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# ON TOTAL $\boldsymbol{H}$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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**Abstract**

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least to one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the *$H$-irregular total $k$-labeling* of a graph $G$ admitting

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an $H$-covering if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The *total $H$-irregularity strength* of a graph $G$ is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling.

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

**Keywords:** $H$-covering, $H$-irregular labeling, total $H$-irregularity strength, copies of graphs, union of graphs.

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## 1. Introduction

Consider a simple and finite graph $G$ with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a *total labeling*. For a total $k$-labeling $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated total vertex-weight of a vertex $x$ is

$$wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

and the associated total edge-weight of an edge $xy$ is

$$wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y).$$

A total $k$-labeling $\psi$ is defined to be an *edge irregular total $k$-labeling* of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ there is $wt_\psi(xy) \neq wt_\psi(x'y')$ and to be a *vertex irregular total $k$-labeling* of $G$ if for every two distinct vertices $x$ and $y$ of $G$ there is $wt_\psi(x) \neq wt_\psi(y)$. This concept was given by Bača, Jendrol', Miller and Ryan in [8].

The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the *total edge irregularity strength* of the graph $G$, $\mathrm{tes}(G)$. Analogously, we define the *total vertex irregularity strength* of $G$, $\mathrm{tvs}(G)$, as the minimum $k$ for which there exists a vertex irregular total $k$-labeling of $G$.

The following lower bound on the total edge irregularity strength of a graph $G$ is given in [8].

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of $G$. This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$.

Ivančo and Jendrol' [12] posed a conjecture that for an arbitrary graph $G$ different from $K_5$ with maximum degree $\Delta(G)$, $\text{tes}(G) = \max\{\lceil(|E(G)|+2)/3\rceil, \lceil(\Delta(G)+1)/2\rceil\}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)|+2)/3 \leq (\Delta(G)+1)/2$ in [10].

The bounds for the total vertex irregularity strength are given in [8] as follows.

$$\left\lceil \tfrac{|V(G)|+\delta(G)}{\Delta(G)+1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1, \tag{2}$$

where $\delta(G)$ is the minimum degree of $G$.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for $r$-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

$$\text{tvs}(G) \leq 3\left\lceil \tfrac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \tfrac{3|V(G)|}{\delta(G)} + 4. \tag{3}$$

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq (|V(G)|)^{0.5} \ln |V(G)|$, then

$$\text{tvs}(G) \leq \tfrac{(2+o(1))|V(G)|}{\delta(G)} + 4. \tag{4}$$

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An *edge-covering* of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering*. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-*covering*.

Let $G$ be a graph admitting an $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\psi$, we define the associated $H$-weight as

$$wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).$$

A total $k$-labeling $\psi$ is called to be an $H$-*irregular total* $k$-*labeling* of the graph $G$ if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The *total* $H$-*irregularity strength* of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. This definition was introduced by Ashraf, Bača, Lascsáková and Semaničová-Feňovčíková [7]. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total $H$-irregularity strength.

**Theorem 1** [7]**.** *Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then*

$$\text{ths}(G,H) \geq \left\lceil 1 + \tfrac{t-1}{|V(H)|+|E(H)|} \right\rceil .$$

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

**Theorem 2** [7]**.** *Let $r, s$, $2 \leq s \leq r$, be positive integers. Then*

$$\text{ths}(P_r, P_s) = \left\lceil \tfrac{s+r-1}{2s-1} \right\rceil .$$

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

## 2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

**Corollary 3.** *Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then*

$$\text{ths}(mG,H) \geq \left\lceil 1 + \tfrac{mt-1}{|V(H)|+|E(H)|} \right\rceil .$$

In the next theorem we give an upper bound for $\text{ths}(mG,H)$.

**Theorem 4.** *Let $G$ be a graph having an $H$-irregular total* $\text{ths}(G,H)$*-labeling $f$. Let $m$ be a positive integer. Then*

$$\text{ths}(mG,H) \leq \text{ths}(G,H) + (m-1) \left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)|+|E(H)|} \right\rceil ,$$

*where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total* $\text{ths}(G,H)$*-labeling $f$ of $G$.*

***Proof.*** Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G,H) = k$. The smallest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^{\min}(H)$. Evidently

$$wt_f^{\min}(H) \geq |V(H)| + |E(H)|. \tag{5}$$

Analogously, the largest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^{\max}(H)$. It holds that

$$wt_f^{\max}(H) \geq wt_f^{\min}(H) + t - 1 \tag{6}$$

and

$$wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k. \tag{7}$$

Thus $f : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ and

$$\left\{wt_f(H^j) : j = 1, 2, \dots, t\right\} \subset \left\{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\max}(H)\right\}. \tag{8}$$

By the symbol $x_i$, $i = 1, 2, \dots, m$, we denote an element (a vertex or an edge) in the $i^{\text{th}}$ copy of $G$, denoted by $G_i$, corresponding to the element $x$ in $G$, i.e., $x \in V(G) \cup E(G)$. Analogously, let $H_i^j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, be the subgraph in the $i^{\text{th}}$ copy of $G$ corresponding to the subgraph $H^j$ in $G$.

Let us define the total labeling $g$ of $mG$ in the following way. For $i = 1, 2, \dots, m$ let

$$g(x_i) = f(x) + (i-1)\left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

Evidently, all the labels are at most

$$k + (m-1)\left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

For the weight of every subgraph $H_i^j$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, isomorphic to the graph $H$ under the labeling $g$ we have

$$\begin{aligned} wt_g(H_i^j) &= \sum_{v \in V(H_i^j)} g(v) + \sum_{e \in E(H_i^j)} g(e) \\ &= \sum_{v \in V(H^j)} \left( f(v) + (i-1)\left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\ &+ \sum_{e \in E(H^j)} \left( f(e) + (i-1)\left\lceil \tfrac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \end{aligned}$$

$$= \sum_{v\in V(H^j)} f(v) + \sum_{e\in E(H^j)} f(e) + |V(H)|(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ |E(H)|(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$= wt_f(H^j) + (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

This means that in the given copy of $G$ the $H$-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to $H$ under the total labeling $g$ in the $i^{\text{th}}$ copy of $G$, $i = 1, 2, \ldots, m$, denoted by $wt_g^{\max}(H : H \subset G_i)$, is at most

$$wt_g^{\max}(H : H \subset G_i) \leq wt_f^{\max}(H)+(|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$

and the smallest weight of a subgraph isomorphic to $H$ under the total labeling $g$ in the $(i+1)^{\text{th}}$ copy of $G$, $i = 1, 2, \ldots, m-1$, denoted by $wt_g^{\min}(H : H \subset G_{i+1})$, is at least

$$wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H) + (|V(H)|+|E(H)|)i\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

After some manipulation we get

$$wt_g^{\min}(H : H \subset G_{i+1})$$
$$\geq wt_f^{min}(H) + (|V(H)|+|E(H)|)i\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$= wt_f^{\min}(H) + (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ (|V(H)|+|E(H)|)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil.$$

As

$$\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil \geq \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}$$

we obtain

$$wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H)$$
$$+ (|V(H)|+|E(H)|)(i-1)\left\lceil \frac{wt_f^{\max}(H)-wt_f^{\min}(H)+1}{|V(H)|+|E(H)|}\right\rceil$$
$$+ \big(wt_f^{\max}(H) - wt_f^{\min}(H) + 1\big)$$

$$= wt_f^{\max}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + 1$$
$$\geq wt_g^{\max}(H : H \subset G_i) + 1 > wt_g^{\max}(H : H \subset G_i).$$

Thus in all components the $H$-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

**Corollary 5.** *Let $G$ be a graph admitting an $H$-irregular total* $\mathrm{ths}(G,H)$*-labeling $f$. Let $m$ be a positive integer. Then*

$$\mathrm{ths}(mG, H) \leq m\,\mathrm{ths}(G, H).$$

***Proof.*** Let $f$ be a $\mathrm{ths}(G,H)$-labeling of a graph $G$ and let $\mathrm{ths}(G,H) = k$. As $wt_f^{\min}(H) \geq |V(H)| + |E(H)|$ and $wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k$ we get

$$\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq \left\lceil \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rceil$$
$$= \left\lceil k - 1 + \frac{1}{|V(H)| + |E(H)|} \right\rceil = k.$$

Hence, by Theorem 4,

$$\mathrm{ths}(mG, H) \leq \mathrm{ths}(G, H) + (m-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq k + (m-1)k = mk.$$
■

Let $\{H^1, H^2, \dots, H^t\}$ be the set of all subgraphs of $G$ isomorphic to $H$. Let $f$ be an $H$-irregular total $k$-labeling of a graph $G$ with $\mathrm{ths}(G,H) = k$ such that

$$\begin{aligned} &\left\{wt_f(H^j) : j = 1, 2, \dots, t\right\} \\ &= \left\{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + t - 1\right\}. \end{aligned} \tag{9}$$

Evidently, if the fraction

$$\frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} = \frac{t}{|V(H)| + |E(H)|}$$

is an integer then the weights of all $H$-weights in $mG$ under the total labeling $g$ of $mG$ defined in the proof of Theorem 4 constitute the set

$$\left\{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + mt - 1\right\}.$$

In particular, this implies that the upper bound for $\mathrm{ths}(mG, H)$ given in Theorem 4 is tight if $G$ is a graph that satisfies the conditions mentioned above.

**Theorem 6.** *Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total* $\operatorname{ths}(G,H)$*-labeling of $G$ such that*

$$\big\{wt_f(H^j) : j = 1,2,\dots,t\big\} = \big\{wt_f^{\min}(H), wt_f^{\min}(H)+1,\dots,wt_f^{\min}(H)+t-1\big\}.$$

*If the fraction $\frac{t}{|V(H)|+|E(H)|}$ is an integer then*

$$\operatorname{ths}(mG,H) \le \operatorname{ths}(G,H) + \frac{(m-1)t}{|V(H)|+|E(H)|}.$$

*Moreover, if* $\operatorname{ths}(G,H) = \left\lceil 1+\frac{t}{|V(H)|+|E(H)|}\right\rceil = 1+\frac{t}{|V(H)|+|E(H)|}$ *then*

$$\operatorname{ths}(mG,H) = \operatorname{ths}(G,H) + \frac{(m-1)t}{|V(H)|+|E(H)|} = 1 + \frac{mt}{|V(H)|+|E(H)|}.$$

Theorem 2 gives the exact value for the total $P_s$-irregularity strength for a path $P_r$. Moreover, the $P_s$-irregular total $(\lceil (s+r-1)/(2s-1)\rceil)$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_s$-weights consists of $t$ consecutive integers, where $t = r-s+1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_s)| = s$ and $|E(P_s)| = s-1$ and if the number $(r-s+1)/(2s-1)$ is an integer then according to Theorem 6 we get that

$$\begin{aligned}
\operatorname{ths}(mP_r,P_s) &= \operatorname{ths}(P_r,P_s) + (m-1)\tfrac{r-s+1}{2s-1} = \left\lceil \tfrac{s+r-1}{2s-1}\right\rceil + (m-1)\tfrac{r-s+1}{2s-1}\\
&= \left\lceil \tfrac{r-s+1+2s-1-1}{2s-1}\right\rceil + (m-1)\tfrac{r-s+1}{2s-1}\\
&= \left\lceil \tfrac{r-s+1}{2s-1} + 1 - \tfrac{1}{2s-1}\right\rceil + (m-1)\tfrac{r-s+1}{2s-1}\\
&= \tfrac{r-s+1}{2s-1} + 1 + (m-1)\tfrac{r-s+1}{2s-1} = m\tfrac{r-s+1}{2s-1} + 1.
\end{aligned}$$

Thus we obtain the following result.

**Corollary 7.** *Let $m, r, s$, $m \ge 1$, $2 \le s \le r$, be positive integers. If $2s-1$ divides $r-s+1$, then*

$$\operatorname{ths}(mP_r,P_s) = \frac{m(r-s+1)}{(2s-1)} + 1.$$

If $H$ is isomorphic to $K_2$ then $\operatorname{ths}(G,K_2) = \operatorname{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

**Corollary 8.** *Let $m$ be a positive integer. Then*

$$\left\lceil \frac{m|E(G)|+2}{3}\right\rceil \le \operatorname{ths}(mG,K_2) = \operatorname{tes}(mG) \le \operatorname{tes}(G) + (m-1)\left\lceil \frac{wt_f^{\max}-wt_f^{\min}+1}{3}\right\rceil,$$

*where $wt_f^{\max}$ and $wt_f^{\min}$ are the largest and smallest edge weights under a total* $\operatorname{tes}(G)$*-labeling $f$ of $G$.*

## 3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total $H$-irregularity strength of two graphs $G_1$ and $G_2$ admitting an $H$-covering. From Theorem 1 we immediately obtain

**Corollary 9.** *Let $G_i$, $i = 1, 2$, be a graph admitting an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. Then*

$$\mathrm{ths}(G_1 \cup G_2, H) \geq \left\lceil 1 + \tfrac{t_1+t_2-1}{|V(H)|+|E(H)|} \right\rceil .$$

The next theorem gives an upper bound for $\mathrm{ths}(G_1 \cup G_2, H)$.

**Theorem 10.** *Let $G_i$, $i = 1, 2$, be a graph having an $H$-irregular total* $\mathrm{ths}(G_i, H)$*-labeling $f_i$. Then*

$$\begin{aligned}&\mathrm{ths}(G_1 \cup G_2, H)\\ &\leq \min\left\{ \max\left\{ \mathrm{ths}(G_2, H), \mathrm{ths}(G_1, H) + \left\lceil \tfrac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\}, \right.\\ &\qquad \left. \max\left\{ \mathrm{ths}(G_1, H), \mathrm{ths}(G_2, H) + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\}\right\},\end{aligned}$$

*where $wt_{f_i}^{\max}(H)$ and $wt_{f_i}^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total* $\mathrm{ths}(G, H)$*-labeling $f_i$ of $G_i$.*

***Proof.*** Let $G_i$, $i = 1, 2$, be a graph that admits an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. We denote these subgraphs as $H_i^1, H_i^2, \ldots, H_i^{t_i}$. Assume that $f_i$ is an $H$-irregular total $k_i$-labeling of a graph $G_i$ with $\mathrm{ths}(G_i, H) = k_i$. The smallest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $wt_{f_i}^{\min}(H)$. Evidently

$$wt_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|. \tag{10}$$

Analogously, the largest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $wt_{f_i}^{\max}(H)$. It holds that

$$wt_{f_i}^{\max}(H) \geq wt_{f_i}^{\min}(H) + t_i - 1 \tag{11}$$

and

$$wt_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i. \tag{12}$$

Thus $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, k_i\}$ and

$$\left\{wt_{f_i}\left(H_i^j\right) : j = 1, 2, \ldots, t_i\right\} \subset \left\{wt_{f_i}^{\min}(H), wt_{f_i}^{\min}(H)+1, \ldots, wt_{f_i}^{\max}(H)\right\}. \tag{13}$$

Let us define the total labeling $g$ of $G_1 \cup G_2$ in the following way.

$$\begin{aligned} g(x) =& f_1(x) && \text{if } x \in V(G_1) \cup E(G_1), \\ g(x) =& f_2(x) + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil && \text{if } x \in V(G_2) \cup E(G_2). \end{aligned}$$

Evidently, all the labels are not greater than

$$\max\left\{ k_1, k_2 + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\}.$$

For the weight of the subgraph $H_1^j$, $j = 1, 2, \dots, t_1$, isomorphic to the graph $H$ under the labeling $g$ we get

$$wt_g(H_1^j) = \sum_{v \in V(H_1^j)} g(v) + \sum_{e \in E(H_1^j)} g(e) = \sum_{v \in V(H_1^j)} f_1(v) + \sum_{e \in E(H_1^j)} f_1(e) = wt_{f_1}\big(H_1^j\big).$$

For the weight of the subgraph $H_2^j$, $j = 1, 2, \dots, t_2$, isomorphic to the graph $H$ under the labeling $g$ we get

$$\begin{aligned} wt_g(H_2^j) =& \sum_{v \in V(H_2^j)} g(v) + \sum_{e \in E(H_2^j)} g(e) \\ =& \sum_{v \in V(H_2^j)} \left( f_2(v) + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right) \\ +& \sum_{e \in E(H_2^j)} \left( f_2(e) + \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right) \\ =& \sum_{v \in V(H_2^j)} f_2(v) + \sum_{e \in E(H_2^j)} f_2(e) + |V(H)| \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \\ +& \ |E(H)| \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \\ =& \, wt_{f_2}(H_2^j) + (|V(H)| + |E(H)|) \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil . \end{aligned}$$

According to (13) we get that the largest weight of a subgraph $H$ under the total labeling $g$ in $G_1$, denoted by $wt_g^{\max}(H : H \subset G_1)$, is at most

$$wt_g^{\max}(H : H \subset G_1) = wt_{f_1}^{\max}(H)$$

and the smallest weight of a subgraph $H$ under the total labeling $g$ in $G_2$, denoted by $wt_g^{\min}(H : H \subset G_2)$, is at least

$$wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left\lceil \tfrac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil .$$

Note, that when writting $H_i$ we only consider subgraphs of $G_i$ isomorphic to $H$. As

$$\left\lceil \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \geq \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|}$$

we get

$$\begin{aligned} wt_g^{\min}(H : H \subset G_2) &\geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|)\frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \\ &\geq wt_{f_2}^{\min}(H) + \big(wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1\big) = wt_{f_1}^{\max}(H)+1 \\ &> wt_{f_1}^{\max}(H) = wt_g^{\max}(H : H \subset G_1). \end{aligned}$$

Thus all the $H$-weights under the labeling $g$ in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling $h$ of $G_1 \cup G_2$ such that

$$\begin{aligned} h(x) &= f_2(x) && \text{if } x \in V(G_2) \cup E(G_2), \\ h(x) &= f_1(x) + \left\lceil \frac{wt_{f_2}^{\max}(H)-wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil && \text{if } x \in V(G_1) \cup E(G_1). \end{aligned}$$

Using similar arguments we can also show that under the total labeling $h$ the $H$-weights in $G_1 \cup G_2$ are distinct.

Thus $g$ and $h$ are $H$-irregular total labelings of $G$. Immediately from this fact we get

$$\begin{aligned} &\mathrm{ths}(G_1 \cup G_2, H) \\ &\leq \min\left\{ \max\left\{ \mathrm{ths}(G_2, H), \mathrm{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H)-wt_{f_1}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\}, \right. \\ &\qquad \left. \max\left\{ \mathrm{ths}(G_1, H), \mathrm{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H)-wt_{f_2}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil \right\} \right\}. \end{aligned}$$

■

Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

**Theorem 11** [18]**.** *The total edge irregularity strength of the disjoint union of graphs* $G_1, G_2, \ldots, G_m$, $m \geq 2$, *is*

$$\mathrm{tes}\left(\bigcup_{i=1}^{m} G_i\right) \leq \sum_{i=1}^{m} \mathrm{tes}(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$

If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that

$$\begin{aligned}\text{ths}(G_1 \cup G_2, K_2) &= \text{tes}(G_1 \cup G_2)\\ &\le \min\left\{ \max\left\{\text{tes}(G_2), \text{tes}(G_1) + \left\lceil \tfrac{3\text{tes}(G_2)-2}{3} \right\rceil \right\},\right.\\ &\qquad \left.\max\left\{\text{tes}(G_1), \text{tes}(G_2) + \left\lceil \tfrac{3\text{tes}(G_1)-2}{3} \right\rceil \right\}\right\}\\ &= \text{tes}(G_1) + \text{tes}(G_2)\end{aligned}$$

which is equal to the result from Theorem 11.

## 4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total ths$(G, H)$-labeling $f$ and $m$ is a positive integer then

$$\text{ths}(mG, H) \le \text{ths}(G, H) + (m-1)\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil,$$

where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total ths$(G, H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

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