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DEFICIENCY AND FORBIDDEN SUBGRAPHS OF CONNECTED, LOCALLY-CONNECTED GRAPHS[1]

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Abstract

A graph G is *locally-connected* if the neighbourhood $N_G(v)$ induces a connected subgraph for each vertex v in G. For a graph G, the *deficiency* of G is the number of vertices unsaturated by a maximum matching, denoted by $\mathrm{def}(G)$. In fact, the deficiency of a graph measures how far a maximum matching is from being perfect matching. Saito and Xiong have studied subgraphs, the absence of which forces a connected and locally-connected graph G of sufficiently large order to satisfy $\mathrm{def}(G) \leq 1$. In this paper, we extend this result to the condition of $\mathrm{def}(G) \leq k$, where k is a positive integer. Let $\beta_0 = \left\lceil \frac{1}{2}(3+\sqrt{8k+17}) \right\rceil - 1$, we show that $K_{1,2}, K_{1,3}, \ldots, K_{1,\beta_0}$, K_3 or $K_2 \vee 2K_1$ is the required forbidden subgraph. Furthermore, we obtain some similar results about 3-connected, locally-connected graphs.

Keywords: deficiency, locally-connected graph, matching, forbidden subgraph.

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1. Introduction

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph

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G, respectively. For a vertex $v \in V(G)$, the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of v in G; the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S in G. A vertex v in a graph G is said to be *locally-connected* if $G[N_G(v)]$ is connected. If every vertex v of G is locally-connected, we say that G is *locally-connected*.

For a graph G, we use $\alpha(G)$ to denote the independence number of G, and use nG to denote the union of n disjoint copies of G. For graphs G_1 and G_2, we denote the join of G_1 and G_2 by $G_1 \vee G_2$, and the union of G_1 and G_2 by $G_1 \cup G_2$. For any real number x_0, we use $\lceil x_0 \rceil$ to denote the smallest integer that is greater than or equal to x_0, and $\lfloor x_0 \rfloor$ to denote the greatest integer that is less than or equal to x_0. For a given graph H, a graph G is called *H-free* if G has no induced subgraph isomorphic to H. We say that H is *forbidden* in G if G is H-free. The bipartite graph $K_{1,n}$ is called a *star*, and the star $K_{1,3}$ is commonly called a *claw*. The claw-free graphs have a striking effect in the study of local connectivity and Hamiltonian properties. Oberly and Sumner [9] proved that every connected and locally-connected claw-free graph of order at least three is Hamiltonian. For additional remarks on claw-free graphs and their properties, we refer the readers to see the survey [6]. Brause *et al.* [3] studied cycle extendability of connected, locally-connected graphs defined by several classes of forbidden subgraphs. Let G be a connected, locally-connected graph of order at least three. If G is $\big(K_1 \vee (K_2 \cup K_3)\big)$-free, then G is weakly pancyclic, i.e., G has a cycle of order t for every integer t between the girth and circumference of G. If G is $\{K_1 \vee K_1 \vee K_3, K_1 \vee P_4\}$-free or $\{K_1 \vee K_1 \vee K_3, K_1 \vee (K_1 \cup P_3)\}$-free, then G is fully cycle extendable, i.e., every vertex of G lies on a triangle, and every cycle in G of order less than $|V(G)|$ is extendable.

A subset M of $E(G)$ is called a *matching* in G if its elements are edges and no two are adjacent in G. A matching M *saturates* a vertex v, and v is said to be *M-saturated*, if some edge of M is incident with v; otherwise, v is *M-unsaturated*. M is called a *maximum matching* if G has no matching M' with $|M'| > |M|$. If every vertex of G is M-saturated, the matching M is called *perfect matching*. Now we introduce a parameter which measures how far a maximum matching is from being perfect matching. For a graph G, the *deficiency* of G is the number of vertices unsaturated by a maximum matching, denoted by $\mathrm{def}(G)$. It means $\mathrm{def}(G) = n - 2m(G)$, where $n = |V(G)|$ and $m(G)$ is the size of any maximum matching in G. Thus, G has a perfect matching if and only if $\mathrm{def}(G) = 0$. If $\mathrm{def}(G) = 1$, we say that G has a *near-perfect matching*. For a graph G and $S \subseteq V(G)$, we denote by $w_o(G-S)$ the number of the odd components of $G-S$, and Berge's Formula says $\mathrm{def}(G) = \max_{S\subseteq V(G)} \big(w_o(G-S) - |S|\big)$. Las Vergnas [8] in 1975 and Sumner [13] in 1976 independently proved the first well-known result about deficiency in claw-free graphs of even order. In 1985, Jünger *et al.* [7] proved the parallel result for odd graphs. We obtain the following theorem by combining these two results.

Theorem 1 [7, 8, 13]. *Every connected claw-free graph G of even* (*respectively, odd*) *order contains a perfect* (*respectively, near-perfect*) *matching, i.e., satisfying* $\mathrm{def}(G) = 0$ (*respectively,* $\mathrm{def}(G) = 1$).

In 2005, Plummer and Saito [10] proved that the forbidden subgraph which forces a connected graph G of sufficiently large order to satisfy $\mathrm{def}(G) \leq 1$ is $K_{1,2}$ or $K_{1,3}$. And they proved the following theorem.

Theorem 2 [10]. *Let k be a positive integer, and let H be a connected graph of order at least* 3. *Suppose there exist real constants α, β with $0 \leq \alpha < 1$ and a positive integer p_0 such that every k-connected H-free graph G of order greater than p_0 satisfies* $\mathrm{def}(G) \leq \alpha|V(G)| + \beta$. *Then,*

(1) *if $k \geq 2$, then $H = K_{1,n}$ for some n, where $2 \leq n \leq 1 + \left(\frac{1+\alpha}{1-\alpha}\right) k$, and*

(2) *if $k = 1$, then $H = K_{1,n}$ for some n, where $2 \leq n \leq \frac{3-\alpha}{1-\alpha}$.*

In 2016, Saito and Xiong [12] considered what forbidden subgraph forces a connected, locally-connected graph G of sufficiently large order to satisfy $\mathrm{def}(G) \leq 1$, and proved that $K_{1,2}$, $K_{1,3}$, K_3 or $K_2 \vee 2K_1$ does.

Theorem 3 [12]. *Let H be a connected graph of order at least* 3. *If there exists a positive integer p_0 such that every connected and locally-connected H-free graph G of order at least p_0 satisfies* $\mathrm{def}(G) \leq 1$, *then H is isomorphic to $K_{1,2}$, $K_{1,3}$, K_3 or $K_2 \vee 2K_1$.*

A natural question is whether we can change the value of the deficiency in Theorem 3 and get some results about the forbidden subgraph H. We first consider the same problem with $\mathrm{def}(G) \leq 4$ and prove that H is isomorphic to $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, K_3 or $K_2 \vee 2K_1$. In fact, when $\mathrm{def}(G) \leq 2$ or 3, we also can show that H is isomorphic to $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, K_3 or $K_2 \vee 2K_1$. Then we generalize the result and prove Theorem 5. Theorems 3 and 4 will follow as corollaries.

Theorem 4. *Let H be a connected graph of order at least* 3. *If there exists a positive integer p_0 such that every connected and locally-connected H-free graph G of order at least p_0 satisfies* $\mathrm{def}(G) \leq 4$, *then H is isomorphic to $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, K_3 or $K_2 \vee 2K_1$.*

Theorem 5. *Let H be a connected graph of order at least* 3. *If there exists a positive integer p_0 and k such that every connected, locally-connected H-free graph G of order at least p_0 satisfies* $\mathrm{def}(G) \leq k$, *then H is isomorphic to $K_{1,2}$, $K_{1,3}$, $\ldots$, K_{1,β_0}, K_3 or $K_2 \vee 2K_1$, where $\beta_0 = \left\lceil \frac{1}{2}\left(3 + \sqrt{8k+17}\right)\right\rceil - 1$.*

Note that every 2-connected, locally-connected graph obviously is a connected, locally-connected graph, and every connected, locally-connected graph is 2-connected. Hence, the results remain true if we change the condition of connected, locally-connected graph G to 2-connected, locally-connected graph in Theorems 4 and 5. Our other result is about 3-connected, locally-connected graphs. Namely, we prove the following. If every 3-connected and locally-connected H-free graph G of sufficiently large order satisfies $\operatorname{def}(G) \leq 4$, then we have $H \in \{K_{1,2}, K_{1,3}, K_{1,4}, K_{1,5}, K_{1,6}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1\}$. Let a be an integer with $a > 4$, such that $k = \frac{1}{6}\left(a^3 - 3a^2 - 4a - 6\right)$ is a positive integer. If every 3-connected and locally-connected H-free graph G of sufficiently large order satisfies $\operatorname{def}(G) \leq k$, then we get $H \in \{K_{1,2}, K_{1,3}, \ldots, K_{1,\frac{1}{2}(a^2-3a+2)}, K_3, K_2 \vee 2K_1, \ldots, K_2 \vee (a-2)K_1\}$.

We use [2] for terminology and notation not defined here. In Section 2, we prove Theorems 4 and 5, and then we will discuss the trivial and nontrivial cases of the forbidden subgraphs in Theorem 5. In Section 3, we study the relationship between the forbidden subgraphs and deficiency in the class of 3-connected, locally-connected graphs. In Section 4, we close this paper by mentioning some related problems.

2. Proofs of Theorems 4 and 5

***Proof of Theorem* 4.** Using the integer p_0 in the statement of theorem, we set $n = \max\{2p_0 + 1, 7\}$. Note that n is an odd number. Let G_1 be a graph isomorphic to $K_2 \vee nK_1$. Then G_1 is a connected and locally-connected graph of order greater than p_0. On the other hand, since $n \geq 7$, $\operatorname{def}(G_1) = n - 2 \geq 5$. Therefore, G_1 is not H-free and it contains an induced subgraph which is isomorphic to H. Since H is a connected graph of order at least three, $H \cong K_{1,m}$ or $H \cong K_2 \vee mK_1$ for some positive integer m (see Figure 1).

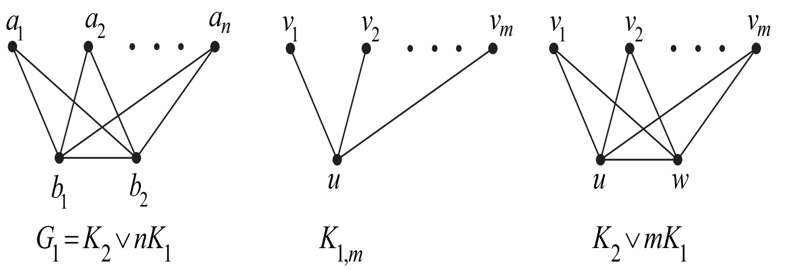

Figure 1. Graphs G_1, $K_{1,m}$ and $K_2 \vee mK_1$.

For an integer t with $t \geq 2$, let A_1, A_2, A_3 and A_4 be 4 disjoint copies of

K_{t+1}. Then choose one vertex a_i in A_i, $1 \leq i \leq 4$, and add edges a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_2a_4 and a_3a_4. Let $A(t+1)$ be the resulting graph (see Figure 2). Note that $A(t+1)$ is a connected graph with independence number $\alpha\big(A(t+1)\big) = 4$.

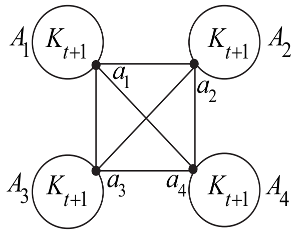

Figure 2. Graph $A(t+1)$ in the proof of Theorem 4.

Let B_0 be a copy of K_5 with $V(B_0) = \{v_1, v_2, \ldots, v_5\}$. For each i, j with $1 \leq i < j \leq 5$, we introduce a new graph $B_{i,j}$ which is a copy of K_n. Then add edges $\big\{v_ix, v_jx : 1 \leq i < j \leq 5, x \in V(B_{i,j})\big\}$. Let B be the resulting graph. Note that B is a connected graph of order greater than p_0.

For $v \in V(B)$, let B_v be the subgraph of B induced by $N_B(v)$. If $v \in V\big(B_{i,j}\big)$ for some i, j with $1 \leq i < j \leq 5$, then $B_v \cong K_{n+1}$. If $v = v_i$ for some i with $1 \leq i \leq 5$, then $N_B(v_i) = \bigcup_{j \in J} V\big(B_{i,j}\big) \cup \big\{v_j : j \in J\big\}$, where $J = \{1, 2, 3, 4, 5\} \setminus \{i\}$, and $B_v \cong A(n+1)$. Therefore, B is a locally-connected graph. Since $B - \big\{v_1, v_2, v_3, v_4, v_5\big\}$ has 10 odd components, $\operatorname{def}(B) \geq 5$. Therefore, B is not H-free and it contains an induced subgraph H which is isomorphic to $K_{1,m}$ or $K_2 \vee mK_1$ for some positive integer m. Since $\alpha(B_v) \leq 4$ for every $v \in V(B)$, we have $m \leq 4$. Therefore, H is one of $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, K_3, $K_2 \vee 2K_1$, $K_2 \vee 3K_1$ and $K_2 \vee 4K_1$.

Assume $H \cong K_2 \vee 3K_1$. Then H contains three independent vertices u_1, u_2, u_3 which have two common neighbors w_1 and w_2. Since $N_B(w_1)$ and $N_B(w_2)$ have three independent vertices, $\{w_1, w_2\} \subset \{v_1, v_2, \ldots, v_5\}$. By symmetry, we may assume $w_1 = v_1, w_2 = v_2$. However, $N_B(v_1) \cap N_B(v_2) = \{v_3, v_4, v_5\} \cup V(B_{1,2})$, which does not contain three independent vertices. This is a contradiction and hence H is not isomorphic to $K_2 \vee 3K_1$. Similarly, H is not isomorphic to $K_2 \vee 4K_1$. ∎

In the following, we give an example of Theorem 4. Let G' be a copy of complete graph K_4 with vertices v_1, v_2, v_3 and v_4. For each i, j with $1 \leq i < j \leq 4$, let $G_{i,j}$ be a copy of complete graph K_{2p_0+1} with $V\big(G_{i,j}\big) = \Big\{v_1^{(i,j)}, v_2^{(i,j)}, \ldots, v_{2p_0+1}^{(i,j)}\Big\}$. Then add edges $\Big\{v_iv_t^{(i,j)}, v_jv_t^{(i,j)} : 1 \leq t \leq 2p_0+1, 1 \leq i < j \leq 4\Big\}$. Let G^* be the resulting graph (see Figure 3). Note that G^* is a connected, locally-connected graph of order greater than p_0. On the one hand, since $G^* - \{v_1, v_2, v_3, v_4\}$ has 6 odd components, we get $\operatorname{def}(G^*) \geq 6 - 4 = 2$ by Berge's Formula. On the other hand, we can find a matching in G^*.

$$M = \{v_1^{(1,2)}v_2^{(1,2)},\ v_3^{(1,2)}v_4^{(1,2)}, \ldots,\ v_{2p_0-1}^{(1,2)}v_{2p_0}^{(1,2)},$$
$$v_1^{(1,3)}v_2^{(1,3)},\ v_3^{(1,3)}v_4^{(1,3)}, \ldots,\ v_{2p_0-1}^{(1,3)}v_{2p_0}^{(1,3)},$$
$$\cdots\cdots$$
$$v_1^{(3,4)}v_2^{(3,4)},\ v_3^{(3,4)}v_4^{(3,4)}, \ldots,\ v_{2p_0-1}^{(3,4)}v_{2p_0}^{(3,4)},$$
$$v_{2p_0+1}^{(1,2)}v_1,\ v_{2p_0+1}^{(1,3)}v_3,\ v_{2p_0+1}^{(1,4)}v_4,\ v_{2p_0+1}^{(2,3)}v_2\}$$

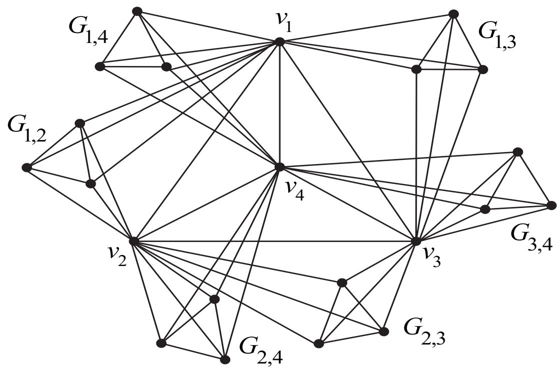

Figure 3. The graph G^* when $p_0 = 1$.

Note that the vertices unsaturated by M are $v_{2p_0+1}^{(2,4)}$ and $v_{2p_0+1}^{(3,4)}$, hence $\mathrm{def}(G^*) \le 2$. Therefore, we have $\mathrm{def}(G^*) = 2 \le 4$. It is easy to see that G^* is $K_{1,4}$-free, and G^* is not $K_{1,2}$-free, $K_{1,3}$-free, K_3-free and $(K_2 \vee 2K_1)$-free. Therefore, G^* is a connected, locally-connected H-free graph of order greater than p_0 satisfies $\mathrm{def}(G) \le 4$, where H is isomorphic to $K_{1,4}$.

***Proof of Theorem* 5.** *Case* 1. k is an odd number. We set $n = \max\{2p_0, k + 3\}$. Note that n is an even number. Let G_1 be a graph isomorphic to $K_2 \vee nK_1$. Then G_1 is a connected, locally-connected graph of order greater than p_0. On the other hand, since $n \ge k+3$, $\mathrm{def}(G_1) = n - 2 \ge k + 1$. Therefore, G_1 is not H-free and hence H is an induced subgraph of $K_2 \vee nK_1$. Since $|V(H)| \ge 3$, $H \cong K_{1,m}$ or $H \cong K_2 \vee mK_1$ for some positive integer m.

For an integer t with $t \ge 2$, let $A_1, A_2, A_3, \ldots, A_{\beta_0}$ be β_0 disjoint copies of K_t, where $\beta_0 = \left\lceil \frac{1}{2}\left(3 + \sqrt{8k+17}\right) \right\rceil - 1$. Then choose one vertex a_i in A_i, $1 \le i \le \beta_0$, and add edges a_ia_j, $1 \le i < j \le \beta_0$. Let $A(t)$ be the resulting graph. Note that $A(t)$ is a connected graph with independence number $\alpha(A(t)) = \beta_0$.

Let B_0 be a copy of K_{β_0+1} with $V(B_0) = \{v_1, v_2, \ldots, v_{\beta_0+1}\}$. For each i, j with $1 \le i < j \le \beta_0 + 1$, we introduce a new graph $B_{i,j}$ that is a copy of K_{n+1}. Then add edges $\{v_i x, v_j x : 1 \le i < j \le \beta_0 + 1, x \in V(B_{i,j})\}$. Let B be the resulting graph. Note that B is a connected graph of order greater than p_0.

For $v \in V(B)$, let B_v be the subgraph of B induced by $N_B(v)$. If $v \in V(B_{i,j})$ for some i, j with $1 \le i < j \le \beta_0 + 1$, then $B_v \cong K_{n+2}$. If $v = v_i$ for some i with $1 \le i \le \beta_0 + 1$, then $N_B(v_i) = \bigcup_{j \in J} V(B_{i,j}) \cup \{v_j : j \in J\}$, where $J = \{1, 2, 3, \ldots, \beta_0 + 1\} \setminus \{i\}$, and $B_v \cong A(n+2)$. Therefore, B is a locally-connected graph. Since $B - \{v_1, v_2, v_3, \ldots, v_{\beta_0+1}\}$ has $\binom{\beta_0+1}{2}$ odd components, by Berge's Formula we have

$$
\begin{aligned}
\operatorname{def}(B) &\ge \binom{\beta_0+1}{2} - (\beta_0+1) = \binom{\left\lceil \frac{1}{2}\left(3+\sqrt{8k+17}\right)\right\rceil}{2} - \left\lceil \frac{1}{2}\left(3+\sqrt{8k+17}\,\right)\right\rceil \\
&= \frac{\left\lceil \frac{3+\sqrt{8k+17}}{2} \right\rceil \left(\left\lceil \frac{3+\sqrt{8k+17}}{2} \right\rceil - 1 \right)}{2} - \left\lceil \frac{3+\sqrt{8k+17}}{2} \right\rceil \\
&= \frac{\left\lceil \frac{3+\sqrt{8k+17}}{2} \right\rceil \left(\left\lceil \frac{3+\sqrt{8k+17}}{2} \right\rceil - 3 \right)}{2} \ge \frac{\frac{3+\sqrt{8k+17}}{2} \left(\frac{3+\sqrt{8k+17}}{2} - 3 \right)}{2} \\
&= \frac{\left(3+\sqrt{8k+17}\right)\left(\sqrt{8k+17}-3\right)}{8} = k+1.
\end{aligned}
$$

Therefore, B is not H-free and it contains an induced subgraph isomorphic to H. Since $\alpha(B_v) \le \beta_0$ for every $v \in V(B)$, we have $m \le \beta_0$.

If $H \cong K_2 \vee mK_1$, we claim that $\alpha(H) \le 2$. Otherwise, assume that $\alpha(H) \ge 3$, then H should contain $K_2 \vee 3K_1$ as an induced subgraph. Recall that H is an induced subgraph of B, B also contains $K_2 \vee 3K_1$, say $w_1 w_2 \vee \{u_1, u_2, u_3\}$, as an induced subgraph. Then $\{w_1, w_2\} \subset \{v_1, v_2, \ldots, v_{\beta_0+1}\}$. By symmetry, we may assume that $w_1 = v_1, w_2 = v_2$. Then $u_1, u_2, u_3 \in N_B(v_1) \cap N_B(v_2)$. However, $N_B(v_1) \cap N_B(v_2) = \{v_3, v_4, \ldots, v_{\beta_0+1}\} \cup V(B_{1,2})$, which does not contain three independent vertices, a contradiction. Therefore, H is isomorphic to $K_{1,2}$, $K_{1,3}$, $\ldots$, K_{1,β_0}, K_3 or $K_2 \vee 2K_1$.

Case 2. k is an even number. We set $n = \max\{2p_0 + 1, k + 3\}$. Note that n is an odd number. Let G_1 be a graph isomorphic to $K_2 \vee nK_1$. By a similar argument to the proof of Case 1, we have $H \cong K_{1,m}$ or $H \cong K_2 \vee mK_1$ for some positive integer m. For an integer t with $t \ge 2$, let A_1, A_2, A_3, $\ldots$, A_{β_0} be β_0 disjoint copies of K_{t+1}. Then choose one vertex a_i in A_i, $1 \le i \le \beta_0$, and add edges $a_i a_j$, $1 \le i < j \le \beta_0$. Let $A(t+1)$ be the resulting graph. Note that $A(t+1)$ is a connected graph with independence number $\alpha(A(t+1)) = \beta_0$.

Let B_0 be a copy of K_{β_0+1} with $V(B_0) = \{v_1, v_2, \ldots, v_{\beta_0+1}\}$. For each i, j with $1 \le i < j \le \beta_0+1$, we introduce a new graph $B_{i,j}$ that is a copy of K_n. Then add edges $\{v_i x, v_j x : 1 \le i < j \le \beta_0 + 1, x \in V(B_{i,j})\}$. Let B be the resulting graph. Note that B is a connected graph of order greater than p_0. For $v \in V(B)$, let B_v be the subgraph of B induced by $N_B(v)$. If $v \in V\big(B_{i,j}\big)$ for some i, j with $1 \le i < j \le \beta_0 + 1$, then $B_v \cong K_{n+1}$. If $v = v_i$ for some i with $1 \le i \le \beta_0 + 1$, then $N_B(v_i) = \bigcup_{j \in J} V\big(B_{i,j}\big) \cup \big\{v_j : j \in J\big\}$, where $J = \{1, 2, 3, \ldots, \beta_0 + 1\} \setminus \{i\}$, and $B_v \cong A(n + 1)$. Therefore, B is a locally-connected graph. Similarly as above, we have that $m \le \beta_0$. If $H \cong K_2 \vee mK_1$, we can also get that $\alpha(H) \le 2$. Therefore, H is isomorphic to $K_{1,2}$, $K_{1,3}$, $\ldots$, K_{1,β_0}, K_3 or $K_2 \vee 2K_1$. ∎

Theorem 5 provides us with $\left\lceil \frac{1}{2}(3 + \sqrt{8k + 17}) \right\rceil$ candidates for a forbidden subgraph H which guarantees that every connected, locally-connected H-free graph G of order at least p_0 satisfies $\operatorname{def}(G) \le k$. In the following, we will show that $K_{1,2}$, K_3 and $K_2 \vee 2K_1$ are trivial cases in Theorem 5. First, we consider $K_{1,2}$-free graphs. It is easy to see that every connected $K_{1,2}$-free graph is a complete graph. Thus, every connected, locally-connected $K_{1,2}$-free graph G satisfies $\operatorname{def}(G) \le 1 \le k$. Therefore, in such case the problem is trivial. Next, we consider K_3-free graphs. Let G be a connected, locally-connected K_3-free graph. Since G is locally-connected and has no triangle as an induced subgraph, every vertex has degree 0 or 1 in G. Since G is connected, G is isomorphic to K_1 or K_2. Therefore, the problem is trivial when $H \cong K_3$. Finally, we consider $\big(K_2 \vee 2K_1\big)$-free graphs. Saito *et al.* [12] proved the following theorem. We give a proof here for completeness.

Theorem 6 [12]. *A connected and locally-connected $\big(K_2 \vee 2K_1\big)$-free graph is a complete graph.*

Proof. Let G be a connected and locally-connected $\big(K_2 \vee 2K_1\big)$-free graph. Then $N_G(v)$ induces a connected $K_{1,2}$-free graph for every $v \in V(G)$. Since every connected $K_{1,2}$-free graph is complete, $N_G(v)$ induces a complete graph for every $v \in V(G)$. Since G is a connected graph, we have that G is a complete graph. ∎

Therefore, the connected and locally-connected $\big(K_2 \vee 2K_1\big)$-free graph G of order at least p_0 satisfies $\operatorname{def}(G) \le k$, trivially.

Hence, only $K_{1,3}, K_{1,4}, \ldots, K_{1,\left\lceil \frac{1}{2}(3+\sqrt{8k+17}) \right\rceil - 1}$ are the possible forbidden subgraphs which force a connected, locally-connected graph G to satisfy $\operatorname{def}(G) \le k$ in a nontrivial manner.

In particular, from the above discussion, we know that if H is isomorphic to $K_{1,2}$, K_3 or $K_2 \vee 2K_1$, then every connected, locally-connected H-free graph G satisfies $\operatorname{def}(G) \le 1$. On the other hand, since every connected, locally-connected $K_{1,3}$-free graph G of order at least three is Hamiltonian, we have $\operatorname{def}(G) \le 1$. Hence, we can rewrite Theorem 3 into a necessary and sufficient form.

Theorem 7. *Let H be a connected graph of order at least 3. There exists a positive integer p_0 such that every connected and locally-connected H-free graph*

G of order at least p_0 satisfies $\mathrm{def}(G) \leq 1$*, if and only if H is isomorphic to $K_{1,2}$, $K_{1,3}$, K_3 or $K_2 \vee 2K_1$.*

Now we consider Theorem 4. If the following conjecture is true, then the result of Theorem 4 would be a necessary and sufficient condition.

Conjecture 8. *There exists a positive integer p_0, such that if a connected and locally-connected $K_{1,4}$-free graph G has order at least p_0, then* $\mathrm{def}(G) \leq 4$.

3. Some Results About 3-Connected, Locally-Connected Graphs

We begin with the following simple proposition, which is a special case of a result by Chartrand and Pippert [4], and we give a proof here for completeness.

Proposition 9 [4]**.** *A graph G is connected and locally-connected, if and only if G is 2-connected and locally-connected.*

Proof. Since a 2-connected and locally-connected graph is obviously connected and locally-connected, we only need to prove that every connected and locally-connected graph is 2-connected.

Let G'' be a connected, locally-connected graph. Suppose that G'' is not 2-connected, then there is a cut vertex v in G''. The deletion of v disconnects G'' into several components $G_1, G_2, \ldots, G_s$, as shown in Figure 4. Then vertex v is not a locally-connected vertex, which is a contradiction. Hence every connected and locally-connected graph G is 2-connected. Obviously, G is 2-connected and locally-connected. ∎

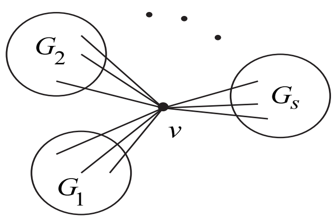

Figure 4. Graph G''.

By Proposition 9, we can change the condition from connected and locally-connected graph into 2-connected and locally-connected graph in Theorems 4 and 5. In the following, we will give some similar results about 3-connected and locally-connected graphs.

Theorem 10. *Let H be a connected graph of order at least 3. If there exists a positive integer p_0 such that every 3-connected and locally-connected H-free graph G of order at least p_0 satisfies* $\mathrm{def}(G) \leq k$ *where* $k \in \{1,2,3,4\}$, *then* $H \in \big\{K_{1,2}, K_{1,3}, K_{1,4}, K_{1,5}, K_{1,6}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1\big\}$.

Proof. Using the integer p_0 in the statement of theorem, we set $n = \max\big\{2p_0, 8\big\}$. Note that n is an even number. Let G_1 be a graph isomorphic to $P_3 \vee nK_1$. Then G_1 is a 3-connected and locally-connected graph of order greater than p_0. Since $n \geq 8$, we have $\mathrm{def}\big(G_1\big) = n - 3 \geq 5$. Therefore, G_1 is not H-free and it contains an induced subgraph which is isomorphic to H. Since H is a connected graph of order at least three, $H \in \mathcal{H}_1 = \big\{K_{1,m}, K_2 \vee mK_1, K_{2,m}, P_3 \vee mK_1\big\}$ for some positive integer m. Let G_2 be a graph isomorphic to $K_3 \vee nK_1$. Similar to G_1, G_2 is a 3-connected and locally-connected graph of order greater than p_0, and $\mathrm{def}\big(G_2\big) = n - 3 \geq 5$. Therefore, G_2 is not H-free and it contains an induced subgraph which is isomorphic to H. Since H is a connected graph of order at least three, $H \in \mathcal{H}_2 = \big\{K_{1,m}, K_2 \vee mK_1, K_3 \vee mK_1\big\}$ for some positive integer m. Hence, $H \in \mathcal{H}_1 \cap \mathcal{H}_2 = \big\{K_{1,m}, K_2 \vee mK_1\big\}$, for some positive integer m.

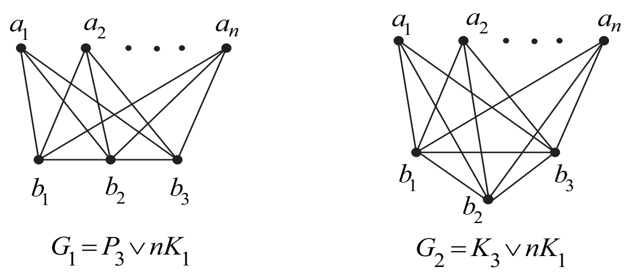

Figure 5. Graphs G_1 and G_2 in the proof of Theorem 10.

Let A_0 be a copy of K_5 with $V(A_0) = \{v_1, v_2, \ldots, v_5\}$. For each i, j, k with $1 \leq i < j < k \leq 5$, we introduce a new graph $A_{i,j,k}$ which is a copy of K_{n+1}. Then add edges $\big\{v_i x, v_j x, v_k x : 1 \leq i < j < k \leq 5, x \in V\big(A_{i,j,k}\big)\big\}$. Let A be the resulting graph. Note that A is a 3-connected graph of order greater than p_0. Let B_0 be a copy of K_4 with $V(B_0) = \big\{u_1, u_2, u_3, u_4\big\}$. For each i, j with $1 \leq i < j \leq 4$, we introduce a new graph $B_{i,j}$ which is a copy of K_{n+1}. Then add edges $\big\{u_i x, u_j x : 1 \leq i < j \leq 4, x \in V\big(B_{i,j}\big)\big\}$. Let B be the resulting graph. Note that B is a connected graph with independence number 6.

For $v \in V(A)$, let $A_v = A\big[N_A(v)\big]$. If $v \in V\big(A_{i,j,k}\big)$ for some i, j, k with $1 \leq i < j < k \leq 5$, then $A_v \cong K_{n+3}$. If $v = v_i$ for some i with $1 \leq i \leq 5$, then $A_v \cong B$. Therefore, A is a locally-connected graph. Since $A - \big\{v_1, v_2, v_3, v_4, v_5\big\}$ has 10 odd components, $\mathrm{def}(A) \geq 10 - 5 = 5$. Therefore, A is not H-free and there is an induced subgraph H of A which is isomorphic to $K_{1,m}$ or $K_2 \vee mK_1$ for some positive integer m. Since $\alpha(A_v) \leq 6$ for every $v \in V(A)$, we have $m \leq 6$. Therefore, $H \in \{K_{1,2}, K_{1,3}, \ldots, K_{1,6}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1, \ldots, K_2 \vee 6K_1\}$.

Assume $H \cong K_2 \vee 4K_1$. Then H contains four independent vertices x_1, x_2, x_3, x_4 which have two common neighbors w_1 and w_2. Since $N_A(w_1)$ and $N_A(w_2)$ have independence number greater than one, $\{w_1, w_2\} \subset \{v_1, v_2, \ldots, v_5\}$. By symmetry, we may assume $w_1 = v_1, w_2 = v_2$. However, $N_A(v_1) \cap N_A(v_2) = \{v_3, v_4, v_5\} \cup V(A_{1,2,3}) \cup V(A_{1,2,4}) \cup V(A_{1,2,5})$, which does not contain four independent vertices. This is a contradiction and hence H is not isomorphic to $K_2 \vee 4K_1$. Similarly, H is not isomorphic to $K_2 \vee 5K_1$ or $K_2 \vee 6K_1$. Therefore, $H \in \{K_{1,2}, K_{1,3}, K_{1,4}, K_{1,5}, K_{1,6}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1\}$. ■

Theorem 11. *Let H be a connected graph of order at least 3. If there exists a positive integer p_0 such that every 3-connected and locally-connected H-free graph G of order at least p_0 satisfies* $\mathrm{def}(G) \leq k$, *where* $k = \lfloor \frac{1}{6}(a^3 - 3a^2 - 4a - 6) \rfloor$, *and a is an integer with $a > 4$, then* $H \in \{K_{1,2}, K_{1,3}, \ldots, K_{1,\frac{1}{2}(a^2-3a+2)}, K_3, K_2 \vee 2K_1, \ldots, K_2 \vee (a-2)K_1\}$.

Proof. *Case* 1. k is an odd number. We set $n = \max\{2p_0 + 1, k + 4\}$. Note that n is an odd number. Let $G_1 \cong P_3 \vee nK_1$ and $G_2 \cong K_3 \vee nK_1$. Analogous to the proof of Theorem 10, $H \in \{K_{1,m}, K_2 \vee mK_1\}$ for some positive integer m.

Let A_0 be a copy of K_a with $V(A_0) = \{v_1, v_2, \ldots, v_a\}$. For each i, j, l with $1 \leq i < j < l \leq a$, we introduce a new graph $A_{i,j,l}$ which is a copy of K_n. Then add edges $\{v_i x, v_j x, v_l x : 1 \leq i < j < l \leq a, x \in V(A_{i,j,l})\}$. Let A be the resulting graph. Note that A is a 3-connected graph of order greater than p_0. Let B_0 be a copy of K_{a-1} with $V(B_0) = \{u_1, u_2, \ldots, u_{a-1}\}$. For each i, j with $1 \leq i < j \leq a - 1$, we introduce a new graph $B_{i,j}$ which is a copy of K_n. Then add edges $\{u_i x, u_j x : 1 \leq i < j \leq a - 1, x \in V(B_{i,j})\}$. Let B be the resulting graph. Note that B is a connected graph with independence number $\alpha(B) = \binom{a-1}{2} = \frac{1}{2}(a-1)(a-2) = \frac{1}{2}(a^2 - 3a + 2)$.

For $v \in V(A)$, let $A_v = A[N_A(v)]$. If $v \in V(A_{i,j,l})$ for some i, j, l with $1 \leq i < j < l \leq a$, then $A_v \cong K_{n+2}$. If $v = v_i$ for some i with $1 \leq i \leq a$, then $A_v \cong B$. Therefore, A is a locally-connected graph. Note that $A - \{v_1, v_2, \ldots, v_a\}$ has $\binom{a}{3}$ odd components. When $a > 4$, we have

$$\begin{aligned}\mathrm{def}(A) &\geq \binom{a}{3} - a = \frac{1}{6}a(a-1)(a-2) - a = \frac{1}{6}(a^3 - 3a^2 - 4a)\\ &= \frac{1}{6}(a^3 - 3a^2 - 4a - 6) + 1 \geq \left\lfloor \frac{1}{6}(a^3 - 3a^2 - 4a - 6) \right\rfloor + 1 = k + 1.\end{aligned}$$

Therefore, A is not H-free and it contains an induced subgraph isomorphic to H. Since $\alpha(A_v) \leq \frac{1}{2}(a^2 - 3a + 2)$ for every $v \in V(A)$, we have $m \leq \frac{1}{2}(a^2 - 3a + 2)$.

Therefore, $H \in \big\{K_{1,2}, K_{1,3}, \ldots, K_{1,\frac{1}{2}(a^2-3a+2)}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1, \ldots, K_2 \vee (\frac{1}{2}(a^2-3a+2))K_1\big\}$.

Assume $H \cong K_2 \vee (a-1)K_1$. Then H contains $a-1$ independent vertices $x_1, x_2, \ldots, x_{a-1}$ which have two common neighbors w_1 and w_2. Since $N_A(w_1)$ and $N_A(w_2)$ have $a-1$ independent vertices, $\big\{w_1, w_2\big\} \subset \big\{v_1, v_2, \ldots, v_a\big\}$. By symmetry, we may assume $w_1 = v_1, w_2 = v_2$. However, $N_A(v_1) \cap N_A(v_2) = \big\{v_3, v_4, \ldots, v_a\big\} \cup V\big(A_{1,2,3}\big) \cup V\big(A_{1,2,4}\big) \cup \cdots \cup V\big(A_{1,2,a}\big)$, which does not contain $a-1$ independent vertices. This is a contradiction and hence H is not isomorphic to $K_2 \vee (a-1)K_1$. Similarly, $H \notin \big\{K_2 \vee aK_1, K_2 \vee (a+1)K_1, \ldots, K_2 \vee \big(\frac{1}{2}(a^2 - 3a+2)\big)K_1\big\}$.

Case 2. k is an even number. We set $n = \max\big\{2p_0, k+4\big\}$. Note that n is an even number. Let $G_1 \cong P_3 \vee nK_1$ and $G_2 \cong K_3 \vee nK_1$. Similarly, $H \in \big\{K_{1,m}, K_2 \vee mK_1\big\}$ for some positive integer m.

Let A_0 be a copy of K_a with $V(A_0) = \big\{v_1, v_2, \ldots, v_a\big\}$. For each i, j, l with $1 \le i < j < l \le a$, we introduce a new graph $A_{i,j,l}$ which is a copy of K_{n+1}. Then add edges $\big\{v_ix, v_jx, v_lx : 1 \le i < j < l \le a, x \in V\big(A_{i,j,l}\big)\big\}$. Let A be the resulting graph. Note that A is a 3-connected graph of order greater than p_0. Let B_0 be a copy of K_{a-1} with $V(B_0) = \big\{u_1, u_2, \ldots, u_{a-1}\big\}$. For each i, j with $1 \le i < j \le a-1$, we introduce a new graph $B_{i,j}$ which is a copy of K_{n+1}. Then add edges $\big\{u_ix, u_jx : 1 \le i < j \le a-1, x \in V\left(B_{i,j}\right)\big\}$. Let B be the resulting graph. Then B is a connected graph with independence number $\alpha(B) = \binom{a-1}{2} = \frac{1}{2}\big(a^2 - 3a + 2\big)$.

For $v \in V(A)$, let $A_v = A\big[N_A(v)\big]$. If $v \in V(A_{i,j,l})$ for some i, j, l with $1 \le i < j < l \le a$, then $A_v \cong K_{n+3}$. If $v = v_i$ for some i with $1 \le i \le a$, then $A_v \cong B$. Therefore, A is a locally-connected graph. Similarly, $\mathrm{def}(A) \ge \binom{a}{3} - a \ge k+1$. Therefore, A is not H-free and it contains an induced subgraph isomorphic to H. Since $\alpha(A_v) \le \frac{1}{2}\big(a^2 - 3a + 2\big)$ for every $v \in V(A)$, we have $m \le \frac{1}{2}(a^2 - 3a + 2)$. Therefore, $H \in \big\{K_{1,2}, K_{1,3}, \ldots, K_{1,\frac{1}{2}(a^2-3a+2)}, K_3, K_2 \vee 2K_1, K_2 \vee 3K_1, \ldots, K_2 \vee \big(\frac{1}{2}(a^2-3a+2)\big)K_1\big\}$.

Similar to Case 1, $H \notin \big\{K_2 \vee (a-1)K_1, K_2 \vee aK_1, \ldots, K_2 \vee \big(\frac{1}{2}(a^2 - 3a+2)\big) K_1\big\}$. Therefore, $H \in \big\{K_{1,2}, K_{1,3}, \ldots, K_{1,\frac{1}{2}(a^2-3a+2)}, K_3, K_2 \vee 2K_1, \ldots, K_2 \vee (a-2)K_1\big\}$. ■

4. Concluding Remarks

In Theorem 11 we prove some results about forbidden subgraphs in 3-connected, locally-connected graphs satisfying $\mathrm{def}(G) \le k$. For any positive integer l, the situation of l-connected and locally-connected graphs is more complicated. We were not able to resolve this question and leave it as an open problem. For more problems about locally highly connected graphs, we refer the readers to [1] and [4].

For a graph G, let $B(G)$ denote the set of vertices of G which are not locally-connected. If $B(G)$ is an independent set and for any $v \in B(G)$, there exists a vertex u in $V(G) \setminus \{v\}$ such that $N_G(v) \cup \{u\}$ induces a connected subgraph of G, then G is called *almost locally connected*. Another natural question is whether we can get a counterpart of our results for connected, almost locally connected graphs. For more discussion and other related problems about almost locally connected graphs, we refer the readers to [5] and [11].

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