown in Figures 2 and 3.

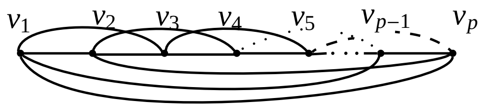


Figure 2. $G = G_1 \oplus G_2$.

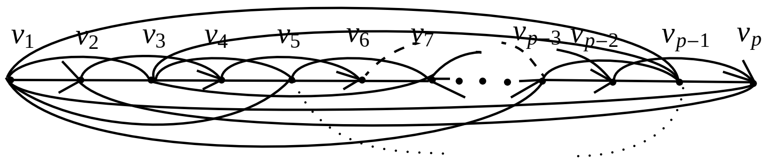


Figure 3. $H = G_1 \oplus G_2 \oplus G_3$.

**Remark 8** [16]**.** Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$, $p$ is a prime. For $1 \leq i \leq (p-1)/2$, let $H_i = v_1 v_{(2+(i-1))} v_{(3+[2(i-1)])} v_{(4+[3(i-1)])} v_{(5+[4(i-1)])} \cdots v_{(p+[(p-1)(i-1)])} v_1$, where the subscripts are taken modulo $p$ with residues $1, 2, 3, \ldots, p$. Note that each $H_i$ is a Hamilton cycle of $K_p$ and $\{H_1, H_2, \ldots, H_{(p-1)/2}\}$ gives a Hamilton cycle decomposition of $K_p$, $p$ is a prime. Further, $\{H_1, H_2, \ldots H_{(p-1)/2}\}$ can be partitioned into sets of 2 or 3 cycles such that the sum of the cycles of each set is isomorphic to $G$ or $H$, respectively.

## 3. Gregarious Kite Factorization of $K_m \times K_n$

**Lemma 9.** *There exists a gregarious kite factorization of* $K_4 \times K_3$.

***Proof.*** Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(K_3) = \{1, 2, 3\}$. Then $V(K_4 \times K_3) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 3\}$. Now we construct a gregarious kite factorization of $K_4 \times K_3$ as follows: For $0 \leq s \leq 2$, let $F_s^1 = \{1_{1+s}, 2_{2+s}, 4_{3+s}; 4_{3+s}3_{1+s}\}$; $F_s^2 = \{2_{1+s}, 4_{3+s}, 3_{2+s}; 3_{2+s}1_{3+s}\}$; $F_s^3 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\}$, where the subscripts are taken modulo 3 with residues $1, 2, 3$. Clearly each $F_i = \bigcup_{s=0}^{2} F_s^i$, $1 \leq i \leq 3$, is a gregarious kite factor of $K_4 \times K_3$ and $\{F_1, F_2, F_3\}$ gives a gregarious kite factorization of $K_4 \times K_3$. ■

**Lemma 10.** *For* $n \equiv 3 \pmod 6$, *there exists a gregarious kite factorization of* $K_4 \times K_n$.

***Proof.*** By Theorem 6, we have a $K_3$-factorization of $K_n, n = 6s+3, s \geq 1$ (since the case $s = 0$ follows from Lemma 9). Since tensor product is distributive over an edge-disjoint union of subgraphs, corresponding to each $K_3$-factor of $K_n$, we have a $(K_4 \times K_3)$-factor of $K_4 \times K_n$. Hence a $K_3$-factorization of $K_n$ gives a $(K_4 \times K_3)$-factorization of $K_4 \times K_n$. By Lemma 9, we have a gregarious kite factorization of $K_4 \times K_3$. Thus combining all these we get a gregarious kite factorization of $K_4 \times K_n$. ∎

**Lemma 11.** *For $|V(G)| = p$, $p \geq 5$ is a prime, there exists a gregarious kite factorization of $K_4 \times G$, where $G$ is described as in Note 7.*

***Proof.*** Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(G) = \{1, 2, \ldots, p\}$, $p \geq 5$. Then $V(K_4 \times G) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_4 \times G$ as follows: For $0 \leq s \leq p-1$, let

$$\begin{aligned} F_s^1 &= \{3_{1+s}, 2_{p+s}, 1_{p-1+s}; 1_{p-1+s}4_{1+s}\}; F_s^2 = \{3_{1+s}, 4_{2+s}, 2_{3+s}; 2_{3+s}1_{1+s}\}; \\ F_s^3 &= \{3_{3+s}, 1_{2+s}, 4_{1+s}; 4_{1+s}2_{p-1+s}\}; F_s^4 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\}; \\ F_s^5 &= \{3_{1+s}, 4_{p+s}, 2_{p-1+s}; 2_{p-1+s}1_{1+s}\}; F_s^6 = \{3_{p+s}, 1_{1+s}, 4_{2+s}; 4_{2+s}2_{4+s}\}. \end{aligned}$$

In all the above constructions the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \leq i \leq 6$, is a gregarious kite factor of $K_4 \times G$ and $\{F_1, F_2, \ldots, F_6\}$ gives a gregarious kite factorization of $K_4 \times G$. ∎

**Lemma 12.** *There exists a gregarious kite factorization of $K_4 \times K_7$.*

***Proof.*** Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(K_7) = \{1, 2, \ldots, 7\}$. Then $V(K_4 \times K_7) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 7\}$. Now we construct a gregarious kite factorization of $K_4 \times K_7$ as follows: For $0 \leq s \leq 6$, let

$$\begin{aligned} F_s^1 &= \{3_{1+s}, 2_{7+s}, 1_{6+s}; 1_{6+s}4_{2+s}\}; F_s^2 = \{4_{1+s}, 2_{2+s}, 3_{7+s}; 3_{7+s}1_{4+s}\}; \\ F_s^3 &= \{1_{1+s}, 3_{2+s}, 4_{7+s}; 4_{7+s}2_{4+s}\}; F_s^4 = \{3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{1+s}\}; \\ F_s^5 &= \{3_{1+s}, 4_{5+s}, 2_{4+s}; 2_{4+s}1_{6+s}\}; F_s^6 = \{1_{1+s}, 2_{4+s}, 3_{7+s}; 3_{7+s}4_{2+s}\}; \\ F_s^7 &= \{4_{1+s}, 2_{4+s}, 1_{7+s}; 1_{7+s}3_{4+s}\};\ F_s^8 = \{1_{1+s}, 2_{3+s}, 4_{5+s}; 4_{5+s}3_{6+s}\}; \\ F_s^9 &= \{2_{1+s}, 3_{3+s}, 4_{6+s}; 4_{6+s}1_{4+s}\}. \end{aligned}$$

In all the above constructions the subscripts are taken modulo 7 with residues $1, 2, \ldots, 7$. Clearly each $F_i = \bigcup_{s=0}^{6} F_s^i$, $1 \leq i \leq 9$, is a gregarious kite factor of $K_4 \times K_7$ and $\{F_1, F_2, \ldots, F_9\}$ gives a gregarious kite factorization of $K_4 \times K_7$. ∎

**Lemma 13.** *For $|V(H)| = p$, $p \geq 11$ is a prime, there exists a gregarious kite factorization of $K_4 \times H$, where $H$ is described as in Note 7.*

***Proof.*** Let $V(K_4) = \{1, 2, 3, 4\}$ and $V(H) = \{1, 2, \ldots, p\}$, $p \geq 11$. Then $V(K_4 \times H) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_4 \times H$ as follows: For $0 \le s \le p-1$, let

$$\begin{aligned}
F_s^1 &= \{4_{p-1+s}, 1_{p+s}, 2_{1+s}; 2_{1+s}3_{5+s}\}; F_s^2 = \{4_{2+s}, 3_{3+s}, 1_{1+s}; 1_{1+s}2_{5+s}\};\\
F_s^3 &= \{1_{p+s}, 4_{2+s}, 3_{1+s}; 3_{1+s}2_{5+s}\}; F_s^4 = \{1_{3+s}, 2_{p-1+s}, 3_{1+s}; 3_{1+s}4_{5+s}\};\\
F_s^5 &= \{2_{p-1+s}, 1_{p-3+s}, 4_{1+s}; 4_{1+s}3_{5+s}\}; F_s^6 = \{4_{p-1+s}, 2_{p+s}, 1_{1+s}; 1_{1+s}3_{5+s}\};\\
F_s^7 &= \{3_{2+s}, 1_{3+s}, 2_{1+s}; 2_{1+s}4_{5+s}\}; F_s^8 = \{2_{3+s}, 4_{p-1+s}, 3_{1+s}; 3_{1+s}1_{5+s}\};\\
F_s^9 &= \{2_{p+s}, 3_{p-1+s}, 4_{1+s}; 4_{1+s}1_{5+s}\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \le i \le 9$, is a gregarious kite factor of $K_4 \times H$ and $\{F_1, F_2, \ldots, F_9\}$ gives a gregarious kite factorization of $K_4 \times H$. ■

**Lemma 14.** *For all odd prime $p$, there exists a gregarious kite factorization of $K_4 \times K_p$.*

***Proof.*** By Remark 8, we have a factorization of $K_p$ into graphs isomorphic to $G$ or $H$. A gregarious kite factorization of $K_4 \times K_p$ follows from Lemmas 9, 11, 12 and 13. ■

**Lemma 15.** *For all odd prime $p$ and $s > 1$, there exists a gregarious kite factorization of $K_4 \times K_{p^s}$.*

***Proof.*** For $s > 1$, $K_4 \times K_{p^s} = K_4 \times \left[pK_{p^{s-1}} \oplus K_p(p^{s-1})\right] = p\left(K_4 \times K_{p^{s-1}}\right) \oplus \left[K_4 \times K_p(p^{s-1})\right]$ (since the case $s = 1$ follows from Lemma 14).

For $s = 2$, $K_4 \times K_{p^2} = p\,(K_4 \times K_p) \oplus [K_4 \times K_p(p)]$. By Lemma 14, we have a gregarious kite factorization of $p\,(K_4 \times K_p)$. By Theorem 4, we have a $K_p$-factorization of $K_p(p)$. Corresponding to each $K_p$-factor of $K_p(p)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(p)$. Hence a $K_p$-factorization of $K_p(p)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(p)$. Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14.

For $s = 3$, $K_4 \times K_{p^3} = p\left(K_4 \times K_{p^2}\right) \oplus \left[K_4 \times K_p(p^2)\right]$. Now the gregarious kite factorization of $p\left(K_4 \times K_{p^2}\right)$ follows from the case $s = 2$. By Theorem 4, we have a $K_p$-factorization of $K_p(p^2)$. Corresponding to each $K_p$-factor of $K_p(p^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(p^2)$. Hence a $K_p$-factorization of $K_p(p^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(p^2)$. Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14.

For $s > 1$, $K_4 \times K_{p^s} = p\left(K_4 \times K_{p^{s-1}}\right) \oplus \left[K_4 \times K_p(p^{s-1})\right]$. By the induction hypothesis on $s$, we have a gregarious kite factorization of $p\left(K_4 \times K_{p^{s-1}}\right)$. By Theorem 4, we have a $K_p$-factorization of $K_p(p^{s-1})$. Corresponding to each $K_p$-factor of $K_p\left(p^{s-1}\right)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p\left(p^{s-1}\right)$. Thus a $K_p$-factorization of $K_p\left(p^{s-1}\right)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p\left(p^{s-1}\right)$.

Now the existence of a gregarious kite factorization of $(K_4 \times K_p)$ follows from Lemma 14. Hence combining all the above we have a gregarious kite factorization of $K_4 \times K_{p^s}$, for all $s > 1$. ■

**Lemma 16.** *For all odd primes $p, q$ $(p < q)$ and all integers $s, t \geq 1$, there exists a gregarious kite factorization of $K_4 \times K_{p^s q^t}$.*

***Proof.*** For $s, t \geq 1$ and $p < q$, $K_4 \times K_{p^s q^t} = K_4 \times K_{p.p^{s-1}q^t} = K_4 \times \left[pK_{p^{s-1}q^t} \oplus K_p\left(p^{s-1}q^t\right)\right] = p\left[K_4 \times K_{p^{s-1}q^t}\right] \oplus \left[K_4 \times K_p(p^{s-1}q^t)\right]$.

*Case* 1. (a) For $s = 1,\ t = 1$, $K_4 \times K_{pq} = K_4 \times (pK_q \oplus K_p(q)) = p\left[K_4 \times K_q\right] \oplus \left[K_4 \times K_p(q)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q)$. Corresponding to each $K_p$-factor of $K_p(q)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q)$. Thus a $K_p$-factorization of $K_p(q)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left(K_4 \times K_q\right)$ follows from Lemma 14.

(b) For $s = 1,\ t = 2$, $K_4 \times K_{pq^2} = K_4 \times \left[pK_{q^2} \oplus K_p(q^2)\right] = p\left[K_4 \times K_{q^2}\right] \oplus \left[K_4 \times K_p(q^2)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q^2)$. Corresponding to each $K_p$-factor of $K_p(q^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q^2)$. Thus a $K_p$-factorization of $K_p(q^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q^2)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left(K_4 \times K_{q^2}\right)$ follows from Lemmas 14 and 15, respectively.

(c) For $s = 1,\ t \geq 3$, $K_4 \times K_{pq^t} = K_4 \times \left[pK_{q^t} \oplus K_p(q^t)\right] = p\left[K_4 \times K_{q^t}\right] \oplus \left[K_4 \times K_p(q^t)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q^t)$. Corresponding to each $K_p$-factor of $K_p(q^t)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(q^t)$. Thus a $K_p$-factorization of $K_p(q^t)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(q^t)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left(K_4 \times K_{q^t}\right)$ follows from Lemmas 14 and 15, respectively.

*Case* 2. (a) For $s = 2,\ t = 1$, $K_4 \times K_{p^2 q} = K_4 \times K_{p.pq} = K_4 \times \left[pK_{pq} \oplus K_p(pq)\right] = p\left[K_4 \times K_{pq}\right] \oplus \left[K_4 \times K_p(pq)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq)$. Corresponding to each $K_p$-factor of $K_p(pq)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq)$. Thus a $K_p$-factorization of $K_p(pq)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq}\right]$ follows from Lemma 14 and Case 1(a), respectively.

(b) For $s = 2,\ t = 2$, $K_4 \times K_{p^2 q^2} = K_4 \times K_{p.pq^2} = K_4 \times \left[pK_{pq^2} \oplus K_p(pq^2)\right] = p\left[K_4 \times K_{pq^2}\right] \oplus \left[K_4 \times K_p(pq^2)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq^2)$. Corresponding to each $K_p$-factor of $K_p(pq^2)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq^2)$. Thus a $K_p$-factorization of $K_p(pq^2)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq^2)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq^2}\right]$ follows from Lemma 14 and Case 1(b), respectively.

(c) For $s = 2, t \geq 3$, $K_4 \times K_{p^2 q^t} = K_4 \times K_{p.pq^t} = K_4 \times \left[pK_{pq^t} \oplus K_p(pq^t)\right] = p\left[K_4 \times K_{pq^t}\right] \oplus \left[K_4 \times K_p(pq^t)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq^t)$. Corresponding to each $K_p$-factor of $K_p(pq^t)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p(pq^t)$. Thus a $K_p$-factorization of $K_p(pq^t)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p(pq^t)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ and $p\left[K_4 \times K_{pq^t}\right]$ follows from Lemma 14 and Case 1(c), respectively.

(d) For $s, t \geq 1$, $K_4 \times K_{p^s q^t} = p\left[K_4 \times K_{p^{s-1}q^t}\right] \oplus \left[K_4 \times K_p(p^{s-1}q^t)\right]$. By induction hypothesis on $s$, we have a gregarious kite factorization of $p\left[K_4 \times K_{p^{s-1}q^t}\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p\left(p^{s-1}q^t\right)$. Corresponding to each $K_p$-factor of $K_p\left(p^{s-1}q^t\right)$, we have a $(K_4 \times K_p)$-factor of $K_4 \times K_p\left(p^{s-1}q^t\right)$. Thus a $K_p$-factorization of $K_p\left(p^{s-1}q^t\right)$ implies a $(K_4 \times K_p)$-factorization of $K_4 \times K_p\left(p^{s-1}q^t\right)$. Now the existence of a gregarious kite factorization of $K_4 \times K_p$ follows from Lemma 14. Hence combining all the above we have a gregarious kite factorization of $K_4 \times K_{p^s q^t}$, for all $s, t \geq 1$ and $p < q$. ∎

**Lemma 17.** *For all odd $n > 1$, there exists a gregarious kite factorization of $K_4 \times K_n$.*

***Proof.*** By fundamental theorem of arithmetic, any integer $n > 1$ can be uniquely written as prime powers or product of prime powers. Consider $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each $p_i$ is a distinct odd prime and $\alpha_i \geq 1, i = 1, 2, \ldots, t$. Fix $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_t^{\alpha_t}$. Now,

$$\begin{aligned} K_4 \times K_n &= K_4 \times K_{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}} = K_4 \times \left[p_1^{\alpha_1} K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}} \oplus K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right] \\ &= p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus \left[K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]. \end{aligned}$$

It is enough to show that there exists a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$ and $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right]$.

*Case* 1. Consider $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. By Theorem 4, we have a $K_{p_1^{\alpha_1}}$-factorization of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. Corresponding to each $K_{p_1^{\alpha_1}}$-factor of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$, we have a $\left(K_4 \times K_{p_1^{\alpha_1}}\right)$-factor of $K_4 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t})$. Now the existence of a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}}$ follows from Lemma 15.

*Case* 2. Consider $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right]$. We write

$$\begin{aligned} p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}\left\{K_4 \times \left[p_2^{\alpha_2} K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}} \oplus K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1}\left\{p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus \left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1} p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3} \cdots p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3} \cdots p_t^{\alpha_t})\right]. \end{aligned}$$

Now we have to show the existence of gregarious kite factorization of $p_1^{\alpha_1}p_2^{\alpha_2}$ $\left[K_4 \times K_{p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right]$ and $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$. The existence of gregarious kite factorization of $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1.

Now we can write

$$\begin{aligned} p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}p_2^{\alpha_2}\left\{K_4 \times \left[p_3^{\alpha_3}K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}} \oplus K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]\right\} \\ &= p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_4 \times K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}}\right] \\ &\oplus p_1^{\alpha_1}p_2^{\alpha_2}\left[K_4 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]. \end{aligned}$$

The existence of gregarious kite factorization of the second term $p_1^{\alpha_1}p_2^{\alpha_2}$ $\left[K_4 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1.

Now we consider the first term $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_4 \times K_{p_4^{\alpha_4}\cdots p_t^{\alpha_t}}\right]$ and repeat the above process until we end up with $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-1}^{\alpha_{t-1}}\left[K_4 \times K_{p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-2}^{\alpha_{t-2}}$ $\left[K_4 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})\right]$. Now the existence of a gregarious kite factorization of $K_4 \times K_{p_t^{\alpha_t}}$ and hence the first term follows from Lemma 15 and the existence of gregarious kite factorization of $K_4 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})$ and hence the second term is similar to Case 1. Thus we have a gregarious kite factorization of $p_1^{\alpha_1}\left[K_4 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right]$.

Hence from Cases 1 and 2, we have a gregarious kite factorization of $K_4 \times K_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}} = K_4 \times K_n$. ■

**Lemma 18.** *There exists a gregarious kite factorization of $K_8 \times K_3$.*

***Proof.*** Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(K_3) = \{1, 2, 3\}$. Then $V(K_8 \times K_3) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 3\}$. Now we construct a gregarious kite factorization of $K_8 \times K_3$ as follows: For $0 \leq s \leq 2$, let

$$\begin{aligned} F_s^1 &= \{(1_{1+s}, 2_{3+s}, 3_{2+s}; 3_{2+s}6_{1+s})\,(8_{1+s}, 5_{3+s}, 4_{2+s}; 4_{2+s}7_{3+s})\}; \\ F_s^2 &= \{(1_{1+s}, 5_{3+s}, 7_{2+s}; 7_{2+s}2_{3+s})\,(3_{1+s}, 6_{2+s}, 4_{3+s}; 4_{3+s}8_{1+s})\}; \\ F_s^3 &= \{(4_{1+s}, 2_{3+s}, 1_{2+s}; 1_{2+s}3_{1+s})\,(5_{1+s}, 8_{3+s}, 6_{2+s}; 6_{2+s}7_{1+s})\}; \\ F_s^4 &= \{(6_{1+s}, 2_{2+s}, 8_{3+s}; 8_{3+s}1_{1+s})\,(7_{1+s}, 5_{3+s}, 3_{2+s}; 3_{2+s}4_{3+s})\}; \\ F_s^5 &= \{(2_{1+s}, 5_{3+s}, 6_{2+s}; 6_{2+s}1_{1+s})\,(3_{1+s}, 8_{3+s}, 7_{2+s}; 7_{2+s}4_{3+s})\}; \\ F_s^6 &= \{(1_{1+s}, 8_{2+s}, 7_{3+s}; 7_{3+s}6_{2+s})\,(2_{1+s}, 4_{3+s}, 5_{2+s}; 5_{2+s}3_{3+s})\}; \\ F_s^7 &= \{(6_{1+s}, 4_{3+s}, 1_{2+s}; 1_{2+s}5_{3+s})\,(8_{1+s}, 3_{3+s}, 2_{2+s}; 2_{2+s}7_{3+s})\}. \end{aligned}$$

In all the above constructions the subscripts are taken modulo 3 with residues $1, 2, 3$. Clearly each $F_i = \bigcup_{s=0}^{2} F_s^i$, $1 \leq i \leq 7$, is a gregarious kite factor of $K_8 \times K_3$ and $\{F_1, F_2, \ldots, F_7\}$ gives a gregarious kite factorization of $K_8 \times K_3$. ■

**Lemma 19.** *For* $n \equiv 3 \pmod 6$, *there exists a gregarious kite factorization of* $K_8 \times K_n$.

***Proof.*** By Theorem 6, we have a $K_3$-factorization of $K_n$, $n = 6s+3$, $s \geq 1$ (since the case $s = 0$ follows from Lemma 18). Corresponding to each $K_3$-factor of $K_n$, we have a $(K_8 \times K_3)$-factor of $K_8 \times K_n$. Hence a $K_3$-factorization of $K_n$ implies a $(K_8 \times K_3)$-factorization of $K_8 \times K_n$. By Lemma 18, we have a gregarious kite factorization of $K_8 \times K_3$. Thus combining all these we get a gregarious kite factorization of $K_8 \times K_n$, $n = 6s+3$, $s \geq 1$. ∎

**Lemma 20.** *For* $|V(G)| = p$, $p \geq 5$ *is a prime, there exists a gregarious kite factorization of* $K_8 \times G$, *where* $G$ *is described as in Note* 7.

***Proof.*** Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(G) = \{1, 2, \ldots, p\}$, $p \geq 5$. Then $V(K_8 \times G) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq p\}$. Now we construct a gregarious kite factorization of $K_8 \times G$ as follows: For $0 \leq s \leq p-1$, let

$$
\begin{aligned}
F_s^1 &= \{(2_{1+s}, 3_{2+s}, 1_{p+s}; 1_{p+s}7_{2+s})\,(8_{1+s}, 5_{3+s}, 4_{2+s}; 4_{2+s}6_{4+s})\};\\
F_s^2 &= \{(1_{1+s}, 5_{3+s}, 7_{2+s}; 7_{2+s}2_{4+s})\,(3_{2+s}, 6_{p+s}, 4_{1+s}; 4_{1+s}8_{p-1+s})\};\\
F_s^3 &= \{(1_{3+s}, 4_{2+s}, 2_{1+s}; 2_{1+s}8_{3+s})\,(3_{2+s}, 5_{3+s}, 6_{1+s}; 6_{1+s}7_{3+s})\};\\
F_s^4 &= \{(1_{3+s}, 6_{2+s}, 8_{1+s}; 8_{1+s}2_{3+s})\,(3_{3+s}, 5_{1+s}, 7_{2+s}; 7_{2+s}4_{p+s})\};\\
F_s^5 &= \{(2_{1+s}, 5_{3+s}, 6_{2+s}; 6_{2+s}1_{4+s})\,(4_{1+s}, 7_{2+s}, 8_{3+s}; 8_{3+s}3_{1+s})\};\\
F_s^6 &= \{(1_{2+s}, 8_{3+s}, 7_{1+s}; 7_{1+s}6_{3+s})\,(5_{1+s}, 4_{2+s}, 2_{3+s}; 2_{3+s}3_{5+s})\};\\
F_s^7 &= \{(1_{2+s}, 5_{1+s}, 4_{3+s}; 4_{3+s}2_{1+s})\,(6_{2+s}, 7_{3+s}, 3_{1+s}; 3_{1+s}8_{p-1+s})\};\\
F_s^8 &= \{(3_{2+s}, 4_{3+s}, 1_{1+s}; 1_{1+s}7_{p-1+s})\,(2_{2+s}, 5_{1+s}, 8_{3+s}; 8_{3+s}6_{1+s})\};\\
F_s^9 &= \{(1_{3+s}, 2_{2+s}, 3_{1+s}; 3_{1+s}6_{3+s})\,(5_{1+s}, 8_{2+s}, 7_{3+s}; 7_{3+s}4_{5+s})\};\\
F_s^{10} &= \{(2_{3+s}, 6_{2+s}, 4_{1+s}; 4_{1+s}1_{3+s})\,(8_{2+s}, 5_{3+s}, 3_{1+s}; 3_{1+s}7_{p-1+s})\};\\
F_s^{11} &= \{(1_{3+s}, 3_{2+s}, 5_{1+s}; 5_{1+s}4_{p-1+s})\,(6_{1+s}, 8_{2+s}, 2_{3+s}; 2_{3+s}7_{5+s})\};\\
F_s^{12} &= \{(5_{2+s}, 6_{3+s}, 1_{1+s}; 1_{1+s}2_{3+s})\,(3_{1+s}, 4_{3+s}, 7_{2+s}; 7_{2+s}8_{p+s})\};\\
F_s^{13} &= \{(1_{2+s}, 8_{1+s}, 6_{3+s}; 6_{3+s}4_{5+s})\,(7_{1+s}, 5_{3+s}, 2_{2+s}; 2_{2+s}3_{p+s})\};\\
F_s^{14} &= \{(2_{1+s}, 7_{2+s}, 6_{3+s}; 6_{3+s}5_{1+s})\,(4_{1+s}, 3_{3+s}, 8_{2+s}; 8_{2+s}1_{p+s})\}.
\end{aligned}
$$

In all the above constructions the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \leq i \leq 14$, is a gregarious kite factor of $K_8 \times G$ and $\{F_1, F_2, \ldots, F_{14}\}$ gives a gregarious kite factorization of $K_8 \times G$. ∎

**Lemma 21.** *There exists a gregarious kite factorization of* $K_8 \times K_7$.

***Proof.*** Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(K_7) = \{1, 2, \ldots, 7\}$. Then $V(K_8 \times K_7) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq 7\}$. Now we construct a gregarious kite factorization of $K_8 \times K_7$ as follows: For $0 \le s \le 6$, let

$$\begin{aligned}
F_s^1 &= \{(6_{1+s}, 2_{5+s}, 3_{3+s}; 3_{3+s}1_{6+s})\,(8_{1+s}, 7_{3+s}, 4_{5+s}; 4_{5+s}5_{1+s})\};\\
F_s^2 &= \{(8_{3+s}, 4_{5+s}, 2_{1+s}; 2_{1+s}5_{4+s})\,(7_{1+s}, 6_{4+s}, 1_{6+s}; 1_{6+s}3_{2+s})\};\\
F_s^3 &= \{(8_{1+s}, 2_{4+s}, 3_{6+s}; 3_{6+s}5_{3+s})\,(4_{1+s}, 6_{3+s}, 7_{5+s}; 7_{5+s}1_{1+s})\};\\
F_s^4 &= \{(4_{1+s}, 1_{2+s}, 2_{6+s}; 2_{6+s}8_{2+s})\,(7_{1+s}, 3_{3+s}, 6_{5+s}; 6_{5+s}5_{1+s})\};\\
F_s^5 &= \{(2_{1+s}, 4_{4+s}, 7_{6+s}; 7_{6+s}5_{3+s})\,(8_{1+s}, 3_{3+s}, 6_{6+s}; 6_{6+s}1_{3+s})\};\\
F_s^6 &= \{(4_{1+s}, 1_{3+s}, 8_{5+s}; 8_{5+s}5_{2+s})\,(2_{1+s}, 6_{5+s}, 7_{3+s}; 7_{3+s}3_{7+s})\};\\
F_s^7 &= \{(1_{1+s}, 2_{3+s}, 7_{6+s}; 7_{6+s}8_{2+s})\,(5_{1+s}, 6_{3+s}, 4_{6+s}; 4_{6+s}3_{2+s})\};\\
F_s^8 &= \{(2_{2+s}, 3_{3+s}, 1_{1+s}; 1_{1+s}6_{5+s})\,(8_{1+s}, 4_{2+s}, 5_{3+s}; 5_{3+s}7_{7+s})\};\\
F_s^9 &= \{(7_{2+s}, 5_{3+s}, 1_{1+s}; 1_{1+s}2_{4+s})\,(4_{1+s}, 3_{2+s}, 6_{6+s}; 6_{6+s}8_{3+s})\};\\
F_s^{10} &= \{(1_{1+s}, 8_{6+s}, 6_{7+s}; 6_{7+s}4_{4+s})\,(7_{1+s}, 5_{7+s}, 3_{2+s}; 3_{2+s}2_{6+s})\};\\
F_s^{11} &= \{(8_{1+s}, 1_{5+s}, 2_{3+s}; 2_{3+s}4_{4+s})\,(6_{1+s}, 5_{3+s}, 3_{2+s}; 3_{2+s}7_{6+s})\};\\
F_s^{12} &= \{(2_{1+s}, 6_{2+s}, 5_{3+s}; 5_{3+s}1_{7+s})\,(4_{1+s}, 7_{2+s}, 8_{3+s}; 8_{3+s}3_{7+s})\};\\
F_s^{13} &= \{(7_{1+s}, 1_{2+s}, 8_{3+s}; 8_{3+s}6_{7+s})\,(5_{1+s}, 2_{3+s}, 4_{2+s}; 4_{2+s}3_{6+s})\};\\
F_s^{14} &= \{(5_{1+s}, 4_{3+s}, 1_{2+s}; 1_{2+s}8_{6+s})\,(6_{2+s}, 7_{3+s}, 3_{1+s}; 3_{1+s}2_{4+s})\};\\
F_s^{15} &= \{(1_{1+s}, 3_{2+s}, 4_{3+s}; 4_{3+s}6_{2+s})\,(5_{1+s}, 8_{3+s}, 2_{2+s}; 2_{2+s}7_{6+s})\};\\
F_s^{16} &= \{(3_{1+s}, 2_{2+s}, 1_{3+s}; 1_{3+s}4_{7+s})\,(8_{2+s}, 7_{3+s}, 5_{1+s}; 5_{1+s}6_{4+s})\};\\
F_s^{17} &= \{(6_{2+s}, 2_{3+s}, 4_{1+s}; 4_{1+s}7_{4+s})\,(3_{1+s}, 8_{2+s}, 5_{3+s}; 5_{3+s}1_{6+s})\};\\
F_s^{18} &= \{(1_{3+s}, 3_{2+s}, 5_{1+s}; 5_{1+s}4_{4+s})\,(6_{1+s}, 2_{3+s}, 8_{2+s}; 8_{2+s}7_{5+s})\};\\
F_s^{19} &= \{(1_{1+s}, 6_{3+s}, 5_{2+s}; 5_{2+s}2_{5+s})\,(7_{2+s}, 4_{3+s}, 3_{1+s}; 3_{1+s}8_{5+s})\};\\
F_s^{20} &= \{(8_{1+s}, 6_{3+s}, 1_{2+s}; 1_{2+s}4_{5+s})\,(7_{1+s}, 2_{2+s}, 5_{3+s}; 5_{3+s}3_{7+s})\};\\
F_s^{21} &= \{(2_{1+s}, 6_{3+s}, 7_{2+s}; 7_{2+s}1_{6+s})\,(4_{1+s}, 3_{3+s}, 8_{2+s}; 8_{2+s}5_{5+s})\}.
\end{aligned}$$

In all the above constructions the subscripts are taken modulo 7 with residues $1, 2, \ldots, 7$. Clearly each $F_i = \bigcup_{s=0}^{6} F_s^i$, $1 \le i \le 21$, is a gregarious kite factor of $K_8 \times K_7$ and $\{F_1, F_2, \ldots, F_{21}\}$ gives a gregarious kite factorization of $K_8 \times K_7$.

■

**Lemma 22.** *For $|V(H)| = p$, $p \ge 11$ is a prime, there exists a gregarious kite factorization of $K_8 \times H$, where $H$ is described as in Note 7.*

***Proof.*** Let $V(K_8) = \{1, 2, \ldots, 8\}$ and $V(H) = \{1, 2, \ldots, p\}$, $p \ge 11$. Then $V(K_8 \times H) = \bigcup_{i=1}^{8} V_i$, where $V_i = \{i_j \mid 1 \le j \le p\}$. Now we construct a gregarious kite factorization of $K_8 \times H$ as follows: For $0 \le s \le p-1$, let

$$\begin{aligned}
F_s^1 &= \{(7_{3+s}, 4_{5+s}, 1_{1+s}; 1_{1+s}2_{5+s})\,(5_{p-1+s}, 8_{p+s}, 6_{1+s}; 6_{1+s}3_{5+s})\};\\
F_s^2 &= \{(4_{3+s}, 2_{2+s}, 1_{1+s}; 1_{1+s}3_{5+s})\,(8_{3+s}, 6_{2+s}, 5_{1+s}; 5_{1+s}7_{5+s})\};
\end{aligned}$$

$$F_s^3 = \{(6_{p-1+s}, 4_{p+s}, 1_{1+s}; 1_{1+s}7_{5+s})\,(5_{p-1+s}, 2_{p+s}, 3_{1+s}; 3_{1+s}8_{5+s})\};$$
$$F_s^4 = \{(8_{p-3+s}, 1_{p-1+s}, 2_{1+s}; 2_{1+s}3_{5+s})\,(7_{p-1+s}, 6_{p+s}, 5_{1+s}; 5_{1+s}4_{5+s})\};$$
$$F_s^5 = \{(1_{3+s}, 2_{2+s}, 3_{1+s}; 3_{1+s}7_{5+s})\,(4_{2+s}, 5_{3+s}, 8_{1+s}; 8_{1+s}6_{5+s})\};$$
$$F_s^6 = \{(7_{2+s}, 4_{3+s}, 2_{1+s}; 2_{1+s}1_{5+s})\,(8_{p-1+s}, 5_{p+s}, 3_{1+s}; 3_{1+s}6_{5+s})\};$$
$$F_s^7 = \{(3_{2+s}, 1_{p+s}, 6_{1+s}; 6_{1+s}4_{5+s})\,(7_{1+s}, 2_{3+s}, 8_{5+s}; 8_{5+s}5_{1+s})\};$$
$$F_s^8 = \{(1_{3+s}, 7_{2+s}, 5_{1+s}; 5_{1+s}3_{5+s})\,(2_{2+s}, 6_{3+s}, 4_{1+s}; 4_{1+s}8_{5+s})\};$$
$$F_s^9 = \{(6_{p+s}, 5_{2+s}, 2_{1+s}; 2_{1+s}7_{5+s})\,(8_{p+s}, 4_{2+s}, 3_{1+s}; 3_{1+s}1_{5+s})\};$$
$$F_s^{10} = \{(8_{p+s}, 2_{p-1+s}, 1_{1+s}; 1_{1+s}5_{5+s})\,(3_{3+s}, 4_{5+s}, 7_{1+s}; 7_{1+s}6_{p-3+s})\};$$
$$F_s^{11} = \{(1_{3+s}, 6_{5+s}, 4_{1+s}; 4_{1+s}3_{5+s})\,(7_{3+s}, 8_{5+s}, 2_{1+s}; 2_{1+s}5_{5+s})\};$$
$$F_s^{12} = \{(8_{3+s}, 6_{p-1+s}, 4_{1+s}; 4_{1+s}2_{p-3+s})\,(1_{3+s}, 7_{4+s}, 5_{5+s}; 5_{5+s}3_{1+s})\};$$
$$F_s^{13} = \{(2_{3+s}, 5_{1+s}, 6_{5+s}; 6_{5+s}7_{1+s})\,(8_{2+s}, 1_{p+s}, 3_{1+s}; 3_{1+s}4_{5+s})\};$$
$$F_s^{14} = \{(7_{2+s}, 5_{p+s}, 4_{1+s}; 4_{1+s}1_{5+s})\,(6_{3+s}, 2_{p-1+s}, 3_{1+s}; 3_{1+s}8_{p-3+s})\};$$
$$F_s^{15} = \{(2_{p-1+s}, 6_{p-3+s}, 5_{1+s}; 5_{1+s}1_{5+s})\,(3_{p-1+s}, 7_{p+s}, 8_{1+s}; 8_{1+s}4_{5+s})\};$$
$$F_s^{16} = \{(1_{3+s}, 8_{p-1+s}, 7_{1+s}; 7_{1+s}5_{5+s})\,(6_{1+s}, 3_{3+s}, 2_{5+s}; 2_{5+s}4_{1+s})\};$$
$$F_s^{17} = \{(8_{2+s}, 2_{3+s}, 4_{1+s}; 4_{1+s}5_{5+s})\,(3_{2+s}, 6_{3+s}, 7_{1+s}; 7_{1+s}1_{5+s})\};$$
$$F_s^{18} = \{(1_{p+s}, 6_{p-1+s}, 8_{1+s}; 8_{1+s}5_{5+s})\,(7_{3+s}, 4_{p-1+s}, 3_{1+s}; 3_{1+s}2_{5+s})\};$$
$$F_s^{19} = \{(5_{2+s}, 3_{p+s}, 1_{1+s}; 1_{1+s}8_{5+s})\,(4_{p-1+s}, 6_{p+s}, 7_{1+s}; 7_{1+s}2_{5+s})\};$$
$$F_s^{20} = \{(4_{p-1+s}, 3_{p+s}, 5_{1+s}; 5_{1+s}2_{5+s})\,(7_{3+s}, 8_{p-1+s}, 6_{1+s}; 6_{1+s}1_{5+s})\};$$
$$F_s^{21} = \{(4_{2+s}, 5_{p+s}, 1_{1+s}; 1_{1+s}6_{5+s})\,(2_{2+s}, 8_{p+s}, 7_{1+s}; 7_{1+s}3_{5+s})\}.$$

In all the above constructions the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$. Clearly each $F_i = \bigcup_{s=0}^{p-1} F_s^i$, $1 \le i \le 21$, is a gregarious kite factor of $K_8 \times H$ and $\{F_1, F_2, \ldots, F_{21}\}$ gives a gregarious kite factorization of $K_8 \times H$. ■

**Lemma 23.** *For all odd prime $p$, there exists a gregarious kite factorization of $K_8 \times K_p$.*

***Proof.*** By Remark 8, $K_p$ has a factorization into graphs isomorphic to $G$ or $H$. Hence a gregarious kite factorization of $K_8 \times K_p$ follows from Lemmas 18, 20, 21 and 22. ■

**Lemma 24.** *For all odd prime $p$ and $s > 1$, there exists a gregarious kite factorization of $K_8 \times K_{p^s}$.*

***Proof.*** For $s > 1$, $K_8 \times K_{p^s} = K_8 \times \left[pK_{p^{s-1}} \oplus K_p(p^{s-1})\right] = p(K_8 \times K_{p^{s-1}}) \oplus \left[K_8 \times K_p(p^{s-1})\right]$ (since the case $s = 1$ follows from Lemma 23).

For $s = 2$, $K_8 \times K_{p^2} = p(K_8 \times K_p) \oplus \left[K_8 \times K_p(p)\right]$. By Lemma 23, we have a gregarious kite factorization of $p(K_8 \times K_p)$. By Theorem 4, we have a $K_p$-factorization of $K_p(p)$. Corresponding to each $K_p$-factor of $K_p(p)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p)$. Thus a $K_p$-factorization of $K_p(p)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p)$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23.

For $s = 3$, $K_8 \times K_{p^3} = p(K_8 \times K_{p^2}) \oplus \left[K_8 \times K_p(p^2)\right]$. Then the gregarious kite factorization of $p(K_8 \times K_{p^2})$ follows from the case $s = 2$. By Theorem 4, we have a $K_p$-factorization of $K_p(p^2)$. Corresponding to each $K_p$-factor of $K_p(p^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^2)$. Thus a $K_p$-factorization of $K_p(p^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^2)$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23.

For $s > 1$, $K_8 \times K_{p^s} = p(K_8 \times K_{p^{s-1}}) \oplus \left[K_8 \times K_p(p^{s-1})\right]$. By the induction hypothesis on $s$, we have a gregarious kite factorization of $p(K_8 \times K_{p^{s-1}})$. By Theorem 4, we have a $K_p$-factorization of $K_p(p^{s-1})$. Corresponding to each $K_p$-factor of $K_p(p^{s-1})$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^{s-1})$. Thus a $K_p$-factorization of $K_p(p^{s-1})$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^{s-1})$. Now the existence of a gregarious kite factorization of $(K_8 \times K_p)$ follows from Lemma 23. Hence combining all the above we have a gregarious kite factorization of $K_8 \times K_{p^s}$, for all $s > 1$. ■

**Lemma 25.** *There exists a gregarious kite factorization of $K_8 \times K_{p^s q^t}$ for all odd primes $p, q$ $(p < q)$ and all integers $s, t \geq 1$.*

***Proof.*** For $s, t \geq 1$ and $p < q$,

$$\begin{aligned} K_8 \times K_{p^s q^t} &= K_8 \times K_{p.p^{s-1}q^t} = K_8 \times \left[pK_{p^{s-1}q^t} \oplus K_p(p^{s-1}q^t)\right] \\ &= p\left[K_8 \times K_{p^{s-1}q^t}\right] \oplus \left[K_8 \times K_p(p^{s-1}q^t)\right]. \end{aligned}$$

*Case* 1. (a) For $s = 1$, $t = 1$, $K_8 \times K_{pq} = K_8 \times [pK_q \oplus K_p(q)] = p\left[K_8 \times K_q\right] \oplus [K_8 \times K_p(q)]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q)$. Corresponding to each $K_p$-factor of $K_p(q)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q)$. Thus a $K_p$-factorization of $K_p(q)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_q)$ follows from Lemma 23.

(b) For $s = 1$, $t = 2$, $K_8 \times K_{pq^2} = K_8 \times \left[pK_{q^2} \oplus K_p(q^2)\right] = p\left[K_8 \times K_{q^2}\right] \oplus \left[K_8 \times K_p(q^2)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q^2)$. Corresponding to each $K_p$-factor of $K_p(q^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q^2)$. Thus a $K_p$-factorization of $K_p(q^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q^2)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_{q^2})$ follows from Lemmas 23 and 24, respectively.

(c) For $s = 1$, $t \geq 3$, $K_8 \times K_{pq^t} = K_8 \times \left[pK_{q^t} \oplus K_p(q^t)\right] = p\left[K_8 \times K_{q^t}\right] \oplus \left[K_8 \times K_p(q^t)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(q^t)$. Corresponding to each $K_p$-factor of $K_p(q^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(q^t)$. Thus a $K_p$-factorization of $K_p(q^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(q^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p(K_8 \times K_{q^t})$ follows from Lemmas 23 and 24, respectively.

*Case* 2. (a) For $s = 2$, $t = 1$, $K_8 \times K_{p^2q} = K_8 \times K_{p.pq} = K_8 \times [pK_{pq} \oplus K_p(pq)] = p\,[K_8 \times K_{pq}] \oplus [K_8 \times K_p(pq)]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq)$. Corresponding to each $K_p$-factor of $K_p(pq)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq)$. Thus a $K_p$-factorization of $K_p(pq)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\,[K_8 \times K_{pq}]$ follows from Lemma 23 and Case 1(a), respectively.

(b) For $s = 2$, $t = 2$, $K_8 \times K_{p^2q^2} = K_8 \times K_{p.pq^2} = K_8 \times \left[pK_{pq^2} \oplus K_p(pq^2)\right] = p\left[K_8 \times K_{pq^2}\right] \oplus \left[K_8 \times K_p(pq^2)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq^2)$. Corresponding to each $K_p$-factor of $K_p(pq^2)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq^2)$. Thus a $K_p$-factorization of $K_p(pq^2)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq^2)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\left[K_8 \times K_{pq^2}\right]$ follows from Lemma 23 and Case 1(b), respectively.

(c) For $s = 2$, $t \geq 3$, $K_8 \times K_{p^2q^t} = K_8 \times K_{p.pq^t} = K_8 \times \left[pK_{pq^t} \oplus K_p(pq^t)\right] = p\left[K_8 \times K_{pq^t}\right] \oplus \left[K_8 \times K_p(pq^t)\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(pq^t)$. Corresponding to each $K_p$-factor of $K_p(pq^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(pq^t)$. Thus a $K_p$-factorization of $K_p(pq^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(pq^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ and $p\left[K_8 \times K_{pq^t}\right]$ follows from Lemma 23 and Case 1(c), respectively.

For $s, t \geq 1$, $K_8 \times K_{p^sq^t} = p\left[K_8 \times K_{p^{s-1}q^t}\right] \oplus \left[K_8 \times K_p(p^{s-1}q^t)\right]$. By the induction hypothesis on $s$, we have a gregarious kite factorization of $p\left[K_8 \times K_{p^{s-1}q^t}\right]$. By Theorem 4, we have a $K_p$-factorization of $K_p(p^{s-1}q^t)$. Corresponding to each $K_p$-factor of $K_p(p^{s-1}q^t)$, we have a $(K_8 \times K_p)$-factor of $K_8 \times K_p(p^{s-1}q^t)$. Thus a $K_p$-factorization of $K_p(p^{s-1}q^t)$ implies a $(K_8 \times K_p)$-factorization of $K_8 \times K_p(p^{s-1}q^t)$. Now the existence of a gregarious kite factorization of $K_8 \times K_p$ follows from Lemma 23. Hence combining all the above we have a gregarious kite factorization of $K_8 \times K_{p^sq^t}$, for all $s, t \geq 1$ and $p < q$. ∎

**Lemma 26.** *There exists a gregarious kite factorization of $K_8 \times K_n$ for all odd $n > 1$.*

***Proof.*** By fundamental theorem of arithmetic, any integer $n > 1$ can be uniquely written as prime powers or product of prime powers.

Consider $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, where each $p_i$ is a distinct odd prime and $\alpha_i \geq 1, i = 1, 2, \ldots, t$. Fix $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_t^{\alpha_t}$. Now,

$$\begin{aligned} K_8 \times K_n = K_8 \times K_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}} &= K_8 \times \left[p_1^{\alpha_1} K_{p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t}} \oplus K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right] \\ &= p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t}}\right] \oplus \left[K_8 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]. \end{aligned}$$

It is enough to show that there exists a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\dots p_t^{\alpha_t})$ and $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right]$.

*Case* 1. Consider $K_8 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})$. By Theorem 4, we have a $K_{p_1^{\alpha_1}}$-factorization of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})$. Corresponding to each $K_{p_1^{\alpha_1}}$-factor of $K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})$, we have a $(K_8 \times K_{p_1^{\alpha_1}})$-factor of $K_8 \times K_{p_1^{\alpha_1}}(p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_t^{\alpha_t})$. Now the existence of a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}}$ follows from Lemma 24.

*Case* 2. Consider $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right]$. We write

$$\begin{aligned}
p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}\left\{K_8 \times \left[p_2^{\alpha_2}K_{p_3^{\alpha_3}\dots p_t^{\alpha_t}} \oplus K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]\right\}\\
&= p_1^{\alpha_1}\left\{p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right] \oplus \left[K_8 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]\right\}\\
&= p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right].
\end{aligned}$$

Now we have to show the existence of gregarious kite factorization of $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right]$ and $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$. The existence of gregarious kite factorization of $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}}(p_3^{\alpha_3}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1. Now we can write

$$\begin{aligned}
p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right] &= p_1^{\alpha_1}p_2^{\alpha_2}\left\{K_8 \times \left[p_3^{\alpha_3}K_{p_4^{\alpha_4}\dots p_t^{\alpha_t}} \oplus K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]\right\}\\
&= p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_8 \times K_{p_4^{\alpha_4}\dots p_t^{\alpha_t}}\right]\\
&\oplus p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right].
\end{aligned}$$

The existence of gregarious kite factorization of second term $p_1^{\alpha_1}p_2^{\alpha_2}\left[K_8 \times K_{p_3^{\alpha_3}}(p_4^{\alpha_4}\cdots p_t^{\alpha_t})\right]$ is similar to Case 1.

Now we consider the first term $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\left[K_8 \times K_{p_4^{\alpha_4}\dots p_t^{\alpha_t}}\right]$ and repeat the above process until we end up with $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-1}^{\alpha_{t-1}}\left[K_8 \times K_{p_t^{\alpha_t}}\right] \oplus p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{t-2}^{\alpha_{t-2}}\left[K_8 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})\right]$. Now the existence of a gregarious kite factorization of $K_8 \times K_{p_t^{\alpha_t}}$ and hence the first term follows from Lemma 24 and the existence of gregarious kite factorization of $K_8 \times K_{p_{t-1}^{\alpha_{t-1}}}(p_t^{\alpha_t})$ and hence the second term is similar to Case 1. Thus we have a gregarious kite factorization of $p_1^{\alpha_1}\left[K_8 \times K_{p_2^{\alpha_2}p_3^{\alpha_3}\dots p_t^{\alpha_t}}\right]$.

Hence from Cases 1 and 2, we have a gregarious kite factorization of $K_8 \times K_{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_t^{\alpha_t}} = K_8 \times K_n$. ■

**Lemma 27.** *For all odd $n > 1$, there exists a gregarious kite factorization of $K \times K_n$, where $K$ is a kite.*

***Proof.*** Let $V(K) = \{1, 2, 3, 4\}$ and $V(K_n) = \{1, 2, \ldots, n\}$. Then $V(K \times K_n) = \bigcup_{i=1}^{4} V_i$, where $V_i = \{i_j \mid 1 \leq j \leq n\}$. Now we construct a gregarious kite factorization of $K \times K_n$ as follows: For $0 \leq s \leq n-2$, let $F_s = \bigoplus_{i=0}^{n-1} \{1_{1+i}, 2_{2+s+i}, 3_{3+2s+i}; 3_{3+2s+i}4_{4+3s+i}\}$, where the subscripts are taken modulo $n$ with residues $1, 2, \ldots, n$. Clearly each $F_s$, $0 \leq s \leq n-2$ is a gregarious kite factor of $K \times K_n$ and all together gives a gregarious kite factorization of $K \times K_n$. ■

**Theorem 28.** *There exists a gregarious kite factorization of $K_m \times K_n$ if and only if $m \equiv 0 \pmod 4$ and $n$ is any odd integer greater than $1$.*

***Proof.*** *Necessity.* It follows by $4|mn$, $\{(m-1)(n-1)\}/2 \in \mathbb{N}$ (respectively, the size of a kite factor and the number of factors in a kite factorization of the graph $K_m \times K_n$).

*Sufficiency.* Let $m = 4s$, $s \geq 1$ and $n$ is odd. The case $s = 1, 2$ follows from Lemmas 17 and 26, respectively. Then for $s \geq 3$, $K_{4s} \times K_n = [sK_4 \oplus K_s(4)] \times K_n = s(K_4 \times K_n) \oplus (K_s(4) \times K_n)$. Now the existence of a gregarious kite factorization of $s\,(K_4 \times K_n)$ follows from Lemma 17. By Theorem 5, we have a kite factorization of $K_s(4), s \geq 3$. Corresponding to each kite factor of $K_s(4)$, we have a $(K \times K_n)$-factor of $(K_s(4) \times K_n)$, where $K$ is a kite. Thus a kite factorization of $K_s(4)$ implies a $(K \times K_n)$-factorization of $(K_s(4) \times K_n)$. Further, the existence of a gregarious kite factorization of $K \times K_n$ follows from Lemma 27. Hence combining all these results we have a gregarious kite factorization of $K_m \times K_n$. ■

**Conclusion.** In this paper, we give a complete solution for the existence of a gregarious kite factorization of $K_m \times K_n$.

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