A NOTE ON UPPER BOUNDS FOR SOME GENERALIZED FOLKMAN NUMBERS

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Abstract

We present some new constructive upper bounds based on product graphs for generalized vertex Folkman numbers. They lead to new upper bounds for some special cases of generalized edge Folkman numbers, including the cases $F_e(K_3, K_4 - e; K_5) \leq 27$ and $F_e(K_4 - e, K_4 - e; K_5) \leq 51$. The latter bound follows from a construction of a K_5-free graph on 51 vertices, for which every edge coloring with two colors contains a monochromatic $K_4 - e$.

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1.   Folkman Numbers

Let $r, s, a_1, \ldots, a_r$ be positive integers such that $r \geq 2$, $s > \max\{a_1, \ldots, a_r\}$ and $\min\{a_1, \ldots, a_r\} \geq 2$. We write $G \to (a_1, \ldots, a_r)^v$ (respectively, $G \to (a_1, \ldots, a_r)^e$) if for every r-coloring of $V(G)$ (respectively, $E(G)$), there exists a monochromatic K_{a_i} in G for some color $i \in \{1, \ldots, r\}$. The Ramsey number $R(a_1, \ldots, a_r)$ is defined as the smallest integer n such that $K_n \to (a_1, \ldots, a_r)^e$. The sets of vertex and edge Folkman graphs are defined as

$$\mathcal{F}_v(a_1, \ldots, a_r; s) = \{G \mid G \to (a_1, \ldots, a_r)^v \text{ and } K_s \nsubseteq G\}, \text{ and}$$

$$\mathcal{F}_e(a_1, \ldots, a_r; s) = \{G \mid G \to (a_1, \ldots, a_r)^e \text{ and } K_s \nsubseteq G\},$$

respectively, and the vertex and edge Folkman numbers are defined as the smallest orders of graphs in these sets, namely

$$F_v(a_1, \ldots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v(a_1, \ldots, a_r; s)\}, \text{ and}$$

$$F_e(a_1, \ldots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e(a_1, \ldots, a_r; s)\}.$$

The generalized vertex and edge Folkman numbers, $F_v(H_1, \ldots, H_r; H)$ and $F_e(H_1, \ldots, H_r; H)$, are defined analogously by considering arrowing graphs H_i while avoiding H, instead of arrowing complete graphs K_{a_i} while avoiding K_s. The edge Folkman number $F_e(a_1, \ldots, a_r; k)$ can be seen as a generalization of the classical Ramsey number $R(a_1, \ldots, a_r)$, since for $k > R(a_1, \ldots, a_r)$ we clearly have $F_e(a_1, \ldots, a_r; k) = R(a_1, \ldots, a_r)$.

In 1970, Folkman [3] proved that for positive integers k and $a_1, \ldots, a_r$, $F_v(a_1, \ldots, a_r; k)$ and $F_e(a_1, a_2; k)$ exist if and only if $k > \max\{a_1, \ldots, a_r\}$. Folkman's method did not work for edge colorings for more than two colors. The existence of $F_e(a_1, \ldots, a_r; k)$ was proved by Nešetřil and Rödl in 1976 [9]. Folkman numbers have been studied by many other authors, in particular in [2], [4]–[7], [11, 15, 16]. The current authors studied chromatic variations of Folkman numbers [12], and some existence questions for $F_v(H_1, \ldots, H_r; H)$ and $F_e(H_1, \ldots, H_r; H)$ [13]. If $H = K_n$, then we may write $F_v(H_1, \ldots, H_r; H)$ and $F_e(H_1, \ldots, H_r; H)$ as $F_v(H_1, \ldots, H_r; n)$ and $F_e(H_1, \ldots, H_r; n)$, respectively.

Perhaps the most wanted Folkman number is $F_e(3, 3; 4)$, for which the currently best known bounds are 20 [1] and 786 (see [13]). Further improvements of the bounds on $F_e(3, 3; 4)$ seem very difficult, but some insights can be made into similar cases involving almost complete graphs $K_k - e$.

For vertex-disjoint graphs G and H, their *join* $G + H$ has the vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H) \cup E(G, H)$, where $E(G, H)$ is the set of all possible edges between $V(G)$ and $V(H)$. Let us also denote $K_k - e$ by J_k. In [13], we proved the existence of $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ and $F_v(K_k, K_k; J_{k+1})$, for all $k \geq 3$. In the same paper we discussed the existence of some generalized Folkman

numbers, especially in the cases of the form $F_e(K_3, K_3; H)$ for some small graphs H. The latter includes proofs of nonexistence of the numbers $F_e(K_3, K_3; J_4)$, $F_e(K_3, K_3; K_2 + 3K_1)$ and $F_e(K_3, K_3; K_1 + P_4)$, and poses some open cases, like that for $F_e(K_3, K_3; K_1 + C_4)$.

In Section 2 we overview some of the prior constructions and related upper bounds, and we present our new constructions. They lead to some new concrete upper bounds, presented in Section 3, for some special cases including $F_e(K_3, J_4; K_5) \leq 27$ and $F_e(J_4, J_4; K_5) \leq 51$.

2. CONSTRUCTIVE UPPER BOUNDS

The multiplicative upper bound inequality for vertex Folkman numbers stated in Theorem 1 below was proved independently in [5] and [16]. A related constructive upper bound for $F_v(k, k; k+1)$ was obtained in [16], which improved earlier known bounds, however it is still much weaker than the best known probabilistic upper bound for these parameters [2].

Theorem 1 [5, 16]. *If* $\max \{a_1, \ldots, a_r\} \leq a$ *and* $\max \{b_1, \ldots, b_r\} \leq b$, *then*

$$F_v(a_1b_1, \ldots, a_rb_r; ab + 1) \leq F_v(a_1, \ldots, a_r; a + 1)F_v(b_1, \ldots, b_r; b + 1).$$

For graphs G and H, we will use their *lexicographic product graph* $G[H]$ defined on the set of vertices $V(G) \times V(H)$ with $\{(u_1, v_1), (u_2, v_2)\} \in E(G[H])$ if and only if $\{u_1, u_2\} \in E(G)$ or ($u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$).

The original proofs of Theorem 1 are very similar to the proof of the following Lemma 2 and Theorem 3. The latter is a simple, but very useful, generalization of Theorem 1.

Lemma 2. *For graphs* G, H *and* H_i, *and integers* $a_j \geq 2$, $1 \leq i, j \leq r$, *if* $G \rightarrow (a_1, \ldots, a_r)^v$ *and* $H \rightarrow (H_1, \ldots, H_r)^v$, *then* $G[H] \rightarrow (K_{a_1}[H_1], \ldots, K_{a_r}[H_r])^v$.

Proof. Let G and H be any graphs as in the assumptions of the lemma. Let their sets of vertices be $U = V(G)$ and $V = V(H)$, respectively, and consider any partition $V(G[H]) = \bigcup_{i=1}^r X_i$, i.e., r-coloring C_v of the vertices of $G[H]$. We need to show that for some color i, $1 \leq i \leq r$, the subgraph induced by X_i contains $K_{a_i}[H_i]$. Note that for each fixed $u \in U$, the vertices $V(u) = \{(u, v) \mid v \in V\}$ induce a graph isomorphic to H in $G[H]$. Hence, for each $u \in U$ there exists a color $i(u)$, $1 \leq i(u) \leq r$, such that the subgraph induced by $V(u)$ contains $H_{i(u)}$ in color $i(u)$.

Next, consider the r-coloring C_v' of vertices of G defined by $i(u)$. Since $G \rightarrow (a_1, \ldots, a_r)^v$, then there exists j such C_v' contains K_{a_j} in color j in G, or equivalently, in the vertex r-coloring C_v of $G[H]$ we have a_j isomorphic copies of

H, each of them containing H_j in color j, and all of them are interconnected by edges in $G[H]$. ∎

Let $cl(H)$ denote the clique number of graph H, i.e., the largest integer s such that $K_s \subset H$. The following generalizes Theorem 1.

Theorem 3. *If* $\max\{a_1, \ldots, a_r\} \le a$ *and* $\max\{cl(H_1), \ldots, cl(H_r)\} \le b$, *then*

$$F_v(K_{a_1}[H_1], \ldots, K_{a_r}[H_r]; ab + 1) \le F_v(a_1, \ldots, a_r; a + 1) F_v(H_1, \ldots, H_r; b + 1).$$

Proof. Consider any graph $G \in \mathcal{F}_v(a_1, \ldots, a_r; a + 1)$ with the set of vertices $V(G) = U = \{u_1, \ldots, u_s\}$, where $s = F_v(a_1, \ldots, a_r; a + 1)$, and any graph H such that $H \in \mathcal{F}_v(H_1, \ldots, H_r; b + 1)$ and $V(H) = V = \{v_1, \ldots, v_t\}$, where $t = F_v(H_1, \ldots, H_r; b + 1)$. Note that $st = |V(G[H])|$ is also equal to the right hand side of the target inequality. By the construction of $G[H]$ one can easily see that $cl(G[H]) \le ab$. Finally, Lemma 2 implies that $G[H] \to (K_{a_1}[H_1], \ldots, K_{a_r}[H_r])^v$, which completes the proof. ∎

We note now, and will also use it later, that Theorem 3 is specially interesting in the cases involving graphs J_k, because we can use the fact that J_{sk+1} is a subgraph of $K_s[J_{k+1}]$. For instance, using Theorem 3 for two colors with $s = a_1 = a_2 = 2$, $k = b$, $F_v(2, 2; 3) = 5$ and $H_1 = H_2 = J_{k+1}$, we obtain

$$F_v(K_{2k+2} - 2K_2, K_{2k+2} - 2K_2; 2k + 1) \le 5F_v(J_{k+1}, J_{k+1}; k + 1).$$

Further, since J_{2k+1} is a subgraph of $K_{2k+2} - 2e$, it also holds that

$$F_v(J_{2k+1}, J_{2k+1}; 2k + 1) \le 5F_v(J_{k+1}, J_{k+1}; k + 1).$$

In fact, we can do a little better on 3 out of 5 blocks of $F_v(J_{k+1}, J_{k+1}; k + 1)$ vertices, as stated in the next theorem.

Theorem 4. *For every integer* $k \ge 2$, *we have that* $F_v(J_{2k+1}, J_{2k+1}; 2k + 1) \le 2F_v(k, k; k + 1) + 2F_v(J_{k+1}, J_{k+1}; k + 1) + F_v(K_k, J_{k+1}; k + 1)$.

Proof. Consider any graphs H_1, H_2, H_3 such that $H_1 \in \mathcal{F}_v(k, k; k + 1)$, $H_2 \in \mathcal{F}_v(K_k, J_{k+1}; k + 1)$, and $H_3 \in \mathcal{F}_v(J_{k+1}, J_{k+1}; k + 1)$, and they have the smallest possible number of vertices, i.e., $|V(H_1)| = F_v(k, k; k+1)$, $|V(H_2)| = F_v(K_k, J_{k+1}; k+1)$, and $|V(H_3)| = F_v(J_{k+1}, J_{k+1}; k+1)$. Let H_4 be an isomorphic copy of H_1, and H_5 an isomorphic copy of H_3. The clique number of all graphs H_i is equal to k.

Our goal is to construct graph $G \in \mathcal{F}_v(J_{2k+1}, J_{2k+1}; 2k + 1)$ on the set of vertices $V(G) = \bigcup_{i=1}^5 V(H_i)$, which has $cl(G) = 2k$. This will suffice to complete the proof. The set of edges of graph G is defined by $E(G) = \bigcup_{i=1}^5 E(H_i) \cup E(1,3) \cup E(1,5) \cup E(2,4) \cup E(2,5) \cup E(3,4)$, where $E(i,j) = \{\{u, v\} \mid u \in$

$V(H_i), v \in V(H_j), i \neq j, 1 \leq i, j \leq 5\}$. One can easily check that G is K_{2k+1}-free, since the edges of types $E(i, j)$ do not form any triangles. It remains to be shown that $G \rightarrow (J_{2k+1}, J_{2k+1})^v$.

For a contradiction, suppose that there exists a partition $V(G) = R \cup B$, i.e. a red-blue coloring of the vertices of G, which has no monochromatic J_{2k+1}. Without loss of generality we may assume that H_1 contains a red K_k. Therefore, there is no red J_{k+1} in H_3 and no red J_{k+1} in H_5, otherwise we would have a red J_{2k+1}. Hence, there are blue J_{k+1}'s in both H_3 and H_5. Therefore, there is no blue K_k in H_2 and no blue K_k in H_4. Hence, there is a red J_{k+1} in H_2 and a red K_k in H_4, and together they form a red J_{2k+1}. Thus $G \rightarrow (J_{2k+1}, J_{2k+1})^v$, which completes the proof. ∎

As an application of the last theorem we consider an interesting case of $F_v(J_5, J_5; 5)$. Likely, it is just somewhat larger (and harder to compute) than the well studied classical case of $F_v(4, 4; 5)$, for which the currently best known bounds are $17 \leq F_v(2, 3, 4; 5) \leq F_v(4, 4; 5) \leq 23$ [14].

Claim 5. $F_v(J_5, J_5; 5) \leq 36$.

Proof. The first three graphs H_i in the proof of Theorem 4 for this case are of orders equal to $F_v(2, 2; 3)$, $F_v(J_3, J_3; 3)$ and $F_v(K_2, J_3; 3)$, respectively. The first of them, equal to 5, is uniquely witnessed by the cycle C_5. Obtaining upper bounds for the other two requires some work when analyzing possible triangle-free graphs arrowing the corresponding graph parameters (note that $J_3 = P_3$ and thus any J_3-free graph must be of the form $sK_1 \cup tK_2$). An example of graph witnessing $F_v(J_3, J_3; 3) \leq 9$ can be constructed by dropping one vertex from the graph $C_5[2K_1]$, and for $F_v(K_2, J_3; 3) \leq 8$ by adding four main diagonals to the cycle C_8. Putting it all together, by Theorem 4 applied to this case we obtain $F_v(J_5, J_5; 5) \leq 2 \cdot 5 + 2 \cdot 9 + 8 = 36$. ∎

We expect that the actual value of $F_v(J_5, J_5; 5)$ is still smaller, but probably not much less so. How to obtain better bound in this and other similar cases by detailed analysis is an interesting and challenging problem.

We end this section with two more upper bounds on $F_v(J_k, J_k; k)$. Let E_s denote the empty graph on s vertices. Thus, for example, $K_s[E_t]$ is the same as the standard complete s-partite graph with all parts of order t, or equivalently, the Turán graph $T_{st,s}$. One can easily show, similarly as in the proof of Lemma 2, that if $G \rightarrow (k-1, k-1)^v$, then $G[E_3] \rightarrow (K_{k-1}[E_2], K_{k-1}[E_2])^v$. Moreover, the same assumption also gives $G[E_3] \rightarrow (J_k, J_k)^v$, because J_k is a subgraph of $K_{k-1}[E_2]$. An even stronger result following from similar considerations is presented in the next theorem.

Theorem 6. *For any integer $k \geq 3$, suppose that $G \in \mathcal{F}_v(k-1, k-1; k)$ and $|V(G)| = F_v(k-1, k-1; k)$. If $f(G)$ is the largest order of any K_{k-1}-free induced subgraph in G, then*

$$F_v(J_k, J_k; k) \leq 3F_v(k-1, k-1; k) - f(G).$$

Proof. Let G be any graph in $\mathcal{F}_v(k-1, k-1; k)$ with the smallest possible number of vertices $|V(G)| = F_v(k-1, k-1; k)$. Let us denote $V(G) = U = \{u_1, \ldots, u_s\}$, so that the vertices $X = \{u_1, \ldots, u_{f(G)}\}$ induce the largest K_{k-1}-free subgraph in G. Note that the vertices of every K_{k-1} in G have nonempty intersection with $Y = \{u_{f(G)+1}, \ldots, u_s\}$. We will construct a graph H on $3|V(G)| - f(G)$ vertices such that $H \in \mathcal{F}_v(J_k, J_k; k)$, which will complete the proof of the theorem.

Let $V = \{v_1, v_2, v_3\}$. First, take the graph $G[E_3]$ with the set of vertices $U \times V$. Then H is obtained from it by dropping $f(G)$ vertices forming the set $\{\{u_i, v_3\} \mid 1 \leq i \leq f(G)\}$ with all incident edges. Clearly, graph H has the right number of vertices and $cl(H) = k-1$. It remains to be shown that $H \to (J_k, J_k)^v$.

Let $V(H) = R \cup B$ be any partition of the vertices of H into two parts, i.e., any red-blue coloring of $V(H)$. Note that for each fixed $u \in U$, there are 2 or 3 vertices in the set $V(u) = \{(u, v) \mid (u, v) \in V(H)\}$. Let $i(u) \in \{R, B\}$ be a color of at least half of vertices in $V(u)$ (1 or 2). Next, consider the red-blue coloring of vertices of G defined by $i(u)$. Since $G \to (k-1, k-1)^v$, then for the coloring $i(u)$ there exists a set of vertices $S \subset V(G)$ containing a monochromatic K_{k-1} in G. Considering the properties of X and Y, S must contain at least one vertex $u \in Y \cap S$, and consequently at least two vertices $(u, x), (u, y) \in V(H)$ are of color $i(u)$, where $x, y \in V$ and x and y are different. Now, S expanded to vertices of H of the same color must contain a monochromatic J_k. ∎

Corollary 7. *If k is an integer no smaller than 3, then*

$$F_v(J_k, J_k; k) \leq \left\lceil \frac{5F_v(k-1, k-1; k)}{2} \right\rceil.$$

Proof. Set $m = \lceil (F_v(k-1, k-1; k) - 1)/2 \rceil$, and let G and $f(G)$ be as in Theorem 6. Observe that for every vertex $v \in V(G)$, $G - v \not\to (k-1, k-1)^v$ and thus there exists a K_{k-1}-free set $S \subset V(G) - \{v\}$ satisfying $|S| \geq m$. Therefore, $f(G) \geq m$ and the corollary easily follows from Theorem 6. ∎

Using arguments similar to those in Lemma 2 and Theorem 6, it is easy to see that $C_5[E_{2t-1}] \to (K_{t,t}, K_{t,t})^v$, and since $|C_5[E_{2t-1}]| = 10t - 5$, we have an upper bound $F_v(K_{t,t}, K_{t,t}; K_3) \leq 10t - 5$. Improving this bound can be difficult, and obtaining a good lower bound even harder but interesting. Hence, we propose the following problem.

Problem 8. *For $t \geq 2$,*

(a) *obtain tight bounds for $F_v(K_{t,t}, K_{t,t}; 3)$, and*

(b) *obtain good bounds for $F_v(K_{t,t}, K_{t,t}; k)$, for $k \geq 4$.*

In another application of Theorem 3 for two colors, with $a_1 = a_2 = 2$, $b = r$ and $H_1 = H_2 = T_{tr,r}$, and for all $t, r \geq 2$, we obtain

$$F_v(T_{2tr,2r}, T_{2tr,2r}; 2r + 1) \leq 5F_v(T_{tr,r}, T_{tr,r}; r + 1).$$

In particular, note that for $r = 2$ we have $T_{tr,r} = K_{t,t}$, and thus the last inequality implies $F_v(T_{4t,4}, T_{4t,4}; K_5) \leq 50t - 25$.

The current authors recently studied the so-called chromatic Folkman numbers [12], which have one additional requirement for their witness Folkman graphs G, namely that their chromatic number $\chi(G)$ is the smallest possible ($\chi(G) = 1 + \sum_{i=1}^{r}(a_i - 1)$ for vertex colorings, and $\chi(G) = R(a_1, \ldots, a_r)$ for edge colorings). Some of the constructions in this section lead to upper bounds involving larger graphs but with the same chromatic number, mainly because $\chi(G[E_s]) = \chi(G)$. Thus, these techniques potentially could lead to stronger claims about upper bounds, where the chromatic number is as small as possible.

3. Two Concrete Upper Bounds

If the graphs we wish to arrow to are not of the form $K_{a_i}[H_i]$ as in Theorem 3, then the constructions for upper bounds may become little more complex. For instance, when we deal with complete but unbalanced multipartite graphs such as $K_{1,2,2}$ or $K_{2,2,3}$. The cases we study in this section also involve edge colorings, and the corresponding generalized edge Folkman numbers. Good upper bounds for the edge cases seem to be even harder to obtain than for vertex colorings. We will focus mainly on two small but puzzling cases of $F_e(J_4, J_4; 5)$ and $F_e(K_3, J_4; 5)$. By the monotonicity of arrowing we have

$$15 = F_e(3, 3; 5) \leq F_e(K_3, J_4; 5) \leq F_e(J_4, J_4; 5) \leq F_e(J_4, J_4; 4) \leq 30193.$$

The equality $F_e(3, 3; 5) = 15$ was obtained in [8, 10], where in the latter it was also shown, with the help of computer algorithms, that $F_v(3, 3; 4) = 14$. Furthermore, in the same work the authors obtained all 153 graphs on 14 vertices in the set $\mathcal{F}_v(3, 3; 4)$. These two parameter scenarios are connected, since it is known that $G + u \in \mathcal{F}_e(3, 3; 5)$ holds whenever $G \in \mathcal{F}_v(3, 3; 4)$ (see Lemma 9(a)). Two examples of such graphs, G_a and G_b, are presented in Figures A and B. They are used in the constructions of the following theorems, which exploit enhancements of the implication in Lemma 9(a). Consult [13] for the discussion of similar bounds and for additional pointers to the literature.

```
0 1 1 0 0  1 1 0 0 0  1 0 0  1
1 0 0 1 0  0 0 0 1 1  0 0 1  1
1 0 0 0 1  0 0 1 0 1  0 1 0  1
0 1 0 0 1  1 0 1 0 0  0 1 0  1
0 0 1 1 0  0 1 0 1 0  0 0 1  1

1 0 0 1 0  0 0 0 1 1  1 1 0  1
1 0 0 0 1  0 0 1 0 1  1 0 1  1
0 0 1 1 0  0 1 0 1 0  1 1 0  1
0 1 0 0 1  1 0 1 0 0  1 0 1  1
0 1 1 0 0  1 1 0 0 0  0 1 1  1

1 0 0 0 0  1 1 1 1 0  0 1 1  0
0 0 1 1 0  1 0 1 0 1  1 0 1  0
0 1 0 0 1  0 1 0 1 1  1 1 0  0
1 1 1 1 1  1 1 1 1 1  0 0 0  0
```

Figure A. Adjacency matrix of 14-vertex graph $G_a \in \mathcal{F}_v(3,3;4)$, with a vertex of maximum degree equal to 10. It is one of 60 such graphs, all of them enumerated in [10]. Graph G_a has a specially nice structure: vertices 1–5 and 6–10 induce C_5's, vertices 11–13 span K_3, 10 neighbors of vertex 14 induce a graph with 320 automorphisms (which is necessarily triangle-free), while the entire G_a has only 2 automorphisms. G_a has independence number 5, it has 41 triangles, and $|E(G_a)| = 48$.

```
0 0 0 0 0 0 0  1 1 1 1 0 0 0
0 0 0 0 0 0 0  1 1 1 0 1 0 0
0 0 0 0 0 0 0  1 1 0 1 0 1 0
0 0 0 0 0 0 0  1 0 1 0 1 0 1
0 0 0 0 0 0 0  0 1 0 1 0 1 1
0 0 0 0 0 0 0  0 0 1 0 1 1 1
0 0 0 0 0 0 0  0 0 0 1 1 1 1

1 1 1 1 0 0 0  0 1 1 0 0 1 1
1 1 1 0 1 0 0  1 0 0 1 1 0 1
1 1 0 1 0 1 0  1 0 0 1 1 1 0
1 0 1 0 1 0 1  0 1 1 0 1 1 0
0 1 0 1 0 1 1  0 1 1 1 0 0 1
0 0 1 0 1 1 1  1 0 1 1 0 0 1
0 0 0 1 1 1 1  1 1 0 0 1 1 0
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Figure B. Adjacency matrix of the Nenov graph G_b [8], which is the unique 14-vertex graph in the set $\mathcal{F}_v(3,3;4)$ with independence number 7. In graph G_b, vertices 1–7 form the only 7-independent set and vertices 8–14 induce $\overline{C_7}$. Graph G_b has 14 automorphisms, 35 triangles, and $|E(G_b)| = 42$, which is the smallest number of edges among all graphs in $\mathcal{F}_v(3,3;4)$.

We will also need some simple facts about arrowing. They are collected in the following lemma.

Lemma 9. *All of the following hold:*

(a) *if $G \to (3,3)^v$ and u is a new vertex, then $G + u \to (3,3)^e$,*

(b) *if $G \to (3,3)^v$, then $G[E_{2k-1}] \to (K_{k,k,k}, K_{k,k,k})^v$ for $k \geq 1$,*

(c) *$K_{1,2,2} \to (J_3, K_3)^e$, and*

(d) *$K_{2,2,3} \to (J_3, J_4)^e$.*

Proof. Part (a) is a basic property of arrowing used by many authors in scenarios similar to ours, for example, in [8]. For part (b) observe that $K_3[E_k] = K_{k,k,k}$ and use Lemma 2 for two colors with $r = 2$, $a_1 = a_2 = 3$ and $H_1 = H_2 = E_k$. For parts (c) and (d), first note that $K_{1,2,2} = K_1 + C_4$ and $K_{2,2,3} = E_3 + C_4$, and then consider possible edges of C_4 in the color avoiding J_3; there are at most two such edges. There are just a few more choices of edges in the J_3-free color since in total there are at most 2 or 3 such edges in $K_{1,2,2}$ and $K_{2,2,3}$, respectively. A routine check easily shows that the edges in the other color must contain a K_3 and J_4, respectively. ∎

Theorem 10. $F_e(J_4, J_4; 5) \leq 51$.

Proof. The skeleton of the proof is as follows. First, we will construct a K_4-free graph H on 50 vertices such that $H \to (K_{2,2,3}, K_{2,2,3})^v$. Then we will claim that the 51-vertex graph $G = K_1 + H$ is in the set $\mathcal{F}_e(J_4, J_4; 5)$, or equivalently, it is a witness of the upper bound in the theorem.

We start with the graph $G_a \in \mathcal{F}_v(3, 3; 4)$ described in Figure A. Let u be the only vertex of degree 10. Consider the partition of the set $V(G_a)$ into $X \cup Y$, where $X = \{v \mid \{u, v\} \in E(G_a)\}$, so that $|X| = 10$ and $|Y| = 4$. Note that every triangle in G_a must have at least one vertex in Y.

Next, define the graph $G_1 = G_a[E_5]$ which has 70 vertices and it is K_4-free. By Lemma 9(b), it holds that $G_1 \to (K_{3,3,3}, K_{3,3,3})^v$, which is too strong for our purpose, so we can drop some vertices from G_1. We define graph H as an induced subgraph of G_1 following the idea of the proof of Theorem 6, which now is to drop $2|X|$ vertices from G_1 formed by two triangle-free parts of G_a. More precisely, if the product graph G_1 uses $V(E_5) = \{v_1, \ldots, v_5\}$, then $V(H) = V(G_1) \setminus X \times \{v_4, v_5\}$ and $|V(H)| = 50$. By an argument very similar to that in the proof of Theorem 6 we can see that $H \to (K_{2,2,3}, K_{2,2,3})^v$. Being a subgraph of G_1, the graph H is K_4-free.

Finally, define graph G to be $K_1 + H$, and let u be the vertex in $V(G) \setminus V(H)$. Clearly, G is K_5-free and we have $V(G) = 51$. It remains to be shown that $G \to (J_4, J_4)^e$. Consider any red-blue coloring C_e of the edges $E(G)$. Define a red-blue vertex coloring C_v by $C_v(x) = C_e(\{u, x\})$ for $x \in V(H)$. Since $H \to (K_{2,2,3}, K_{2,2,3})^v$, then H contains a monochromatic $K_{2,2,3}$ in C_v, say, with the vertex set U. Without loss of generality assume that all vertices in U are red. Now, by Lemma 9(d), the set of edges induced by U must contain a red J_3 or

blue J_4 in C_e. If it is blue J_4, then we are done, otherwise red J_3 together with vertex u induces a red J_4.   ∎

Theorem 11. $F_e(K_3, J_4; K_5) \leq 27$.

Proof. The reasoning is very similar to that in the proof of Theorem 10, just the parameters vary. We start with the graph G_b described in Figure B, and let $G_1 = G_b[E_3]$. By Lemma 9(b), it holds that $G_1 \to (K_{2,2,2}, K_{2,2,2})^v$, so we can drop some vertices from G_1. Note that $J_4 = K_{1,1,2}$. We will define an induced subgraph H of G_1 on 26 vertices such that $H \to (K_{1,1,2}, K_{1,2,2})^v$. Then the graph $K_1 + H$ will be a witness of the upper bound.

Consider a partition of the set $V(G_b)$ into $X \cup Y \cup Z$, where X consists of the vertices of 7-independent set (the first 7 vertices), Y is the pair of vertices of any nonedge contained in the second block of 7 vertices, Z is formed by the remaining vertices, and thus they have orders 7, 2 and 5, respectively. Note that every triangle in G_b must have at least one vertex in Z, and at least two vertices in $Y \cup Z$. Drop $2|X|$ vertices from $V(G_1)$, 2 associated with each vertex in X, and similarly drop $|Y|$ vertices associated with Y. This defines an induced subgraph H on $|X| + 2|Y| + 3|Z| = 26$ vertices.

Now, for every red-blue vertex coloring C_1 of $V(G_1)$ define the coloring C_b of $V(G_b)$ by assigning to $u \in V(G_b)$ color of the majority of vertices in the set $\{\{u, v\} \mid \{u, v\} \in V(H)\}$ under C_1. Note that these sets have the cardinality 1, 2 and 3 for u in X, Y and Z, respectively. Thus, every monochromatic triangle in G_b can be expanded in H to at least 1, 1, and 3 vertices, or at least 1, 2, and 2 vertices, respectively. Hence, similarly as in the proof of Theorem 6, we can see that $H \to (K_{1,1,2}, K_{1,2,2})^v$. Furthermore, being a subgraph of G_1, the graph H is K_4-free.

Finally, define the graph G as $K_1 + H$, so that G is K_5-free and we have $V(G) = 27$. It remains to be shown that $G \to (K_3, J_4)^e$. Let u be the vertex in $V(G) \setminus V(H)$ and consider any red-blue coloring C_e of the edges $E(G)$. Define a red-blue vertex coloring C_v by $C_v(x) = C_e(\{u, x\})$ for $x \in V(H)$. Since $H \to (K_{1,1,2}, K_{1,2,2})^v$, then H contains a red $K_{1,1,2}$ or blue $K_{1,2,2}$ in C_v, say with the vertex set U. Recall that $J_4 = K_{1,1,2}$. First suppose that the vertices in U are forming a red J_4 in C_v. If at least one of the edges with both endpoints in U is red in C_e, then we have red K_3 spanned by this edge and u in C_e. Otherwise, all edges induced by U are blue in C_e, so we have blue J_4. On the other hand, suppose that the vertices in U form a blue $K_{1,2,2}$ in C_v. By Lemma 9(c), the set of edges induced by U must contain a red K_3 or blue J_3 in C_e. If it is red K_3, then we are done, otherwise blue J_3 together with vertex u induces a blue J_4.   ∎

The upper bounds of the last two theorems are likely larger than the actual values, despite that the graphs G_a and G_b where chosen to be the best for our

method, namely: none of the graphs on 14 vertices in $\mathcal{F}_v(3,3;4)$ has any induced triangle-free subgraph on more than 10 vertices and none has two independent sets whose union has more than 9 vertices, respectively. On the other hand, improving on either of the bounds $F_v(K_{2,2,3}, K_{2,2,3}; 4) \leq 50$ or $F_v(K_{1,1,2}, K_{1,2,2}; 4) \leq 26$, currently used in the proofs, would lead to better upper bounds in Theorems 10 and 11, respectively. Finally, let us note that Folkman numbers with other parameters could be studied by the current method, for example, such as when exploiting the inequality $F_e(3,4;7) \leq 1 + F_v(K_4, K_3 + C_5; 6)$, which relies on the well known case of edge arrowing $K_3 + C_5 \rightarrow (3,3)^e$.

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