

## ON THE MINIMUM NUMBER OF SPANNING TREES IN CUBIC MULTIGRAPHS

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### Abstract

Let  $G_{2n}, H_{2n}$  be two non-isomorphic connected cubic multigraphs of order  $2n$  with parallel edges permitted but without loops. Let  $t(G_{2n}), t(H_{2n})$  denote the number of spanning trees in  $G_{2n}, H_{2n}$ , respectively. We prove that for  $n \geq 3$  there is the unique  $G_{2n}$  such that  $t(G_{2n}) < t(H_{2n})$  for any  $H_{2n}$ . Furthermore, we prove that such a graph has  $t(G_{2n}) = 5^2 2^{n-3}$  spanning trees. Based on our results we give a conjecture for the unique  $r$ -regular connected graph  $H_{2n}$  of order  $2n$  and odd degree  $r$  that minimizes the number of spanning trees.

**Keywords:** cubic multigraph, spanning tree, regular graph, enumeration.

**2010 Mathematics Subject Classification:** 05C05, 05C38.

### 1. INTRODUCTION

There is an extensive literature devoted to identifying connected graphs  $G$  on  $V(G)$  vertices and  $E(G)$  edges and with either maximum or minimum number of spanning trees  $t(G)$  when  $|V(G)|$  and  $|E(G)|$  are predetermined. Identifying such graphs allows establishing upper and lower bounds for the number of spanning trees in families of connected graphs when  $|V(G)|$  and  $|E(G)|$  are fixed. Most published papers focused on the maximum number of spanning trees cover just a few restricted families of graphs, e.g., [5, 6, 9]. Determining the graphs with the minimum number of spanning trees was much more successful. In particular, it was determined in [4] that specific threshold graph  $G$  minimizes the number of spanning trees over all connected simple graphs with the same number of vertices and edges. However, it was also determined that  $G$  was not unique. Based on that, it was subsequently proved in [2] that there is a well-defined

class of connected simple graphs that minimize the number of spanning trees among the simple connected graphs on the same number of vertices and edges. Corresponding results for the maximum number of spanning trees in undirected simple graphs have yet to be found.

In addition to identifying the connected simple graphs with minimum number of spanning trees, there were also number of papers recently published devoted to the minimum number of spanning trees in the special families of connected graphs. Kostochka [7] identified the minimum number of spanning trees in a simple cubic graph with fixed number of vertices. In [1] we proved that there is a unique threshold graph that minimizes the number of spanning trees over all 2-connected chordal graphs, and in [3] we identified simple cubic connected graphs that minimize the number of spanning trees over other cubic graphs, on the same number of vertices. Most recently, Ok and Thomassen [8] determined a lower bound on the number of spanning trees in a  $k$ -edge-connected graph and identified the extremal  $k$ -edge-connected graph.

In this paper we consider all connected cubic graphs of given order  $2n$  without loops, and prove/identify that there is the unique graph  $M_{2n}$  belonging to this family that minimizes the number of spanning trees. For convenience, throughout the rest of this paper by graph we mean either a multigraph without loops and with at least one pair of parallel edges, or a simple graph. Hence, if  $G_{2n}$  is a cubic graph, then either  $G_{2n}$  contains induced  $C_2$  or it is simple.

## 2. CONNECTED CUBIC MULTIGRAPHS WITH MINIMAL SPANNING TREES

Let  $M_3$  be a multigraph constructed from a  $C_2$  cycle on two vertices  $v_1, v_2$  by joining a third vertex  $v_3$  with two single edges to vertices  $v_1$  and  $v_2$ . Let  $M_{2n} = M_{2(3+k)}$ ,  $n \geq 3$ , be a connected cubic multigraph on  $2n$  vertices that consists of two  $M_3$  subgraphs and  $k$   $C_2$  cycles, all joined with one another by single edges—see Figure 1.

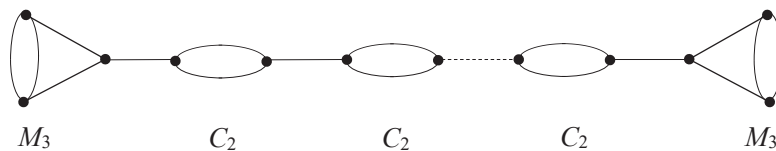


Figure 1. Graph  $M_{2n}$ .

For parallel edges  $e_1, e_2$  we assume that two spanning trees containing  $e_1, e_2$  respectively are distinct. In addition, if  $G$  is isomorphic to  $H$ , then we write

$G \simeq H$ , otherwise we write  $G \not\simeq H$ . The proof of our main result in Theorem 3 is based on graph transformations derived from the following simple lemma.

**Lemma 1.** *Let  $T(G)$  be a spanning tree of connected  $G$  that includes an edge  $e$ . Let  $H$  be a graph obtained from  $G$  by contracting  $e$  into a vertex. Then contracting  $e$  into a vertex in  $T(G)$  produces a spanning tree  $T(H)$ . Furthermore,  $t(H)$  equals the number of spanning trees in  $G$  that contain  $e$ .*

**Proof.** Clearly, contracting  $e$  into a vertex in  $T(G)$  does not produce a cycle and results in a connected spanning subgraph of  $H$ , which is  $T(H)$ . Hence, to every unique spanning tree of  $H$  there corresponds a unique spanning tree of  $G$  that contains edge  $e$ . ■

We also need the following lemma.

**Lemma 2.** *Connected cubic graph  $G_6$  minimizes  $t(G_6)$  if and only if  $G_6 \simeq M_6$ .*

**Proof.** It's easy to verify that there are only six pairwise non-isomorphic connected cubic graphs on six vertices (Figure 2): (1) Möbius ladder  $H_6$ , (2) prism  $P_6 \simeq C_2 \square C_3$ , (3) multigraph  $C_2 \times 1$  with one induced  $C_2$  cycle, (4) multigraph  $C_2 \times 2$  with two induced  $C_2$  cycles, (5) multigraph  $C_2 \times 3$  with three induced  $C_2$  cycles, and (6)  $M_6$ .

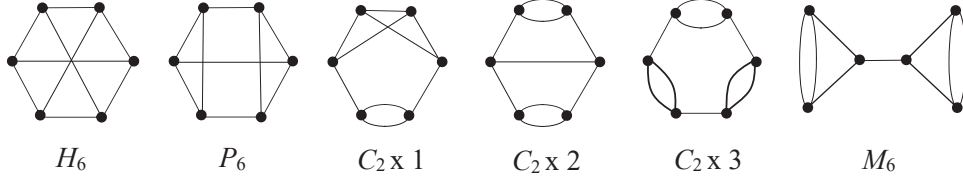


Figure 2. All distinct connected cubic graphs on six vertices.

Furthermore, it's trivial to verify based on a well-known Kirchhoff's matrix-tree theorem that  $t(H_6) = 81 > t(P_6) = 75 > t(C_2 \times 1) = 56 > t(C_2 \times 2) = 45 > t(C_2 \times 3) = 36 > t(M_6) = 25$ . ■

We can now state the main result as follows.

**Theorem 3.** *Connected cubic graph  $G_{2n}$  minimizes  $t(G_{2n})$  for given  $n \geq 3$  if and only if  $G_{2n} \simeq M_{2n}$ .*

**Proof.** For  $n = 3$ , according to Lemma 2,  $G_{2n}$  minimizes  $t(G_{2n})$  if and only if  $G_6 \simeq M_6$ . Suppose there exists  $G_{2n}$  for  $n \geq 4$  such that  $t(G_{2n}) \leq t(M_{2n})$  and

$G_{2n} \not\cong M_{2n}$ . Without loss of generality, assume  $G_{2n}$  to be with minimum  $n \geq 4$  that satisfies  $t(G_{2n}) \leq t(M_{2n})$  and  $G_{2n} \not\cong M_{2n}$ .

Suppose  $G_{2n}$  contains a simple cycle  $C_i$  on  $i$  vertices, where  $C_i$  is not included in any  $M_3$  component of  $G_{2n}$ . If replacing in  $G_{2n}$  component  $X$  with component  $Y$  produces connected cubic graph  $H_{2m}$ , then we denote it by  $G_{2n}(X \rightarrow Y) \rightarrow H_{2m}$ . If  $i$ -th spanning tree in a graph  $G$  induces a spanning tree in a subgraph  $S$  of  $G$ , then such a spanning tree in  $G$  we denote by  $T_i(G, S)$ . Otherwise,  $i$ -th spanning tree in  $G$  we denote by  $\hat{T}_i(G, S)$ . Then we have the following:

**Claim 1.**  $G_{2n}$  does not contain  $C_2$  that is not included in  $M_3$ .

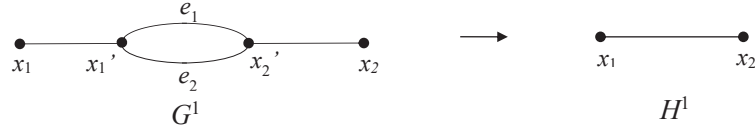


Figure 3. Transformation based on  $C_2$  component.

**Proof.** If  $G_{2n}$  contains  $C_2$  outside  $M_3$  components, then there is a transformation illustrated in Figure 3. The subgraph  $G^1$  does not have to be an induced subgraph of  $G$  (e.g., there might be an edge between  $x_1$  and  $x_2$  in  $G^1$ ). If  $x_1 = x_2$  in Figure 3, then  $C_2$  belongs to  $M_3$ —a contradiction. So, transformation in Figure 3 does not produce a loop. Consequently, we can transform  $G_{2n}$  as follows  $G_{2n}(G^1 \rightarrow H^1) \rightarrow H_{2n-2}$ . Furthermore, for every spanning tree  $T_i(H_{2n-2}, H^1)$  there are two unique spanning trees:

1.  $T_{i_1}(G_{2n}, G^1)$  with edges  $(x_1, x_1'), e_1, (x_2', x_2)$ ,
2.  $T_{i_2}(G_{2n}, G^1)$  with edges  $(x_1, x_1'), e_2, (x_2', x_2)$ ,

and for every spanning tree  $\hat{T}_i(H_{2n-2}, H^1)$  there are five unique spanning trees:

1.  $\hat{T}_{i_1}(G_{2n}, G^1)$  with edges  $(x_1, x_1'), e_1$ ,
2.  $\hat{T}_{i_2}(G_{2n}, G^1)$  with edges  $(x_1, x_1'), e_2$ ,
3.  $\hat{T}_{i_3}(G_{2n}, G^1)$  with edges  $e_1, (x_2', x_2)$
4.  $\hat{T}_{i_4}(G_{2n}, G^1)$  with edges  $e_2, (x_2', x_2)$ ,
5.  $\hat{T}_{i_5}(G_{2n}, G^1)$  with edges  $(x_1, x_1'), (x_2', x_2)$ .

If there is an edge between  $x_1$  and  $x_2$  in  $G^1$ , then  $3t(H_{2n-2}) < t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$  implying  $t(H_{2n-2}) < t(M_{2n-2})$ —a contradiction. If there is no edge between  $x_1$  and  $x_2$  in  $G^1$ , then  $2t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$  and  $H_{2n-2} \not\cong M_{2n-2}$ —a contradiction. These contradictions prove Claim 1.  $\square$

**Claim 2.**  $G_{2n}$  does not contain induced  $C_3$ .

**Proof.** Suppose a subgraph of  $G_{2n}$  exists that includes induced cycle  $C_3 = x'_1 x'_2 x'_3$ . Let  $(x_1, x'_1), (x_2, x'_2), (x_3, x'_3)$  be the edges not in  $E(C_3)$ . There are two cases to consider based on the vertices  $x_1, x_2, x_3$ .

*Case 1.*  $x_1, x_2, x_3$  are pairwise distinct. In this case there is a transformation  $G_{2n}(C_3 \rightarrow x'_1) \rightarrow H_{2n-2}$ , which is a contraction of  $C_3$  in  $G_{2n}$  into a vertex  $x'_1$ . Hence, for every spanning tree  $T_i(H_{2n-2}, x'_1) = T_i(H_{2n-2})$  there are three unique spanning trees:

1.  $T_{i_1}(G_{2n}, C_3)$  with edges  $(x'_1, x'_2), (x'_2, x'_3)$ ,
2.  $T_{i_2}(G_{2n}, C_3)$  with edges  $(x'_2, x'_3), (x'_3, x'_1)$ ,
3.  $T_{i_3}(G_{2n}, C_3)$  with edges  $(x'_3, x'_1), (x'_1, x'_2)$ .

Consequently,  $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ , implying  $t(H_{2n-2}) < t(M_{2n-2})$ —a contradiction.

*Case 2.*  $x_1, x_2, x_3$  are not pairwise distinct. If  $x_1 = x_2 = x_3$ , then  $G_{2n} \simeq K_4$ —a contradiction ( $2n = 4 < 6$ ). So, without loss of generality assume  $x_1 \neq x_2 = x_3$ . In this case there is a transformation illustrated in Figure 4.

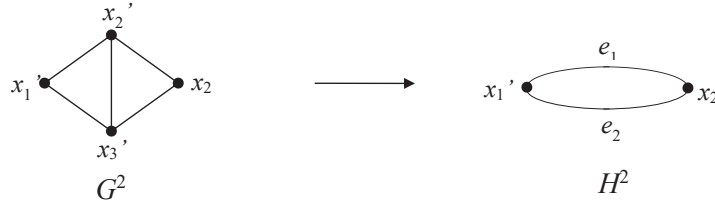


Figure 4. Transformation based on  $C_3$  when  $x_1 \neq x_2 = x_3$ .

So, there is a transformation  $G_{2n}(G^2 \rightarrow H^2) \rightarrow H_{2n-2}$ . Clearly,  $t(G^2) = 8$  and  $t(H^2) = 2$ . This means that there are four times more  $T_i(G_{2n}, G^2)$  spanning trees than  $T_i(H_{2n-2}, H^2)$  spanning trees. In addition, for every spanning tree  $\hat{T}_i(H_{2n-2}, H^2)$  there are eight unique spanning trees:

1.  $\hat{T}_{i_1}(G_{2n}, G^2)$  with edges  $(x'_1, x'_2), (x'_1, x'_3)$ ,
2.  $\hat{T}_{i_2}(G_{2n}, G^2)$  with edges  $(x'_1, x'_2), (x'_2, x'_3)$ ,
3.  $\hat{T}_{i_3}(G_{2n}, G^2)$  with edges  $(x'_2, x'_3), (x'_1, x'_3)$ ,
4.  $\hat{T}_{i_4}(G_{2n}, G^2)$  with edges  $(x_2, x'_2), (x_2, x'_3)$ ,
5.  $\hat{T}_{i_5}(G_{2n}, G^2)$  with edges  $(x_2, x'_2), (x'_2, x'_3)$ ,
6.  $\hat{T}_{i_6}(G_{2n}, G^2)$  with edges  $(x'_2, x'_3), (x_2, x'_3)$ ,
7.  $\hat{T}_{i_7}(G_{2n}, G^2)$  with edges  $(x'_1, x'_2), (x_2, x'_3)$ ,
8.  $\hat{T}_{i_8}(G_{2n}, G^2)$  with edges  $(x_2, x'_2), (x'_1, x'_3)$ .

Hence,  $2t(H_{2n-2}) < t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ , implying  $t(H_{2n-2}) < t(M_{2n-2})$ , a contradiction.

Consequently, contradictions of Cases 1–2 prove Claim 2.  $\square$

**Claim 3.**  $G_{2n}$  does not contain induced  $C_4$ .

**Proof.** Suppose that  $G_{2n}$  contains induced square  $C_4$ —Figure 5. In Figure 5 we allow  $x_1 = x_3$  and  $x_2 = x_4$  but due to Claim 2 we do not allow other  $x_i = x_j$  for  $i \neq j$ .

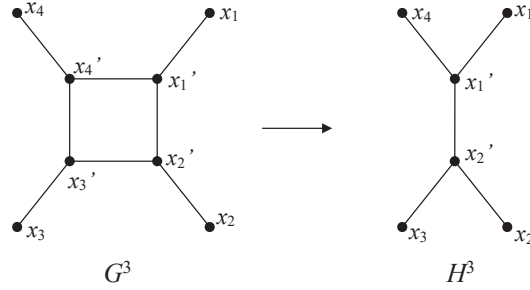


Figure 5. Transformation based on  $C_4$ .

So, there is a transformation  $G_{2n}(G^3 \rightarrow H^3) \rightarrow H_{2n-2}$ . Let  $X$  be a subgraph of  $G_{2n}$  induced by  $\{x'_1, x'_2, x'_3, x'_4\}$ , and let  $Y$  be a subgraph of  $H_{2n-2}$  induced by  $\{x'_1, x'_2\}$  corresponding to Figure 5. Clearly,  $t(X) = 4$  and  $t(Y) = 1$ . This means that there are four times more  $T_i(G_{2n}, X)$  spanning trees than  $T_i(H_{2n-2}, Y)$  spanning trees. In addition, for every spanning tree  $\hat{T}_i(H_{2n-2}, Y)$  there is a path  $P_H = x'_1 \cdots x'_2$  on at least three vertices. Hence, for every spanning tree  $\hat{T}_i(H_{2n-2}, Y)$  there are at least three unique spanning trees of  $G_{2n}$  based on the following four cases.

*Case 1.*  $\hat{T}_i(H_{2n-2}, Y)$  contains  $P_H = x'_1 x_1 \cdots x_2 x'_2$ . For every  $\hat{T}_i(H_{2n-2}, Y)$  there correspond three unique spanning trees  $\hat{T}_{i_j}(G_{2n}, X)$  that contain all edges of  $\hat{T}_i(H_{2n-2}, Y)$  and the following:

1.  $\hat{T}_{i_1}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_2, x'_3)$ ,
2.  $\hat{T}_{i_2}(G_{2n}, X)$  contains additional edges  $(x'_2, x'_3), (x'_3, x'_4)$ ,
3.  $\hat{T}_{i_3}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_3, x'_4)$ .

*Case 2.*  $\hat{T}_i(H_{2n-2}, Y)$  contains  $P_H = x'_1 x_1 \cdots x_3 x'_2$ . For every  $\hat{T}_i(H_{2n-2}, Y)$  there correspond four unique spanning trees  $\hat{T}_{i_j}(G_{2n}, X)$  that contain all edges of  $\hat{T}_i(H_{2n-2}, Y)$  and the following:

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_2, x'_3)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains additional edges  $(x'_2, x'_3), (x'_3, x'_4)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_1, x'_4)$ ,
4.  $\widehat{T}_{i_4}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_3, x'_4)$ .

*Case 3.*  $\widehat{T}_i(H_{2n-2}, Y)$  contains  $P_H = x'_1 x_4 \cdots x_3 x'_2$ . For every  $\widehat{T}_i(H_{2n-2}, Y)$  there correspond three unique spanning trees  $\widehat{T}_{i_j}(G_{2n}, X)$  that contain all edges of  $\widehat{T}_i(H_{2n-2}, Y)$  and the following:

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_2, x'_3)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_2, x'_3)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_1, x'_4)$ .

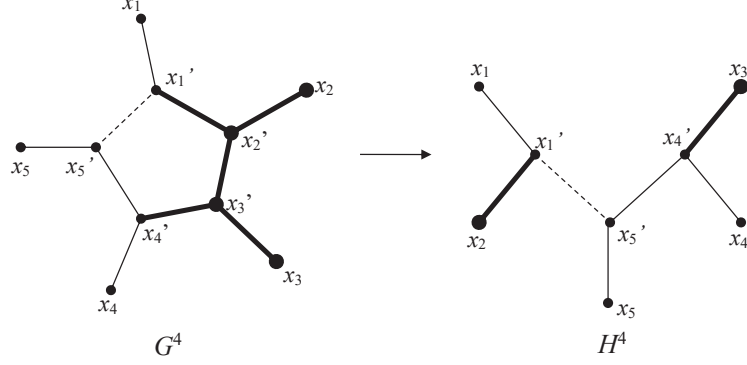
*Case 4.*  $\widehat{T}_i(H_{2n-2}, Y)$  contains  $P_H = x'_1 x_4 \cdots x_2 x'_2$ . For every  $\widehat{T}_i(H_{2n-2}, Y)$  there correspond four unique spanning trees  $\widehat{T}_{i_j}(G_{2n}, X)$  that contain all edges of  $\widehat{T}_i(H_{2n-2}, Y)$  and the following:

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_2, x'_3)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_2, x'_3)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_4), (x'_3, x'_4)$ ,
4.  $\widehat{T}_{i_4}(G_{2n}, X)$  contains additional edges  $(x'_1, x'_2), (x'_3, x'_4)$ .

None of the added edges, or combination of these edges, in Cases 1–4 could result in a cycle in  $\widehat{T}_{i_j}(G_{2n}, X)$  because it would imply a cycle in  $\widehat{T}_i(H_{2n-2}, Y)$  from which it was constructed. So, by Cases 1–4, there are at least three times more  $\widehat{T}_i(G_{2n}, X)$  spanning trees than  $T_i(H_{2n-2}, Y)$  spanning trees. Hence,  $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ , implying  $t(H_{2n-2}) < t(M_{2n-2})$ —a contradiction, which proves Claim 3.  $\square$

**Claim 4.**  $G_{2n}$  does not contain induced  $C_k$  for  $k \geq 5$ .

**Proof.** Suppose that  $G_{2n}$  contains induced cycle  $C_k$  for  $k \geq 5$ —Figure 6. In Figure 6 we allow  $x_1 = x_4$  but all other vertices  $x_i, x_j$  are pairwise distinct for  $i \leq 4$  and  $j \leq 4$ . Otherwise, either Claim 2 or Claim 3 would be violated. In particular  $x_2 \neq x_3$ . Let  $G^4$  be a subgraph of  $G_n$ . So, there is a transformation  $G_{2n}(G^4 \rightarrow H^4) \rightarrow H_{2n-2}$ . Let  $X$  be a subgraph of  $G^4$  induced by  $\{x_2, x_3, x'_1, x'_2, x'_3, x'_4\}$ , and let  $Y$  be a subgraph of  $H^4$  induced by  $\{x_2, x_3, x'_1, x'_4\}$  indicated in Figure 6 with thick solid lines each. There are important properties of the subgraphs  $X, Y, G^4, H^4$  in Figure 6 as follows: (1) edges of  $Y$  do not belong to  $E(G_n)$ , (2) other edges of  $H^4$  than the ones in  $Y$  belong to  $E(G_n)$ , (3) edges of  $X$  do not belong to  $E(H_{n-2})$ , and (4) edges of  $X$  belong to  $E(G_n)$ . We explore these properties in the following four cases.

Figure 6. Transformation based on  $C_k$  for  $k \geq 5$ .

Case 1.  $\widehat{T}_i(H_{2n-2}, Y)$  contains neither  $(x'_1, x_2)$  nor  $(x_3, x'_4)$ .

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains edges  $(x'_1, x'_2), (x'_2, x'_3) \in E(X)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains edges  $(x_2, x'_2), (x'_2, x'_3) \in E(X)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains edges  $(x'_2, x'_3), (x_3, x'_3) \in E(X)$ ,
4.  $\widehat{T}_{i_4}(G_{2n}, X)$  contains edges  $(x'_2, x'_3), (x'_3, x'_4) \in E(X)$ .

In addition, each  $\widehat{T}_{i_j}(G_{2n}, X)$  contains edges  $E(\widehat{T}_i(H_{2n-2}, Y))$ , for  $4 \geq j \geq 1$ , which together represent all edges in  $\widehat{T}_{i_1}(G_{2n}, X)$ .

Case 2.  $\widehat{T}_i(H_{2n-2}, Y)$  contains  $(x'_1, x_2)$  but not  $(x_3, x'_4)$ .

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains edges  $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3) \in E(X)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains edges  $(x'_1, x'_2), (x_2, x'_2), (x_3, x'_3) \in E(X)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains edges  $(x'_1, x'_2), (x_2, x'_2), (x'_3, x'_4) \in E(X)$ .

In addition, each  $\widehat{T}_{i_j}(G_{2n}, X)$  contains edges  $E(\widehat{T}_i(H_{2n-2}, Y)) \setminus \{(x'_1, x_2)\}$ , for  $3 \geq j \geq 1$ , which together represent all edges in  $\widehat{T}_{i_1}(G_{2n}, X)$ .

Case 3.  $\widehat{T}_i(H_{2n-2}, Y)$  contains  $(x_3, x'_4)$  but not  $(x'_1, x_2)$ .

1.  $\widehat{T}_{i_1}(G_{2n}, X)$  contains edges  $(x_3, x'_3), (x'_3, x'_4), (x'_2, x'_3) \in E(X)$ ,
2.  $\widehat{T}_{i_2}(G_{2n}, X)$  contains edges  $(x_3, x'_3), (x'_3, x'_4), (x_2, x'_2) \in E(X)$ ,
3.  $\widehat{T}_{i_3}(G_{2n}, X)$  contains edges  $(x'_2, x'_3), (x'_3, x'_4), (x'_1, x'_2) \in E(X)$ .

In addition, each  $\widehat{T}_{i_j}(G_{2n}, X)$  contains edges  $E(\widehat{T}_i(H_{2n-2}, Y)) \setminus \{(x_3, x'_4)\}$ , for  $3 \geq j \geq 1$ , which together represent all edges in  $\widehat{T}_{i_1}(G_{2n}, X)$ .

Case 4.  $\widehat{T}_i(H_{2n-2}, Y)$  contains  $(x'_1, x_2)$  and  $(x_3, x'_4)$ . In this case there is a  $\widehat{T}_{i_1}(G_{2n}, X)$  that contains edges  $(x'_1, x'_2), (x_2, x'_2), (x_3, x'_3), (x'_3, x'_4) \in E(X)$ . Furthermore, removing either  $(x_3, x'_3)$  or  $(x'_3, x'_4)$  from  $\widehat{T}_{i_1}(G_{2n}, X)$  induces forest  $Z_n$  of two trees. Clearly, vertices  $x_3, x'_4$  must belong to two different trees



in  $Z_n$ . This implies that either  $x_3, x'_1$  or  $x'_4, x'_1$  belong to two separate trees in either case. If  $x_3, x'_1$  belong to two separate trees, then we obtain second  $\widehat{T}_{i_2}(G_{2n}, X)$  with edges  $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3), (x_3, x'_3) \in E(X)$ . Otherwise,  $x'_4, x'_1$  belong to two separate trees and we obtain another second  $\widehat{T}_{i_2}(G_{2n}, X)$  with edges  $(x'_1, x'_2), (x_2, x'_2), (x'_2, x'_3), (x'_3, x'_4) \in E(X)$  instead.

On the other hand, removing either  $(x'_1, x'_2)$  or  $(x_2, x'_2)$  from  $\widehat{T}_{i_1}(G_{2n}, X)$  induces different forest  $Z_n$  of two trees. Clearly, vertices  $x'_1, x_2$  must belong to two different trees in  $Z_n$ . This implies that either  $x'_1, x'_4$  or  $x_2, x'_4$  belong to two separate trees in either case. If  $x'_1, x'_4$  belong to two separate trees, then we obtain third  $\widehat{T}_{i_3}(G_{2n}, X)$  with edges  $(x'_1, x'_2), (x'_2, x'_3), (x_3, x'_3), (x'_3, x'_4) \in E(X)$ . Otherwise,  $x_2, x'_4$  belong to two separate trees and we obtain another third  $\widehat{T}_{i_3}(G_{2n}, X)$  with edges  $(x_2, x'_2), (x'_2, x'_3), (x_3, x'_3), (x'_3, x'_4) \in E(X)$  instead. In addition, each  $\widehat{T}_{i_j}(G_{2n}, X)$  contains edges  $E(\widehat{T}_i(H_{2n-2}, Y)) \setminus \{(x'_1, x_2), (x_3, x'_4)\}$ , for  $3 \geq j \geq 1$ , which together with edges of previous three  $\widehat{T}_{i_1}(G_{2n}, X)$  trees represent all edges in  $\widehat{T}_{i_1}(G_{2n}, X)$ .

In Cases 1–4 we examined all possible spanning trees of  $H_{2n-2}$ . We conclude that for every spanning tree of  $H_{2n-2}$  there are at least three unique spanning trees of  $G_n$ . Hence,  $3t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ , implying  $t(H_{2n-2}) < t(M_{2n-2})$ —a contradiction, which proves Claim 4.  $\square$

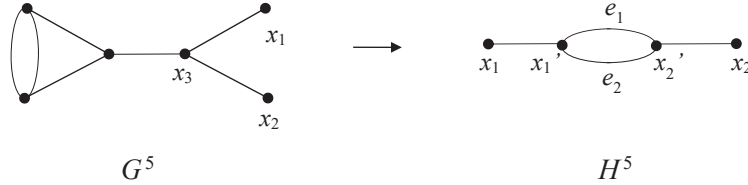


Figure 7. Transformation based on  $M_3$ .

Based on Claims 1–4 we conclude that every cycle of  $G_{2n}$  must belong to some component  $M_3$  in  $G_{2n}$ . So,  $G_{2n}$  must consist of at least three  $M_3$  components. This implies that transformation  $G_{2n}(G^5 \rightarrow H^5) \rightarrow H_{2n-2}$  illustrated in Figure 7 is possible.

Edges  $(x_1, x_3), (x_2, x_3)$  in  $G^5$  do not belong to any cycle. The number of spanning trees in  $G^5$  is  $t(G^5) = 5$ , while in  $H^5$  is  $t(H^5) = 2$ . Consequently,  $\frac{5}{2}t(H_{2n-2}) \leq t(G_{2n}) \leq t(M_{2n}) = 2t(M_{2n-2})$ , implying  $t(H_{2n-2}) < t(M_{2n-2})$ , a contradiction. This contradiction proves Theorem 3.  $\blacksquare$

We note that  $M_{2n}$  is the unique graph as opposed to the simple connected cubic graphs of order  $2n$  that minimize the number of spanning trees, which were identified in [3].

### 3. EXTENSION OF $M_{2n}$ TO ALL CONNECTED, ODD-REGULAR MULTIGRAPHS

We define a regular multigraph  $M_{2n}^{\frac{d-1}{2}}$  of odd degree  $d$ ,  $d \geq 3$ , and on  $2n$  vertices as follows:

- (1)  $M_{2n}^1 := M_{2n}$ , and it consists of components  $M_3^1 := M_3$ ,  $C_2^1 := C_2$ ,
- (2)  $M_{2n}^{i+1}$  is constructed from  $M_{2n}^i$  as follows:
  - (i) add one edge for every pair of vertices in both  $M_3^i$  components of  $M_{2n}^i$ ,
  - (ii) for every component  $C_2^i$  not included in  $M_3^i$  add two parallel edges.

First, consider the number of spanning trees in  $M_{2n}^i$ .

**Theorem 4.**  $t(M_{2n}^k) = k^2(3k+2)^2(k+1)^{n-3}$ , for  $n \geq 3$  and  $k \geq 1$ .

**Proof.** Based on the definition,  $M_{2n}^k$  contains two  $M_3^k$  components and  $n-3$   $C_2^k$  components. It is easy to see that  $t(M_3^k) = (k^2 + k(k+1) + k(k+1)) = k(3k+2)$  and  $t(C_2^k) = k+1$ . Since these components do not belong to any cycle, the number of spanning trees in  $M_{2n}^k$  equals

$$(t(M_3^k))^2 \cdot (t(C_2^k))^{n-3} = k^2(3k+2)^2(k+1)^{n-3}. \quad \blacksquare$$

In particular, for the connected cubic graphs we get lower sharp bound for the number of spanning trees as follows.

**Corollary 4.** Let  $G_{2n}$  be a connected cubic graph of order  $2n \geq 6$ . Then  $t(G_{2n}) \geq 5^2 2^{n-3}$ .

**Proof.** According to Theorem 3,  $t(G_{2n}) \geq t(M_{2n})$ , and according to Theorem 4,  $t(M_{2n}) = 5^2 2^{n-3}$  for  $n \geq 3$ .  $\blacksquare$

Finally, based on our results we propose the following.

**Conjecture 5.** Connected  $r$ -regular graph  $G_{2n}$  of order  $2n$  minimizes  $t(G_{2n})$  for  $r$  odd and  $r, n \geq 3$  if and only if  $G_{2n} \simeq M_{2n}^{\frac{r-1}{2}}$ .

If our conjecture is true then  $((r-1)/2)^2((3r+1)/2)^2((r+1)/2)^{n-3}$  is lower sharp bound for the number of spanning trees in connected  $r$ -regular graphs of order  $2n$  and  $r$  odd.

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