Discussiones Mathematicae
Graph Theory 39 (2019) 951–974
doi:10.7151/dmgt.2119

SPECTRAL RADIUS AND HAMILTONICITY OF GRAPHS

Guidong Yu[1,2], Yi Fang[1],

Yizheng Fan[3] and Gaixiang Cai[1]

[1] *School of Mathematics & Computation Sciences*
Anqing Normal University
Anqing 246133, China

[2] *Basic Department, Hefei Preschool Education College*
Hefei 230013, P.R. China

[3] *School of Mathematical Sciences*
Anhui University
Hefei 230039, China

e-mail: guidongy@163.com
1041098329@qq.com
caigaixiang@sina.com
fanyz@ahu.edu.cn

Abstract

In this paper, we study the Hamiltonicity of graphs with large minimum degree. Firstly, we present some conditions for a simple graph to be Hamilton-connected and traceable from every vertex in terms of the spectral radius of the graph or its complement, respectively. Secondly, we give the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement, respectively.

Keywords: spectral radius, singless Laplacian spectral radius, traceable, Hamiltonian-connected, traceable from every vertex, minimum degree.

2010 Mathematics Subject Classification: 05C50, 05C45, 05C35.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Denote by $e(G) = |E(G)|$ the number of edges of the graph G. Let $N_G(v)$ be the set of vertices which are adjacent to v in G. The degree of v is denoted by $d_G(v) = |N_G(v)|$ (or simply $d(v)$), the minimum degree of G is denoted by $\delta(G)$. Let $X \subseteq V(G)$, $G - X$ is the graph obtained from G by deleting all vertices in X. G is called *k-connected* (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. We note that if G is k-connected, then $\delta(G) \geq k$. A *regular graph* is one graph whose all vertices have the same degrees, and a *bipartite semi-regular graph* is a bipartite graph for which the vertices in the same part have the same degrees. The *complement* of G is denoted by $\overline{G} = (V(\overline{G})E(\overline{G}))$, where $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{xy : \ x, y \in V(G), xy \notin E(G)\}$. Let $G = (X, Y; E)$ be a bipartite graph with two partite sets X, Y. If $|X| = |Y|$, $G = (X, Y; E)$ is called a *balanced bipartite graph.* If $|X| = |Y| - 1$, $G = (X, Y; E)$ is called a *nearly balanced bipartite graph.* The *quasi-complement* of $G = (X, Y; E)$ is denoted by $\widehat{G} := (X, Y; E')$, where $E' = \{xy : x \in X, y \in Y, xy \notin E\}$. For two disjoint graphs G_1 and G_2, the union of G_1 and G_2, denoted by $G_1 + G_2$, is defined as $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$, and the join of G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and $E(G_1 \vee G_2) = E(G_1 + G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote K_n the complete graph on n vertices, $O_n = \overline{K_n}$ the empty graph on n vertices (without edges), $K_{n,m} = O_n \vee O_m$ the complete bipartite graph with two parts having n, m vertices, $G - v$ ($v \in V(G)$) the graph obtained from G by deleting v, respectively.

The *adjacency matrix* of G is defined to be a matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The degree matrix of G is denoted by $D(G) = \operatorname{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$. The matrix $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* (or *Q-matrix*) of G. Obviously, $A(G)$ and $Q(G)$ are real symmetric matrix. So their eigenvalues are real number and can be ordered. The largest eigenvalue of $A(G)$, denoted by $\mu(G)$, and the corresponding eigenvectors (whose all components are positive number) are called the *spectral radius* and the *Perron vector* of G, respectively. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the *signless Laplacian spectral radius* of G.

A *Hamiltonian cycle* of the graph G is a cycle of order n contained in G, and a *Hamiltonian path* of G is a path of order n contained in G, where $|V(G)| = n$. The graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and is said to be *traceable* if it contains a Hamiltonian path. If every two vertices of G are connected by a Hamiltonian path, it is said to be *Hamilton-connected.* A graph G is *traceable from a vertex x* if it has a Hamiltonian x-path. The problem of deciding whether a graph is Hamiltonian is one of the most difficult classical problems in graph theory. Indeed, determining whether a graph is Hamiltonian is NP-complete.

Recently, the spectral theory of graphs has been applied to this problem. Up to now, there are some references on the spectral conditions for a graph to be traceable, Hamiltonian, Hamilton-connected or traceable from every vertex. We refer readers to see [4, 5, 7, 10, 12–17, 19, 21–25]. Particularly, Li and Ning [4, 5] and Nikiforov [19] study spectral sufficient conditions of graphs with large minimum degree. Li and Ning [4] present some (signless Laplacian) spectral radius conditions for a simple graph and a balanced bipartite graph to be traceable and Hamiltonian, respectively. Li and Ning [5] present some spectral radius conditions for a balanced bipartite graph and a nearly balanced bipartite graph to be traceable, respectively. Nikiforov [19] gives some spectral radius conditions for a simple graph to be traceable and Hamiltonian, respectively. Motivated by those papers, in this paper, we also study the graphs with large minimum degree. We will respectively present some conditions for a simple graph to be Hamilton-connected and traceable from every vertex in terms of the spectral radius of the graph or its complement in Section 2, and respectively give the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement in Section 3.

2. Spectral Radius Conditions for a Graph to Be Hamilton-Connected, and Traceable from Every Vertex

For an integer $k \geq 0$, the *k-closure* of a graph G, denoted by $C_k(G)$, is the graph obtained from G by successively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pair remains, see [2]. The k-closure of the graph G is unique, independent of the order in which edges are added. Note that $d_{C_k(G)}(u) + d_{C_k(G)}(v) \leq k-1$ for any pair of nonadjacent vertices u and v of $C_k(G)$.

Lemma 1 (Ore [20], Bondy and Chvátal [1])**.**

(i) *If G is a 2-connected graph of order n and $d_G(u) + d_G(v) \geq n+1$ for any two distant nonadjacent vertices u and v, then G is Hamilton-connected.*

(ii) *A 2-connected graph G is Hamilton-connected if and only if $C_{n+1}(G)$ is so.*

Lemma 2 (Yu, Ye, Cai and Cao [22])**.** *Let G be a simple graph, with degree sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_n)$ and $n \geq 3$. Suppose that there is no integer $2 \leq k \leq \frac{n}{2}$ such that $d_G(v_{k-1}) \leq k$, and $d_G(v_{n-k}) \leq n-k$, then G is Hamilton-connected.*

Lemma 3 (Hong, Shu and Fang [11], Nikiforov [18])**.** *If G is a graph of order n, with m edges and minimum degree δ, then*

$$\mu(G) \leq \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Lemma 4 (Hong, Shu and Fang [11], Nikiforov [18]). *If $2m \leq n(n-1)$, then the function*

$$f(x) = \frac{x-1}{2} + \sqrt{2m - nx + \frac{(x+1)^2}{4}}$$

is decreasing in x for $x \leq n-1$.

Lemma 5 (Bondy and Murty [2]). *Let G be a graph. Then G is traceable from every vertex if and only if $G \vee K_1$ is Hamilton-connected.*

Given a graph G of order n, a vector $\mathbf{x} \in \mathbb{R}^n$ is called to be defined on G, if there is a 1-1 map φ from $V(G)$ to the entries of $\mathbf{x}$; simply written $x_u = \varphi(u)$ for each $u \in V(G)$. If $\mathbf{x}$ is an eigenvector of $A(G)$, then $\mathbf{x}$ is defined on G naturally, x_u is the entry of $\mathbf{x}$ corresponding to the vertex u. One can find that

$$\mathbf{x}^T A(G)\mathbf{x} = 2 \sum_{uv \in E(G)} x_u x_v, \tag{1}$$

when μ is an eigenvalue of G corresponding to the eigenvector $\mathbf{x}$ if and only if $\mathbf{x} \neq \mathbf{0}$,

$$\mu x_v = \sum_{u \in N_G(v)} x_u, \tag{2}$$

for each vertex $v \in V(G)$. Equation (2) is called the eigenvalue-equation for the graph G. In addition, for an arbitrary unit vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mu(G) \geq \mathbf{x}^T A(G)\mathbf{x}, \tag{3}$$

with equality holds if and only if $\mathbf{x}$ is an eigenvector of $A(G)$ according to $\mu(G)$.

Lemma 6 (Li and Ning [4]). *Let G be a graph with non-empty edge set. Then*

$$\mu(G) \geq \min\left\{\sqrt{d(u)d(v)} : uv \in E(G)\right\}. \tag{4}$$

Moreover, if G is connected, then equality holds if and only if G is regular or bipartite semi-regular graph.

Lemma 7. *Let G be a graph of order n. Then*

$$\mu(G \vee K_1) > \frac{n-1}{n}\mu(G) + 2\frac{\sqrt{n-1}}{n}.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ be a unit Perron vector of G. Then by (1) and (3),

$$\mu(G) = \mathbf{x}^T A(G)\mathbf{x} = 2 \sum_{uv \in E(G)} x_u x_v.$$

Let $w \in V(K_1)$, $H = G \vee K_1$, and let $\mathbf{x}' \in \mathbb{R}^{n+1}, x_u' = \sqrt{\frac{n-1}{n}} x_u$, for every $u \in V(G)$, $x_w' = \frac{1}{\sqrt{n}}$. Since $\sum_{u \in V(G)} x_u^2 = 1, x_u > 0$, we have

$$\sum_{u \in V(H)} {x_u'}^2 = \sum_{u \in V(G)} {x_u'}^2 + {x_w'}^2 = \frac{n-1}{n} \sum_{u \in V(G)} x_u^2 + \frac{1}{n} = 1,$$

and $\sum_{u \in V(G)} x_u > \sum_{u \in V(G)} x_u^2 = 1$. Then by (1) and (3)

$$\begin{aligned} \mu(G \vee K_1) = \mu(H) &\geq \mathbf{x}'^T A(H) \mathbf{x}' = 2 \sum_{uv \in E(G)} x_u' x_v' + 2x_w' \sum_{u \in V(G)} x_u' \\ &= 2\frac{n-1}{n} \sum_{uv \in E(G)} x_u x_v + 2\frac{1}{\sqrt{n}} \sqrt{\frac{n-1}{n}} \sum_{u \in V(G)} x_u \\ &> \frac{n-1}{n} \mu(G) + 2\frac{\sqrt{n-1}}{n}. \end{aligned}$$

So the result follows. ∎

Lemma 8 (Tomescu [3]). *Every t-regular graph on $2t$ ($t \geq 3$) vertices not isomorphic to $K_{t,t}$, or of order $2t+1$ for even $t \geq 4$, is Hamilton-connected.*

Lemma 9. *Let $k \geq 2$, $n \geq 2k^2 + 1$, and G be a graph of order n. If G is a subgraph of $K_2 \vee (K_{n-k-1} + K_{k-1})$, with minimum degree $\delta(G) \geq k$, then $\mu(G) < n - k$, unless $G = K_2 \vee (K_{n-k-1} + K_{k-1})$.*

Proof. Set for short $\mu := \mu(G)$, and let $\mathbf{x} = (x_{v_1}, \ldots, x_{v_n})^T$ be a unit Perron vector of G. By (3), we have that

$$\mu = \mathbf{x}^T A(G) \mathbf{x}.$$

Assume that G is a proper subgraph of $K_2 \vee (K_{n-k-1} + K_{k-1})$. By Perron-Frobenius theorem, we can assume that G is obtained by omitting just one edge uv of $K_2 \vee (K_{n-k-1} + K_{k-1})$.

Write X for the set of vertices of $K_2 \vee (K_{n-k-1} + K_{k-1})$ of degree k, let Y be the set of their neighbors not in the set X, and let Z be the set of the remaining $n-k-1$ vertices of $K_2 \vee (K_{n-k-1} + K_{k-1})$.

Since $\delta(G) \geq k$, we can see that G must contain all the edges between X and Y. Therefore, $\{u, v\} \subset Y \cup Z$, with three possible cases: (a) $\{u, v\} \subset Y$; (b) $u \in Y, v \in Z$; (c) $\{u, v\} \subset Z$. We shall show that case (c) yields a graph of no smaller spectral radius than case (b), and that case (b) yields a graph of no smaller spectral radius than case (a).

Indeed, by (2), we have $x_i = x_j$ for any $i, j \in X$; likewise, $x_i = x_j$ for any $i, j \in Y \backslash \{u, v\}$ and for any $i, j \in Z \backslash \{u, v\}$. Thus, let

$$\begin{aligned} x &:= x_i, i \in X, \\ y &:= x_i, i \in Y \backslash \{u, v\}, \\ z &:= x_i, i \in Z \backslash \{u, v\}. \end{aligned}$$

Suppose that case (a) holds, that is, $\{u, v\} \subset Y$. Choose a vertex $w \in Z$, remove the edge vw and add the edge uv. Then the obtained graph G' is covered by case (b).

If $x_w \leq x_u$, we have

$$\mathbf{x}^T A(G')\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} = 2x_v(x_u - x_w) \geq 0.$$

If $x_w > x_u$, swap the entries x_u and x_w, write $\mathbf{x}'$ for the resulting vector. We note that $\mathbf{x}'$ is also a unit vector, and have that

$$\mathbf{x}'^T A(G')\mathbf{x}' - \mathbf{x}^T A(G)\mathbf{x} = 2(x_w - x_u) \sum_{i \in X} x_i \geq 0.$$

Then by (3), $\mu(G') \geq \mu(G)$, as claimed.

Essentially the same argument proves that case (c) yields a graph of no smaller spectral radius than case (b). Therefore, we may assume that $\{u, v\} \subset Z$. Since the vertices u and v are symmetric, so $x_u = x_v$. Set $t := x_u$ and note that the n eigenvalue-equations of G are reduced to four equations involving just the unknowns x, y, z, and t.

$$\begin{aligned} \mu x &= (k-2)x + 2y, \\ \mu y &= (k-1)x + y + (n-k-3)z + 2t, \\ \mu z &= 2y + (n-k-4)z + 2t, \\ \mu t &= 2y + (n-k-3)z. \end{aligned}$$

We find that

$$\begin{aligned} x &= \frac{2y}{\mu - k + 2}, \\ z &= \left(1 - \frac{2(k-1)}{(\mu+1)(\mu-k+2)}\right) y, \\ t &= \frac{\mu+1}{\mu+2} \left(1 - \frac{2(k-1)}{(\mu+1)(\mu-k+2)}\right) y. \end{aligned}$$

Furtherly, note that if we delete all edges incident to vertices in X, and add the edge uv to G, we obtain the graph $K_{n-k+1} + \overline{K_{k-1}}$. Letting $\mathbf{x}''$ be the restriction of $\mathbf{x}$ to K_{n-k+1}, we find that

$$\begin{aligned}\mathbf{x}''^T A(K_{n-k+1})\mathbf{x}'' &= \mathbf{x}^T A(G)\mathbf{x} + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \\ &= \mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2.\end{aligned}$$

But since $\|\mathbf{x}''\|^2 = 1 - (k-1)x^2$, we see that

$$\begin{aligned}&\mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \\ &= \mathbf{x}''^T A(K_{n-k+1})\mathbf{x}'' \le \mu(K_{n-k+1})\|\mathbf{x}''\|^2 \\ &= (n-k)\big(1-(k-1)x^2\big).\end{aligned}$$

Assume for a contradiction that $\mu \ge n-k$. This assumption, together with above inequality, yields

$$\mu + 2t^2 - 4(k-1)xy - (k-1)(k-2)x^2 \le \mu\big(1-(k-1)x^2\big),$$

and therefore

$$2(k-1)xy - \frac{(\mu-k+2)(k-1)x^2}{2} \ge t^2.$$

Now, first combining above equality about x, then combining equality about t, we have

$$\frac{2(k-1)y^2}{\mu-k+2} \ge \left(\frac{\mu+1}{\mu+2}\right)^2 \left(1 - \frac{2(k-1)}{(\mu+1)(\mu-k+2)}\right)^2 y^2.$$

Cancelling y^2 and applying Bernoulli's inequality to the right side, we get

$$\begin{aligned}2(k-1) &\ge (\mu-k+2)\left(1-\frac{1}{\mu+2}\right)^2\left(1-\frac{2(k-1)}{(\mu+1)(\mu-k+2)}\right)^2 \\ &> (\mu-k+2)\left(1-\frac{2}{\mu+2}-\frac{4(k-1)}{(\mu+1)(\mu-k+2)}\right) \\ &= \mu-k+2-\frac{2\mu-2k+4}{\mu+2}-\frac{4(k-1)}{\mu+1} > \mu-k+2-\frac{2\mu+2k}{\mu+1}.\end{aligned}$$

Using the inequalities $\mu \ge n-k \ge 2k^2-k+1$, we easily find that

$$2 < \frac{2\mu+2k}{\mu+1} < 3,$$

and so,

$$2(k-1) > 2k^2-k+1-k+2-3 = 2k^2-2k,$$

a contradiction, completing the proof. ∎

Theorem 10. *Let $k \geq 2$, $n \geq 2k^2+1$ and let G be a graph of order n with minimum degree $\delta(G) \geq k$. If*

$$\mu(G) \geq n-k,$$

then G is Hamilton-connected, unless $G = K_2 \vee \big(K_{n-k-1} + K_{k-1}\big)$.

Proof. Assume that $\mu(G) \geq n-k$, but G is not Hamilton-connected. Let $H = C_{n+1}(G)$. Then H is not Hamilton-connected by Lemma 1, $\delta(H) \geq \delta(G) \geq k$, and $\mu(H) \geq \mu(G) \geq n-k$ by Perron-Frobenius theorem. Note that H is $(n+1)$-closure of G, thus every two nonadjacent vertices u, v have degree sum at most n, i.e.,

$$d_H(u) + d_H(v) \leq n. \tag{5}$$

Since H is not Hamilton-connected, by Lemma 2, there is an integer $2 \leq s \leq \frac{n}{2}$ such that $d_H(v_{s-1}) \leq s$ and $d_H(v_{n-s}) \leq n-s$, obviously, $s \geq \delta(H) \geq k$. Write m for the number of edges of H, set $\delta(H) := \delta$, then we can get

$$2m = \sum_{i=1}^{s-1} d_H(v_i) + \sum_{i=s}^{n-s} d_H(v_i) + \sum_{i=n-s+1}^{n} d_H(v_i) \tag{6}$$

$$\leq s(s-1) + (n-2s+1)(n-s) + s(n-1) \tag{7}$$

$$= 3s^2 + n^2 - 2ns + n - 3s. \tag{8}$$

On the other hand, combining Lemmas 3 and 4, we have

$$n-k \leq \mu(H) \leq \frac{k-1}{2} + \sqrt{2m - nk + \frac{(k+1)^2}{4}},$$

which, after some algebra operations, gives

$$2m \geq n^2 - 2kn + 2k^2 + n - 2k. \tag{9}$$

Next, we will prove that $s = k$. Suppose $k+1 \leq s \leq \frac{n}{2}$. Let $f(x) = 3x^2 + n^2 - 2nx + n - 3x$. We note $f(x)$ is convex in x, then $f(s) \leq f(k+1)$ or $f(s) \leq f\left(\frac{n}{2}\right)$.

Combining (8) and (9), we get

$$\begin{aligned} & n^2 - 2kn + 2k^2 + n - 2k \leq 2m \leq f(s) \leq f(k+1) \\ & = 3(k+1)^2 + n^2 - 2n(k+1) + n - 3(k+1) \end{aligned}$$

or

$$n^2 - 2kn + 2k^2 + n - 2k \leq 2m \leq f(s) \leq f\left(\frac{n}{2}\right) = \frac{3}{4}n^2 - \frac{n}{2}.$$

Then $n \leq \frac{k^2+5k}{2}$ or $n^2+(6-8k)n+8k(k-1) \leq 0$, each of these inequalities leads to a contradiction. So we have $s = k$, and thus $\delta(H) = k$, then,

$$d_H(v_1) = d_H(v_2) = \cdots = d_H(v_{k-1}) = k.$$

Our next goal is to show that $d_H(v_k) \geq n - k^2$. Indeed, suppose that

$$d_H(v_k) < n - k^2.$$

Also using Lemma 2, we get

$$\begin{aligned} 2m &= \sum_{i=1}^{k-1} d_H(v_i) + d_H(v_k) + \sum_{i=k+1}^{n-k} d_H(v_i) + \sum_{i=n-k+1}^{n} d_H(v_i) \\ &< (k-1)k + n - k^2 + (n-2k)(n-k) + k(n-1) \\ &= n^2 - 2kn + 2k^2 + n - 2k, \end{aligned}$$

contradicting (9). Hence $d_H(v_i) \geq n - k^2$ for every $i \in \{k, k+1, \ldots, n\}$.

Next, we shall show that the vertices $v_k, v_{k+1}, \ldots, v_n$ induce a complete graph in H. Indeed, let $v_i, v_j \in \{v_k, v_{k+1}, \ldots, v_n\}$ be two distinct vertices of H. If they are nonadjacent, then

$$d_H(v_i) + d_H(v_j) \geq\ 2n - 2k^2 \geq n + 2k^2 + 1 - 2k^2 = n + 1,$$

contradicting (5).

Write X for the vertex set $\{v_1, v_2, \ldots, v_{k-1}\}$. Write Y for the set of vertices in $\{v_k, v_{k+1}, \ldots, v_n\}$ having neighbors in X. Let Z be the set of remaining vertices of $V(G)$.

Since $|X| = k-1$, and $d_H(v_1) = d_H(v_2) = \cdots = d_H(v_{k-1}) = k$, we get $Y \neq \emptyset$, and any vertex in X must have at least two neighbors in $\{v_k, v_{k+1}, \ldots, v_n\}$.

In fact, every vertex from Y is adjacent to every vertex in X. Indeed, suppose that this is not the case, and let $w \in \{v_k, v_{k+1}, \ldots, v_n\}, u \in X, v \in X$, such that w is adjacent to u, but not to v. We see that

$$d_H(w) + d_H(v) \geq n - k + 1 + k = n + 1,$$

contradicting (5).

Next, let $l = |Y|$ and note that $2 \leq l \leq k$.

If $l = 2$, then $H = K_2 \vee (K_{n-k-1} + K_{k-1})$. Since $G \subseteq H$, by Lemma 9, if G is a proper subgraph of H, $\mu(G) < n - k$, then $G = K_2 \vee (K_{n-k-1} + K_{k-1})$, a contradiction.

If $3 \leq l \leq k-1$, we can get H is Hamilton-connected, which contradicts the assumptions of H.

Indeed, let I be the graph induced by $X \cup Y \backslash \{u\}$, where $u \in Y$. Since $K_{l-1} \vee \overline{K_{k-1}} \subset I$, and $l \geq 3$, we see that I is 2-connected. Furtherly, if x and y are distinct nonadjacent vertices of I,

$$d_I(x) + d_I(y) \geq 2k - 2 \geq k + l - 1,$$

then I is Hamilton-connected by Lemma 1.

Then for any two distinct vertices x, y of H, we can get a Hamilton path of H with x, y as endpoint. So, H is Hamilton-connected. For example, for any $x, y \in X$. Let $xP_1u_1vP_2y$ be a Hamilton path of I, where $v \in Y$. Let M be a subgraph of H, which is induced by $V(H) \backslash V(I)$. We note that M is a complete graph, then M is Hamiltonian. So, there is a Hamilton cycle $C : uP_3v_1u$ of M. Now we delete the edges u_1v, uv_1, and add the edges u_1u, vv_1, then we get a path $xP_1u_1uP_3v_1vP_2y$ be a Hamilton path of H. Similar methods prove the other cases.

If $l = k$, we also can find that H is Hamilton-connected, which contradicts the assumptions of H. For example, for any $x \in X$, $y \in Z$. Because every vertex in Y is adjacent to every vertex in X, there is a path xP_4v, which contains all vertices of $X \cup Y$, where $v \in Y$. Let N be a subgraph of H, which is induced by $Z \cup \{v\}$. We note that N is a complete graph, then N is Hamilton-connected. So, there is a Hamilton path vP_5w of N. Now, we get a path xP_4vP_5w being a Hamilton path of H. Similar methods prove the other cases.

So, the result follows. ■

Theorem 11. *Let $k \geq 1, n \geq 2(k+1)^2$, and let G be a graph of order n with minimum degree $\delta(G) \geq k$. If*

$$\mu(G) \geq \frac{n^2}{n-1} - \frac{nk}{n-1} - \frac{2}{\sqrt{n-2}},$$

then G is traceable from every vertex, unless $G = K_1 \vee (K_{n-k-1} + K_k)$.

Proof. Let $H = G \vee K_1$. Then H be a graph of order $n+1$, with minimum degree $\delta(H) \geq k+1$. By Lemma 7 and the assumption, we have

$$\begin{aligned} \mu(H) &> \frac{n-1}{n}\mu(G) + 2\frac{\sqrt{n-1}}{n} \\ &\geq \frac{n-1}{n}\left(\frac{n^2}{n-1} - \frac{nk}{n-1} - \frac{2}{\sqrt{n-1}}\right) + 2\frac{\sqrt{n-1}}{n} \\ &= (n+1) - (k+1). \end{aligned}$$

Then by Theorem 10, we get H is Hamilton-connected, unless $H = K_2 \vee (K_{n-k-1} + K_k)$.

So, according to Lemma 5, G is traceable from every vertex, unless $G = K_1 \vee (K_{n-k-1} + K_k)$. ■

Let ES_n be the set of following graphs of even order n:

(i) $K_{\frac{n}{2},\frac{n}{2}}$;

(ii) $G_1 \vee G_2$, where G_1 is a regular graph of order $n-r$ with degree $\frac{n}{2}-r$, G_2 has r vertices, $1 \le r \le \frac{n}{2}$.

Let EW_n be the set of following graphs of odd order n:
$G_1 \vee G_2$, where G_1 is a regular graph of order $n+1-r$ with degree $\frac{n+1}{2}-r$, G_2 has $r-1$ vertices, $1 \le r \le \frac{n+1}{2}$.

Theorem 12. *Let G be a graph of order $n \ge 2k$, where $k \ge 2$. If $\delta(G) \ge k$ and*

$$\mu(\overline{G}) \le \sqrt{(k-1)(n-k-1)},$$

then G is Hamilton-connected, unless $G = K_{k-1,n-k-1} \vee K_2$ or $G = K_{k-1,n-k-1} \vee O_2$ or $G \in ES_n$ and $n = 2k$.

Proof. Let $H = C_{n+1}(G)$. If H is Hamilton-connected, then so is G by Lemma 1. Now we assume that H is not Hamilton-connected. Note that H is $(n+1)$-closure of G, thus every two nonadjacent vertices u, v of H have degree sum at most n, i.e.,

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) \ge n-2, \text{ for any edge } uv \in E(\overline{H}). \tag{10}$$

Since $d_G(u) \ge k$ and $d_G(v) \ge k$, we have $d_{\overline{H}}(u) \le n-k-1$ and $d_{\overline{H}}(v) \le n-k-1$. Then combining (10), $k-1 \le d_{\overline{H}}(u) \le n-k-1$, $k-1 \le d_{\overline{H}}(v) \le n-k-1$, this implies that

$$d_{\overline{H}}(u)d_{\overline{H}}(v) \ge d_{\overline{H}}(u)(n-2-d_{\overline{H}}(u)) \ge (k-1)(n-k-1),$$

with equality if and only if (up to symmetry), $d_{\overline{H}}(u) = k-1$ and $d_{\overline{H}}(v) = n-k-1$. By Lemma 6, Perron-Frobenius theorem, and the assumption,

$$\begin{aligned} \sqrt{(k-1)(n-k-1)} \ &\ge\ \mu(\overline{G}) \ge \mu(\overline{H}) \ge \min_{uv \in E(\overline{H})} \sqrt{d_{\overline{H}}(u)d_{\overline{H}}(v)} \\ &\ge\ \sqrt{(k-1)(n-k-1)}. \end{aligned}$$

Therefore, $\mu(\overline{G}) = \mu(\overline{H}) = \sqrt{(k-1)(n-k-1)}$, and $d_{\overline{H}}(u) + d_{\overline{H}}(v) = n-2$ for any edge $uv \in E(\overline{H})$, and $d_{\overline{H}}(u) = k-1$, $d_{\overline{H}}(v) = n-k-1$. Note that every nontrivial component of $\overline{H}$ has a vertex of degree at least $\frac{n}{2}-1$ and hence of order at least $\frac{n}{2}$. This implies that $\overline{H} = K_{\frac{n}{2}} + K_{\frac{n}{2}}$ for $n = 2k$, or $\overline{H}$ contains exactly one nontrivial component F which is either regular or semi-regular, and $\frac{n}{2} \le |V(F)| \le n$.

Noting that $\mu(\overline{G}) = \mu(\overline{H})$, $\overline{G} \supseteq \overline{H}$, if $\overline{H} = K_{\frac{n}{2}} + K_{\frac{n}{2}}$ and $n = 2k$, then $\overline{G} = \overline{H}$ by the Perron-Frobenius theorem. So $G = K_{\frac{n}{2},\frac{n}{2}} \in ES_n$ and $n = 2k$, a contradiction. Therefore we assume that $\overline{H}$ contains exactly one nontrivial component F.

First suppose F is a bipartite semi-regular graph. By the condition of the degree sum of two adjacent vertices, we have F contains at least $n-2$ vertices. If F contains $n-2$ vertices, then $\overline{H} = K_{k-1,n-k-1} + O_2$. Noting that $\mu(\overline{G}) = \mu(\overline{H})$, $\overline{G} \supseteq \overline{H}$, then $\overline{G} = \overline{H}$ or $K_{k-1,n-k-1} + K_2$ by the Perron-Frobenius theorem. So $G = (K_{k-1} + K_{n-k-1}) \vee K_2$ or $(K_{k-1} + K_{n-k-1}) \vee O_2$, a contradiction. If F contains $n-1$ vertices, then the graph F with two partite sets X, Y has $|X| = k-1$, $|Y| = n-k$ or $|X| = k$, $|Y| = n-k-1$. Thus according to the edge number of F, we have $(n-k)(k-1) = (k-1)(n-k-1)$ or $(n-k-1)(k-1) = k(n-k-1)$, a contradiction. If F contains n vertices, then the graph F with two partite sets X, Y has $|X| = k$, $|Y| = n-k$ or $|X| = k+1$, $|Y| = n-k-1$ or $|X| = k-1$, $|Y| = n-k+1$. If $|X| = k$, $|Y| = n-k$, then according to the edge number of F, we have $(n-k-1)k = (k-1)(n-k)$, $n = 2k$, and then $H = \overline{F}$ is Hamilton-connected, a contradiction. If $|X| = k+1$, $|Y| = n-k-1$ or $|X| = k-1$, $|Y| = n-k+1$, then according to the edge number of F, we have $(n-k-1)(k+1) = (k-1)(n-k-1)$ or $(n-k-1)(k-1) = (k-1)(n-k+1)$, a contradiction.

Next we assume F is a regular graph. Then for every $v \in V(F)$, $d_F(v) = \frac{n}{2} - 1$, and $n = 2k$. If $F = \overline{H}$, by a similar discussion as the above, $\overline{G} = \overline{H}$, and hence $G = H$ is regular of degree $\frac{n}{2}$. By Lemma 8, $G = K_{\frac{n}{2},\frac{n}{2}} \in ES_n$, or G is Hamilton-connected, a contradiction. Otherwise, $\overline{H} = F \cup O_r$, where $r = n - |V(F)|$ and $1 \leq r \leq \frac{n}{2}$. Noting that $\mu(\overline{G}) = \mu(\overline{H})$, we have $\overline{G} = F \cup F_1$, where F_1 is obtained from O_r possibly adding some edges. Hence $G = \overline{F} \vee \overline{F_1} \in ES_n$, a contradiction. ■

Theorem 13. *Let G be a graph of order $n \geq 2k+1$, where $k \geq 2$. If $\delta(G) \geq k$ and*

$$\mu(\overline{G}) \leq \sqrt{k(n-k-1)},$$

then G is traceable from every vertex, unless $G = K_{k,n-k-1} \vee K_1$ or $G \in EW_n$ and $n = 2k+1$.

Proof. Let $G' = G \vee K_1$. We note that $|V(G')| = n+1$, $\mu(\overline{G'}) = \mu(\overline{G}) \leq \sqrt{k(n-k-1)}$, $\delta(G') \geq k+1$. By Theorem 12, we get G' is Hamilton-connected, unless $G' = K_{k,n-k-1} \vee K_2$ or $G' = K_{k,n-k-1} \vee O_2$ or $G' \in ES_{n+1}$ and $n = 2k+1$. By Lemma 5 and the construction of G', we have G is traceable from every vertex, unless $G = K_{k,n-k-1} \vee K_1$ or $G \in EW_n$ and $n = 2k+1$. ■

3. (Sigless Laplacian) Spectral Radius Conditions for a Nearly Balanced Bipartite Graph to be Traceable

We note that if a bipartite graph $G = (X, Y; E)$ is traceable, G is a balanced bipartite graph or a nearly balanced bipartite graph. Li and Ning [4] have presented some (signless Laplacian) spectral radius conditions for a balanced bipartite graph to be Hamiltonian. If $G = (X, Y; E)$ is a nearly balanced bipartite graph with $|X| = |Y| - 1$, we can obtained G' from G by adding a vertex which is adjacent to every vertex in Y, then G' is a balanced partite graph. Note that G is traceable if and only if G' is Hamiltonian. Inspired by this, in this section, we will study the conditions for a nearly balanced bipartite graph to be traceable in terms of spectral radius, signless Laplacian spectral radius of the graph or its quasi-complement.

Let G be balanced bipartite graph of order $2n$. The *bipartite closure* of G, denoted by $cl_B(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices in different partite sets whose degree sum is at least $n+1$ until no such pair remains. Note that $d_{cl_B(G)}(u) + d_{cl_B(G)}(v) \le n$ for any pair of nonadjacent vertices u and v in the distant partite sets of $cl_B(G)$.

Lemma 14 (Bondy and Chvátal [1]). *A balanced bipartite graph G is Hamiltonian if and only if $cl_B(G)$ is Hamiltonian.*

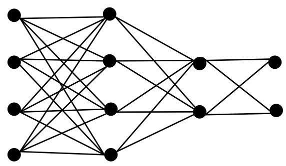
Before introducing our results, we need some notations. In order to facilitate understanding, in this paper, when we mention a bipartite graph, we always fix its partite sets, e.g., $O_{n,m}$ and $O_{m,n}$ are considered as different bipartite graphs, unless $m = n$.

Let G_1, G_2 be two bipartite graphs, with the bipartition $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 + G_2$ by adding all possible edges between X_1 and Y_2 and all possible edges between Y_1 and X_2. We define some classes of graphs as follows.

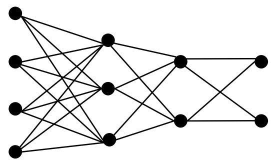
$$\begin{aligned} B_n^k &= O_{k,n-k} \sqcup K_{n-k,k} \ (1 \le k \le n/2), \\ C_n^k &= O_{k,n-k} \sqcup K_{n-k-1,k} \ (1 \le k \le n/2). \end{aligned}$$

Note that $e(B_n^k) = n(n-k) + k^2$, $e(C_n^k) = n(n-k-1) + k^2$, $\mu(\widehat{B_n^k}) = \mu(\widehat{C_n^k}) = \mu(K_{k,n-k}) = \sqrt{k(n-k)}$, and B_n^k is not Hamiltonian, C_n^k is not traceable. By Perron-Frobenius theorem, $\mu(B_n^k) > \mu(K_{n,n-k}) = \sqrt{n(n-k)}$, $\mu(C_n^k) > \mu(K_{n,n-k-1}) = \sqrt{n(n-k-1)}$.

Let $G = (X, Y)$ be a bipartite graph with two partite sets X, Y. Denote by $\mathcal{B}_n^k (1 \le k \le n/2)$ the family of graphs $\{O_{k,n-k} \sqcup G(X, Y)$, where $|X| = n-k$, $|Y| = k\}$. Denote by $\mathcal{C}_n^k (1 \le k \le n/2)$ the family of graphs $\{O_{k,n-k} \sqcup G(X, Y)$, where $|X| = n-k-1$, $|Y| = k\}$.



B_6^2



C_6^2

Figure 1. Graphs B_6^2 and C_6^2.

Lemma 15 (Li and Ning [4]). *Let G be a balanced bipartite graph of order $2n$. If $\delta(G) \geq k \geq 1$, $n \geq 2k+1$ and*

$$e(G) > n(n-k-1) + (k+1)^2,$$

then G is Hamiltonian unless $G \subseteq B_n^k$.

Lemma 16. *Let $G = (X, Y)$ be a nearly balanced bipartite graph of order $2n-1$. If $\delta(G) \geq k \geq 1$, $n \geq 2k+1$, and*

$$e(G) > n(n-k-2) + (k+1)^2,$$

then G is traceable unless $G \subseteq C_n^k$.

Proof. Let $|X| = n-1$, $|Y| = n$, and let G' be obtained from G by adding a vertex which is adjacent to every vertex in Y. Then G' be a balanced bipartite graph. Note that G is traceable if and only if G' is Hamiltonian. We have $|V(G')| = 2n$, $\delta(G') \geq \delta(G) \geq k \geq 1$, $n \geq 2k+1$, and

$$e(G') = e(G) + n > n(n-k-2) + (k+1)^2 + n = n(n-k-1) + (k+1)^2.$$

By Lemma 15, G' is Hamiltonian unless $G' \subseteq B_n^k$. Thus G is traceable unless $G \subseteq C_n^k$. ■

Lemma 17 (Bhattacharya, Friedland and Peled [6]). *Let G be a bipartite graph. Then*

$$\mu(G) \leq \sqrt{e(G)}.$$

Lemma 18 (Ferrara, Jacobson and Powell [9]). *Let G be a non-Hamiltonian balanced bipartite graph of order $2n$. If $d(u)+d(v) \geq n$ for every two nonadjacent vertices u, v in distinct partite sets, then either $G \in \bigcup_{k=1}^{n/2} \mathcal{B}_n^k$, or $G = \Gamma_1$ or Γ_2 for $n = 4$.*

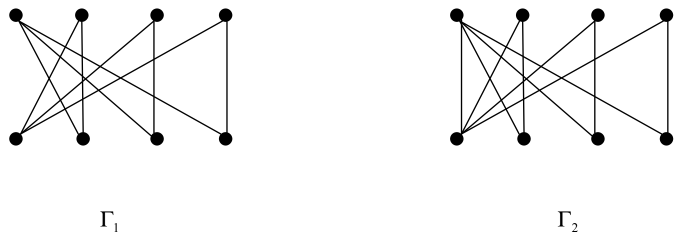


Figure 2. Graphs Γ_1 and Γ_2.

Lemma 19 (Feng and Yu [8], Yu and Fan [21])**.** *Let G be a graph with non-empty edge set. Then*

$$q(G) \leq \max\left\{ d(u) + \frac{\sum_{v\in N(u)} d(v)}{d(u)} : u \in V(G) \right\}.$$

Lemma 20. *Let G be a bipartite graph with two partite sets X, Y, and* $\max\{|X|, |Y|\} = n$. *Then*

$$q(G) \leq \frac{e(G)}{n} + n.$$

Proof. If G is an edgeless graph, then $q(G) = 0$, and the result is trivially true. Now assume G contains at lest one edge. Let $x \in V(G)$, and

$$d(x) + \frac{\sum_{v\in N(x)} d(v)}{d(x)} = \max\left\{ d(u) + \frac{\sum_{v\in N(u)} d(v)}{d(u)} : u \in V(G) \right\}.$$

By Lemma 19, for every $v \in V(G)$, $d_G(v) \leq \max\{|X|, |Y|\} = n$, we get

$$\begin{aligned} \frac{e(G)}{n} + n - q(G) &\geq \left(\frac{\sum_{v\in N(x)} d(v)}{n} + n \right) - \left(d(x) + \frac{\sum_{v\in N(x)} d(v)}{d(x)} \right) \\ &= (n - d(x)) \left(1 - \frac{\sum_{v\in N(x)} d(v)}{nd(x)} \right) \geq 0. \end{aligned}$$

The result follows. ■

Lemma 21. *Let $k \geq 1$, $n \geq \frac{k^3}{2} + k + 2$. If G is a subgraph of C_n^k, $\delta(G) \geq k$, then $\mu(G) < \sqrt{n(n-k-1)}$, unless $G = C_n^k$.*

Proof. The proof is similar to the proof of Lemma 9. Set for short $\mu := \mu(G)$, and let $\mathbf{x} = (x_{v_1}, \ldots, x_{v_{2n-1}})^T$ be a unit Perron vector of G. By (3), we have

$$\mu = \mathbf{x}^T A(G)\mathbf{x}.$$

Assume that G is a proper subgraph of C_n^k. By Perron-Frobenius theorem, we may assume that G is obtained by omitting just one edge uv of C_n^k.

Write X for the set of vertices of C_n^k of degree k, let Y be the set of vertices of C_n^k of degree n, write Z for the set of vertices of C_n^k of degree $n-k-1$, let H be the set of the remaining k vertices of C_n^k of degree $n-1$.

Since $\delta(G) \geq k$, we can see that G must contain all the edges between X and H. Therefore $\{u,v\} \subset Y \cup H$ or $\{u,v\} \subset Y \cup Z$, with two possible cases: (a) $u \in Y, v \in H$; (b) $u \in Y, v \in Z$. We shall show that case (b) yields a graph of no smaller spectral radius than case (a).

Indeed, by (2), we have $x_i = x_j$ for any $i,j \in X$; likewise $x_i = x_j$ for any $i,j \in Y \backslash \{u\}$, for any $i,j \in Z \backslash \{v\}$ and for any $i,j \in H \backslash \{v\}$. Thus, let

$$\begin{aligned} x &:= x_i, i \in X, \\ y &:= y_i, i \in Y \backslash \{u\}, \\ z &:= z_i, i \in Z \backslash \{v\}, \\ h &:= h_i, i \in H \backslash \{v\}. \end{aligned}$$

Suppose that case (a) holds, that is, $u \in Y, v \in H$. Choose a vertex $w \in Z$, remove the edge uw, and add the edge uv. Then the obtained graph G' is covered by case (b).

If $x_w \leq x_v$, we have

$$\mathbf{x}^T A(G')\mathbf{x} - \mathbf{x}^T A(G)\mathbf{x} = 2x_u(x_v - x_w) \geq 0.$$

If $x_w > x_v$, swap the entries x_v and x_w, write $\mathbf{x}'$ for the resulting vector. We note that $\mathbf{x}'$ is also a unit vector, and have that

$$\mathbf{x}'^T A(G')\mathbf{x}' - \mathbf{x}^T A(G)\mathbf{x} = 2(x_w - x_v)\sum_{i \in X} x_i \geq 0.$$

Then by (3), $\mu(G') \geq \mu(G)$, as claimed.

Therefore, we may assume that $u \in Y, v \in Z$, and set $t := x_u, s := x_v$. Note that the $2n-1$ eigenvalue-equations of G are reduced to six equations involving just the unknowns x, y, z, h, t, and s.

$$\begin{aligned} \mu x &= kh, \\ \mu y &= (n-k-1)z + kh + s, \\ \mu z &= (n-k-2)y + t, \end{aligned}$$

$$\begin{aligned}
\mu h &= kx + (n-k-2)y + t,\\
\mu t &= (n-k-1)z + kh,\\
\mu s &= (n-k-2)y.
\end{aligned}$$

We find that

$$\begin{aligned}
x &= \frac{k}{\mu}h,\\
t &= \frac{(n-k-1)\left(\mu^2-k^2\right)+k\mu^2}{\mu^3}h,\\
s &= \frac{\left(\mu^2-(n-k-1)\right)\left(\mu^2-k^2\right)-k\mu^2}{\mu^4}h.
\end{aligned}$$

Furtherly, note that if we remove all edges between X and H, and add the edge uv to G, we obtain the graph $K_{n,n-k-1}+\overline{K_k}$. Letting $\mathbf{x}''$ be the restriction of $\mathbf{x}$ to $K_{n,n-k-1}$, we find that

$$\mathbf{x}''^T A(K_{n,n-k-1})\mathbf{x}'' = \mathbf{x}^T A(G)\mathbf{x} + 2st - 2k^2xh = \mu + 2st - 2k^2xh.$$

But since $\|\mathbf{x}''\|^2 = 1-kx^2$, we see that

$$\begin{aligned}
\mu + 2st - 2k^2xh &= \mathbf{x}''^T A(K_{n,n-k-1})\mathbf{x}'' \leq \mu(K_{n,n-k-1})\|\mathbf{x}''\|^2\\
&= \sqrt{n(n-k-1)}(1-kx^2).
\end{aligned}$$

Assume for a contradiction that $\mu \geq \sqrt{n(n-k-1)}$. This assumption together with the above inequality yields

$$\mu + 2st - 2k^2xh \leq \mu\left(1-kx^2\right),$$

and therefore

$$2st - 2k^2xh \leq -kx^2\mu.$$

Now, first combining above equality about x, then combining above equalities about t and s, we have

$$\begin{aligned}
k^3 &\geq \frac{2\mu}{h^2}st \geq \frac{2(n-k-1)\left(\mu^2-(n-k-1)\right)\left(\mu^2-k^2\right)^2}{\mu^6}\\
&- \frac{2k(n-k-1)\left(\mu^2-k^2\right)}{\mu^4}.
\end{aligned}$$

Applying Bernoulli's inequality to the right side, we get

$$\begin{aligned} k^3 &\geq 2(n-k-1)\left(\frac{\mu^2-(n-k-1)}{\mu^2}\right)\left(\frac{\mu+k}{\mu}\right)^2\left(\frac{\mu-k}{\mu}\right)^2 \\ &- \frac{2k(n-k-1)}{\mu^2}\left(1-\frac{k^2}{\mu^2}\right) \\ &= 2(n-k-1)\left(1-\frac{n-k-1}{\mu^2}\right)\left(1+\frac{k}{\mu}\right)^2\left(1-\frac{k}{\mu}\right)^2 \\ &- \frac{2k(n-k-1)}{\mu^2}+\frac{2k^3(n-k-1)}{\mu^4} \\ &> 2(n-k-1)\left(1-\frac{n-k-1}{\mu^2}\right)-\frac{2k(n-k-1)}{\mu^2}. \end{aligned}$$

Using the inequality $\mu \geq \sqrt{n(n-k-1)}$, we easily find that

$$k^3 > 2(n-k-1) - \frac{2(n-1)}{n} > 2(n-k-2),$$

and then $n < \frac{k^3}{2} + k + 2$, a contradiction. ■

Theorem 22. *Let G be a nearly balanced bipartite graph of order $2n-1$ $\left(n \geq \max\left\{\frac{k^3}{2}+k+2,(k+1)^2\right\}\right)$, where $k \geq 1$. If $\delta(G) \geq k$ and*

$$\mu(G) > \sqrt{n(n-k-1)},$$

then G is traceable, unless $G = C_n^k$.

Proof. By the assumption and Lemma 17,

$$\sqrt{n(n-k-1)} < \mu(G) \leq \sqrt{e(G)}.$$

Thus, we obtain

$$e(G) > n(n-k-1) \geq n(n-k-2) + (k+1)^2,$$

when $n \geq \max\left\{\frac{k^3}{2}+k+2,(k+1)^2\right\} > 2k+1$, by Lemma 16, G is traceable or $G \subseteq C_n^k$. But if $G \subseteq C_n^k$, then $\mu(G) < \sqrt{n(n-k-1)}$, unless $G = C_n^k$ by Lemma 21, a contradiction. ■

Theorem 23. *Let $G=(X,Y)$ be a nearly balanced bipartite graph of order $2n-1$ $(n \geq 2k)$, where $k \geq 1$. If $\delta(G) \geq k$, and*

$$\mu(\widehat{G}) \leq \sqrt{k(n-k)},$$

then G is traceable, unless $G \in \bigcup_{k=1}^{n/2} \mathcal{C}_n^k$ or $\Gamma_2 - v$, where $d_{\Gamma_2}(v) = 4$.

Proof. Let $|X| = n-1$, $|Y| = n$, G' be obtained from G by adding a vertex which is adjacent to every vertex in Y. Then G' is a balanced partite graph. Note that G is traceable if and only if G' is Hamiltonian. Let $H = cl_B(G')$. If H is Hamiltonian, then so is G' by Lemma 14. Now we assume that H is not Hamiltonian. Note that H is bipartite closure of G, thus every two nonadjacent vertices u, v in distant partite sets of H have degree sum at most n, i.e.,

$$d_{\widehat{H}}(u) + d_{\widehat{H}}(v) = n - d_H(u) + n - d_H(v) \geq n, \tag{11}$$

for any edge $uv \in E(\widehat{H})$.

This implies that $\widehat{H}$ contains only one component or $\widehat{H} = K_{s,n-s} + K_{t,n-t}$, $s,t \geq 1$. If $\widehat{H} = K_{s,n-s} + K_{t,n-t}$, $s,t \geq 1$, it contradicts the structure of $\widehat{H}$ (it must contain an isolated vertex). So, $\widehat{H}$ contains only one component.

Since $\delta(H) \geq \delta(G^{'}) \geq \delta(G) \geq k$, we can see that $d_{\widehat{H}}(u) \leq n-k$ and $d_{\widehat{H}}(v) \leq n-k$. Thus by (11), we have $k \leq d_{\widehat{H}}(u) \leq n-k$, $k \leq d_{\widehat{H}}(v) \leq n-k$, which implies that

$$d_{\widehat{H}}(u)d_{\widehat{H}}(v) \geq d_{\widehat{H}}(u)(n - d_{\widehat{H}}(u)) \geq k(n-k),$$

with equality if and only if (up to symmetry) $d_{\widehat{H}}(u) = k$, $d_{\widehat{H}}(v) = n-k$. By Lemma 6,

$$\sqrt{k(n-k)} \geq \mu(\widehat{G}) = \mu(\widehat{G^{'}}) \geq \mu(\widehat{H}) \geq \min_{uv\in E(\widehat{H})} \sqrt{d_{\widehat{H}}(u)d_{\widehat{H}}(v)} \geq \sqrt{k(n-k)},$$

which implies that $\mu(\widehat{H}) = \sqrt{k(n-k)}$ and there is an edge $uv \in E(\widehat{H})$ such that $d_{\widehat{H}}(u) = k$, $d_{\widehat{H}}(v) = n-k$. Let F be the component of $\widehat{H}$ which contains uv. By Lemma 6, F is a bipartite semi-regular graph with partite sets $X' \subseteq X$, and $Y' \subseteq Y$, and for any vertex $x \in X'$, $d_F(x) = k$, and any vertex $y \in Y'$, $d_F(y) = n-k$. Then $d_H(u) + d_H(v) = n$ for every two nonadjacent vertices u, v in distinct partite sets of H. By Lemma 18, $H \in \bigcup_{k=1}^{n/2} \mathcal{B}_n^k$ or $H = \Gamma_1$ or Γ_2 for $n = 4$ and $k = 2$. Then $G^{'} \subseteq B_n^k$ $(1 \leq k \leq n/2)$ or $G^{'} \subseteq \Gamma_1$ or Γ_2 for $n = 4$ and $k = 2$. By Perron-Frobenius theorem, every (spanning) subgraph of Γ_1, Γ_2 or B_n^k, $1 \leq k \leq n/2$, if it is not Γ_1 or Γ_2 or a graph in $\overline{B}_n^k$, $1 \leq k \leq n/2$, then it has the quasi-complement with spectral radius greater than $\sqrt{k(n-k)}$. Thus $G^{'} \in \bigcup_{k=1}^{n/2} \mathcal{B}_n^k$ or Γ_1 or Γ_2 for $n = 4$ and $k = 2$. By the construction of $G^{'}$, we get $G \in \bigcup_{k=1}^{n/2} \mathcal{C}_n^k$ or $\Gamma_2 - v$, where $d_{\Gamma_2}(v) = 4$, a contradiction. ∎

Theorem 24. *Let G be a nearly balanced bipartite graph of order $2n-1$ $\big(n \geq (k+1)^2\big)$, where $k \geq 1$. If $\delta(G) \geq k$ and*

$$q(G) > \frac{n(2n-k-2) + (k+1)^2}{n},$$

then G is traceable, unless $G \subseteq C_n^k$.

Proof. By the assumption and Lemma 20,

$$\frac{n(2n-k-2)+(k+1)^2}{n} < q(G) \le \frac{e(G)}{n} + n.$$

Thus, we obtain

$$e(G) > n(n-k-1) \ge n(n-k-2)+(k+1)^2,$$

when $n \ge (k+1)^2$, by Lemma 16, G is traceable or $G \subseteq C_n^k$. ■

Remark 25. In Theorem 24, we cannot change $G \subseteq C_n^k$ to $G = C_n^k$ like in Theorem 22. In fact, we can find a subgraph $G \subset C_n^k$, which satisfies the conditions of Theorem 24, such that $q(G) > 2n-k-1 \ge \frac{n(2n-k-2)+(k+1)^2}{n}$.

Proof. Assume that G is a proper subgraph of C_n^k, $n \ge (k+1)^2, k \ge 1, \delta(G) \ge k$, and has the maximum signless Laplacian spectral. By Perron-Frobenius theorem, G is obtained by omitting just one edge uv of C_n^k. Set for short $q := q(G)$, and let $\mathbf{x} = (x_{v_1}, \ldots, x_{v_{2n-1}})^T$ be a positive unit eigenvector to q. We have

$$q = \mathbf{x}^T Q(G)\mathbf{x} = 2\sum_{uv\in E(G)} (x_u + x_v)^2, \tag{12}$$

and

$$(q - d_G(v))x_v = \sum_{u\in N_G(v)} x_u, \tag{13}$$

for each vertex $v \in V(G)$. Equation (13) is called the signless Laplacian eigenvalue-equation for the graph G. In addition, for an arbitrary unit vector $\mathbf{x} \in \mathbb{R}^n$,

$$q \ge \mathbf{x}^T Q(G)\mathbf{x}, \tag{14}$$

with equality holds if and only if $\mathbf{x}$ is an eigenvector of $Q(G)$ according to q.

Write X for the set of vertices of C_n^k of degree k, let Y be the set of vertices of C_n^k of degree n, write Z for the set of vertices of C_n^k of degree $n-k-1$, let H be the set of the remaining k vertices of C_n^k of degree $n-1$.

Since $\delta(G) \ge k$, we see that G must contain all the edges between X and H. Therefore $\{u,v\} \subset Y \cup H$ or $\{u,v\} \subset Y \cup Z$, with two possible cases: (a) $u \in Y, v \in H$; (b) $u \in Y, v \in Z$. We shall show that case (b) yields a graph of no smaller signless Laplacian spectral radius than case (a).

Indeed, by (13), we have $x_i = x_j$ for any $i,j \in X$; likewise $x_i = x_j$ for any $i,j \in Y\backslash\{u\}$, for any $i,j \in Z\backslash\{v\}$ and for any $i,j \in H\backslash\{v\}$. Thus, let

$$\begin{aligned} x &:= x_i, i \in X,\\ y &:= y_i, i \in Y\backslash\{u\},\\ z &:= z_i, i \in Z\backslash\{v\},\\ h &:= h_i, i \in H\backslash\{v\}. \end{aligned}$$

Suppose that case (a) holds, that is, $u \in Y, v \in H$. Choose a vertex $w \in Z$, remove the edge uw, and add the edge uv. Then the obtained graph G' is covered by case (b).

If $x_w \le x_v$, we have

$$\mathbf{x}^T Q(G')\mathbf{x} - \mathbf{x}^T Q(G)\mathbf{x} = 2(x_v + x_u)^2 - 2(x_w + x_u)^2 \ge 0.$$

If $x_w > x_v$, swap the entries x_v and x_w, write $\mathbf{x}'$ for the resulting vector. We note that $\mathbf{x}'$ is a unit vector, and have that

$$\mathbf{x}'^T Q(G')\mathbf{x}' - \mathbf{x}^T Q(G)\mathbf{x} = 2\left(x_w + \sum_{i\in X} x_i\right)^2 - 2\left(x_v + \sum_{i\in X} x_i\right)^2 \ge 0.$$

Then by (14), $q(G') \ge q(G)$.

Therefore, G is obtained by omitting just one edge uv of C_n^k, where $u \in Y, v \in Z$. Now set $t := x_u, s := x_v$. Note that the $2n-1$ signless Laplacian eigenvalue-equations of G are reduced to six equations involving just the unknowns x, y, z, h, t, and s. By Equation (13), we have

$$\begin{aligned}
(q-k)x &= kh,\\
(q-n)y &= (n-k-1)z + kh + s,\\
(q-(n-k-1))z &= (n-k-2)y + t,\\
(q-(n-1))h &= kx + (n-k-2)y + t,\\
(q-(n-1))t &= (n-k-1)z + kh,\\
(q-(n-k-2))s &= (n-k-2)y.
\end{aligned}$$

Transform the above equations into a matrix equation $(B - qI)\mathbf{x} = 0$, where $\mathbf{x} = (x, y, z, h, t, s)^T$,

$$B = \begin{pmatrix}
k & 0 & 0 & k & 0 & 0\\
0 & n & n-k-1 & k & 0 & 1\\
0 & n-k-2 & n-k-1 & 0 & 1 & 0\\
k & n-k-2 & 0 & n-1 & 1 & 0\\
0 & 0 & n-k-1 & k & n-1 & 0\\
0 & n-k-2 & 0 & 0 & 0 & n-k-2
\end{pmatrix}.$$

Let

$$\begin{aligned}
f(x) &= \det(B - xI) = -x(n-1-x)(x^4 + (-4n+k+4)x^3\\
&+ (-nk + 6 + 5n^2 - 2k^2 - 11n + k)x^2\\
&+ (7n^2 + 5nk + 2 + 6nk^2 - 2n^2k - 7n - 2n^3 - 6k^2 - 2k^3 - 3k)x\\
&+ 2nk^3 - k^3 - 2k - 3k^2 + 8nk^2 + 7nk + 2n^3k - 4n^2k^2 - 7n^2k).
\end{aligned}$$

Thus, q is the largest root of $f(x) = 0$, and when $x > q$, $f(x)$ is monotonically increasing.

But when $k \geq 1, n \geq (k+1)^2$, we have

$$\begin{aligned} f(2n-k-1) \;=\; & (k+1-2n)(k-n)\big((4-k^2)n^2 \\ & + \big(-6k+k^3-4\big)n + 1 + 3k^2 + 3k + k^3\big) < 0, \end{aligned}$$

which implies that $q > 2n-k-1$, and the result follows. ∎

Acknowledgment

Supported by the National Natural Science Foundation of China under Grant no. 11371028, the Natural Science Foundation of Anhui Province (1808085MA04), and the Natural Science Foundation of Department of Education of Anhui Province (KJ2017A362).

References

[1] J.A. Bondy and V. Chvátal, *A method in graph theory*, Discrete Math. **15** (1976) 111–135.
doi:10.1016/0012-365X(76)90078-9

[2] J.A. Bondy and U.S. Murty, Graph Theory (Springer, New York, 2008).

[3] I. Tomescu, *On Hamiltonian-connected regular graphs*, J. Graph Theory **7** (1983) 429–436.
doi:10.1002/jgt.3190070407

[4] B.L. Li and B. Ning, *Spectral analogues of Erdős' and Moon—Moser's theorems on Hamilton cycles*, Linear Multilinear Algebra **64** (2016) 2252–2269.
doi:10.1080/03081087.2016.1151854

[5] B.L. Li and B. Ning, *Spectral analogues of Moon—Moser's theorem on Hamilton paths in bipartite graphs*, Linear Algebra Appl. **515** (2017) 180–195.
doi:10.1016/j.laa.2016.11.024

The 15,1(2018).

[6] A. Bhattacharya, S. Friedland and U.N. Peled, *On the first eigenvalue of bipartite graphs*, Electron. J. Combin. **15** (2008) #R144.

[7] M. Fiedler and V. Nikiforov, *Spectral radius and Hamiltonicity of graphs*, Linear Algebra Appl. **432** (2010) 2170–2173.
doi:10.1016/j.laa.2009.01.005

[8] L.H Feng and G.H Yu, *On three conjectures involving the signless Laplacian spectral radius of graphs*, Publ. Inst. Math. (Beograd) (N.S.) **85** (2009) 35–38.
doi:10.2298/PIM0999035F

[9] M. Ferrara, M. Jacobson and J. Powell, *Characterizing degree-sum maximal non-hamiltonian bipartite graphs*, Discrete Math. **312** (2012) 459–461. doi:10.1016/j.disc.2011.08.029

[10] Y.Z. Fan and and G.D. Yu, *Spectral condition for a graph to be Hamiltonian with respect to normalized Laplacian*, (2012).

[11] Y. Hong, J. Shu and K. Fang, *A sharp upper bound of the spectral radius of graphs*, J. Combin. Theory Ser. B **81** (2001) 177–183. doi:10.1006/jctb.2000.1997

[12] M. Lu, H.Q. Liu and F. Tian, *Spectral radius and Hamiltonion graphs*, Linear Algebra Appl. **437** (2012) 1670–1674. doi:10.1016/j.laa.2012.05.021

[13] R. Li, *Egienvalues, Laplacian eigenvalues and some Hamiltonian properties of graphs*, Util. Math. **88** (2012) 247–257.

[14] R. Li, *Spectral conditions for some stable properties of graphs*, J. Combin. Math. Combin. Comput. **88** (2014) 199–205.

[15] R. Liu, W.C. Shiu and J. Xue, *Sufficient spectral conditions on Hamiltonian and traceable graphs*, Linear Algebra Appl. **467** (2015) 254–266. doi:10.1016/j.laa.2014.11.017

[16] B. Ning and J. Ge, *Spectral radius and Hamiltonian properties of graphs*, Linear Multilinear Algebra **63** (2015) 1520–1530. doi:10.1080/03081087.2014.947984

[17] B. Ning and B. Li, *Spectral radius and traceability of connected claw-free graphs*, Filomat **30** (2016) 2445–2452. doi:10.2298/FIL1609445N

[18] V. Nikiforov, *Some inequalities for the largest eigenvalue of a graph*, Combin. Probab. Comput. **11** (2002) 179–189. doi:10.1017/S0963548301004928

[19] V. Nikiforov, *Spectral radius and Hamiltonicity of graphs with large minimum degree*, Czechoslovak Math. J. **66** (2016) 925–940. doi:10.1007/s10587-016-0301-y

[20] O. Ore, *Note on Hamilton circuits*, Amer. Math. Monthly **67** (1960) 55. doi:/10.2307/2308928

[21] G.D. Yu and Y.Z. Fan, *Spectral conditions for a graph to be Hamilton-connected*, Appl. Mechanics and Materials **336–338** (2013) 2329–2334. doi:10.4028/www.scientific.net/AMM.336-338.2329

[22] G.D. Yu, M.-L. Ye, G.-X. Cai and J.-D. Cao, *Signless Laplacian spectral conditions for Hamiltonicity of graphs*, J. Appl. Math. **2014** (2014) ID 282053. doi:10.1155/2014/282053

[23] G.D. Yu, *Spectral radius and Hamiltonicity of a graph*, Math. Appl. **27** (2014) 588–595.

[24] G.D. Yu, R. Li and B.H. Xing, *Spectral invariants and some stable properties of a graph*, Ars Combin. **121** (2015) 33–46.

[25] B. Zhou, *Signless Laplacian spectral radius and Hamiltonicity*, Linear Algebra Appl. **432** (2010) 566–570.
doi:10.1016/j.laa.2009.09.004

Received 22 September 2017
Revised 10 January 2018
Accepted 10 January 2018