

ON TOTAL H -IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

FARAHA ASHRAF

*Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan*

e-mail: faraha27@gmail.com

SUSANA CLARA LÓPEZ

*Dept. Matemàtiques, Universitat Politècnica de Catalunya
C/Esteve Terrades 5, 08860 Castelldefels, Spain*

e-mail: susana.clara.lopez@upc.edu

FRANCESC ANTONI MUNTANER-BATLE, AKITO OSHIMA

*Graph Theory and Applications Research Group, School of Electrical Engineering
and Computer Science, Faculty of Engineering and Built Environment
The University of Newcastle, Newcastle, Australia*

e-mail: fambles@yahoo.es
akitoism@yahoo.co.jp

AND

MARTIN BAČA¹, ANDREA SEMANIČOVÁ-FEŇOVČÍKOVÁ

*Department of Applied Mathematics and Informatics
Technical University, Košice, Slovakia*

e-mail: martin.baca@tuke.sk
andrea.fenovcikova@tuke.sk

Abstract

A simple graph G admits an H -covering if every edge in $E(G)$ belongs to at least to one subgraph of G isomorphic to a given graph H . For the subgraph $H \subseteq G$ under a total k -labeling we define the associated H -weight as the sum of labels of all vertices and edges belonging to H . The total k -labeling is called the H -irregular total k -labeling of a graph G admitting

¹Corresponding author.

an H -covering if all subgraphs of G isomorphic to H have distinct weights. The *total H -irregularity strength* of a graph G is the smallest integer k such that G has an H -irregular total k -labeling.

In this paper, we estimate lower and upper bounds on the total H -irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

Keywords: H -covering, H -irregular labeling, total H -irregularity strength, copies of graphs, union of graphs.

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1. INTRODUCTION

Consider a simple and finite graph G with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a *total labeling*. For a total k -labeling $\psi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ the associated total vertex-weight of a vertex x is

$$wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

and the associated total edge-weight of an edge xy is

$$wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y).$$

A total k -labeling ψ is defined to be an *edge irregular total k -labeling* of the graph G if for every two different edges xy and $x'y'$ of G there is $wt_\psi(xy) \neq wt_\psi(x'y')$ and to be a *vertex irregular total k -labeling* of G if for every two distinct vertices x and y of G there is $wt_\psi(x) \neq wt_\psi(y)$. This concept was given by Bača, Jendrol', Miller and Ryan in [8].

The minimum k for which the graph G has an edge irregular total k -labeling is called the *total edge irregularity strength* of the graph G , $tes(G)$. Analogously, we define the *total vertex irregularity strength* of G , $tv_s(G)$, as the minimum k for which there exists a vertex irregular total k -labeling of G .

The following lower bound on the total edge irregularity strength of a graph G is given in [8].

$$(1) \quad tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\},$$

where $\Delta(G)$ is the maximum degree of G . This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$.

Ivančo and Jendrol' [12] posed a conjecture that for an arbitrary graph G different from K_5 with maximum degree $\Delta(G)$, $\text{tes}(G) = \max \{ \lceil (|E(G)| + 2)/3 \rceil, \lceil (\Delta(G) + 1)/2 \rceil \}$. This conjecture has been verified for complete graphs and complete bipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2$ in [10].

The bounds for the total vertex irregularity strength are given in [8] as follows.

$$(2) \quad \left\lceil \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1,$$

where $\delta(G)$ is the minimum degree of G .

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for r -regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

$$(3) \quad \text{tvs}(G) \leq 3 \left\lceil \frac{|V(G)|}{\delta(G)} \right\rceil + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4.$$

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq (|V(G)|)^{0.5} \ln |V(G)|$, then

$$(4) \quad \text{tvs}(G) \leq \frac{(2+o(1))|V(G)|}{\delta(G)} + 4.$$

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i , $i = 1, 2, \dots, t$. Then it is said that G admits an $(H_1, H_2, \dots, H_t)$ -(*edge*) *covering*. If every subgraph H_i is isomorphic to a given graph H , then the graph G admits an H -*covering*.

Let G be a graph admitting an H -covering. For the subgraph $H \subseteq G$ under the total k -labeling ψ , we define the associated H -weight as

$$wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).$$

A total k -labeling ψ is called to be an H -irregular total k -labeling of the graph G if all subgraphs of G isomorphic to H have distinct weights. The *total H -irregularity strength* of a graph G , denoted $\text{ths}(G, H)$, is the smallest integer k such that G has an H -irregular total k -labeling. This definition was introduced by Ashraf, Bača, Lascsóková and Semaničová-Feňovčíková [7]. If H is isomorphic to K_2 , then the K_2 -irregular total k -labeling is isomorphic to the edge irregular total k -labeling and thus the total K_2 -irregularity strength of a graph G is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total H -irregularity strength.

Theorem 1 [7]. *Let G be a graph admitting an H -covering given by t subgraphs isomorphic to H . Then*

$$\text{ths}(G, H) \geq \left\lceil 1 + \frac{t-1}{|V(H)|+|E(H)|} \right\rceil.$$

If H is isomorphic to K_2 then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. *Let r, s , $2 \leq s \leq r$, be positive integers. Then*

$$\text{ths}(P_r, P_s) = \left\lceil \frac{s+r-1}{2s-1} \right\rceil.$$

In this paper, we estimate lower and upper bounds on the total H -irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. COPIES OF GRAPHS

By the symbol mG we denote the disjoint union of m copies of a graph G . Immediately from Theorem 1 we obtain a lower bound for the H -irregularity strength of m copies of a graph G .

Corollary 3. *Let G be a graph admitting an H -covering given by t subgraphs isomorphic to H and let m be a positive integer. Then*

$$\text{ths}(mG, H) \geq \left\lceil 1 + \frac{mt-1}{|V(H)|+|E(H)|} \right\rceil.$$

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. *Let G be a graph having an H -irregular total $\text{ths}(G, H)$ -labeling f . Let m be a positive integer. Then*

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)|+|E(H)|} \right\rceil,$$

where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\text{ths}(G, H)$ -labeling f of G .

Proof. Let G be a graph that admits an H -covering given by t subgraphs isomorphic to H . We denote these subgraphs as $H^1, H^2, \dots, H^t$. Assume that f is an H -irregular total k -labeling of a graph G with $\text{ths}(G, H) = k$. The smallest

weight of a subgraph H under the total k -labeling f is denoted by the symbol $wt_f^{\min}(H)$. Evidently

$$(5) \quad wt_f^{\min}(H) \geq |V(H)| + |E(H)|.$$

Analogously, the largest weight of a subgraph H under the total k -labeling f is denoted by the symbol $wt_f^{\max}(H)$. It holds that

$$(6) \quad wt_f^{\max}(H) \geq wt_f^{\min}(H) + t - 1$$

and

$$(7) \quad wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k.$$

Thus $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$ and

$$(8) \quad \{wt_f(H^j) : j = 1, 2, \dots, t\} \subset \{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\max}(H)\}.$$

By the symbol $x_i, i = 1, 2, \dots, m$, we denote an element (a vertex or an edge) in the i^{th} copy of G , denoted by G_i , corresponding to the element x in G , i.e., $x \in V(G) \cup E(G)$. Analogously, let $H_i^j, i = 1, 2, \dots, m, j = 1, 2, \dots, t$, be the subgraph in the i^{th} copy of G corresponding to the subgraph H^j in G .

Let us define the total labeling g of mG in the following way. For $i = 1, 2, \dots, m$ let

$$g(x_i) = f(x) + (i - 1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

Evidently, all the labels are at most

$$k + (m - 1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

For the weight of every subgraph $H_i^j, i = 1, 2, \dots, m, j = 1, 2, \dots, t$, isomorphic to the graph H under the labeling g we have

$$\begin{aligned} wt_g(H_i^j) &= \sum_{v \in V(H_i^j)} g(v) + \sum_{e \in E(H_i^j)} g(e) \\ &= \sum_{v \in V(H^j)} \left(f(v) + (i - 1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\ &\quad + \sum_{e \in E(H^j)} \left(f(e) + (i - 1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{v \in V(H^j)} f(v) + \sum_{e \in E(H^j)} f(e) + |V(H)|(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&+ |E(H)|(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&= wt_f(H^j) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.
\end{aligned}$$

This means that in the given copy of G the H -weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to H under the total labeling g in the i^{th} copy of G , $i = 1, 2, \dots, m$, denoted by $wt_g^{\max}(H : H \subset G_i)$, is at most

$$wt_g^{\max}(H : H \subset G_i) \leq wt_f^{\max}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil$$

and the smallest weight of a subgraph isomorphic to H under the total labeling g in the $(i+1)^{\text{th}}$ copy of G , $i = 1, 2, \dots, m-1$, denoted by $wt_g^{\min}(H : H \subset G_{i+1})$, is at least

$$wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H) + (|V(H)| + |E(H)|)i \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

After some manipulation we get

$$\begin{aligned}
&wt_g^{\min}(H : H \subset G_{i+1}) \\
&\geq wt_f^{\min}(H) + (|V(H)| + |E(H)|)i \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&= wt_f^{\min}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&+ (|V(H)| + |E(H)|) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.
\end{aligned}$$

As

$$\left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \geq \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|}$$

we obtain

$$\begin{aligned}
&wt_g^{\min}(H : H \subset G_{i+1}) \geq wt_f^{\min}(H) \\
&+ (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\
&+ (wt_f^{\max}(H) - wt_f^{\min}(H) + 1)
\end{aligned}$$

$$\begin{aligned}
 &= wt_f^{\max}(H) + (|V(H)| + |E(H)|)(i-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + 1 \\
 &\geq wt_g^{\max}(H : H \subset G_i) + 1 > wt_g^{\max}(H : H \subset G_i).
 \end{aligned}$$

Thus in all components the H -weights are distinct. This concludes the proof. ■

We obtain the following corollary.

Corollary 5. *Let G be a graph admitting an H -irregular total $\text{ths}(G, H)$ -labeling f . Let m be a positive integer. Then*

$$\text{ths}(mG, H) \leq m \text{ths}(G, H).$$

Proof. Let f be a $\text{ths}(G, H)$ -labeling of a graph G and let $\text{ths}(G, H) = k$. As $wt_f^{\min}(H) \geq |V(H)| + |E(H)|$ and $wt_f^{\max}(H) \leq (|V(H)| + |E(H)|)k$ we get

$$\begin{aligned}
 \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil &\leq \left\lceil \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rceil \\
 &= \left\lceil k - 1 + \frac{1}{|V(H)| + |E(H)|} \right\rceil = k.
 \end{aligned}$$

Hence, by Theorem 4,

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \leq k + (m-1)k = mk. \quad \blacksquare$$

Let $\{H^1, H^2, \dots, H^t\}$ be the set of all subgraphs of G isomorphic to H . Let f be an H -irregular total k -labeling of a graph G with $\text{ths}(G, H) = k$ such that

$$\begin{aligned}
 &\{wt_f(H^j) : j = 1, 2, \dots, t\} \\
 (9) \quad &= \{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + t - 1\}.
 \end{aligned}$$

Evidently, if the fraction

$$\frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} = \frac{t}{|V(H)| + |E(H)|}$$

is an integer then the weights of all H -weights in mG under the total labeling g of mG defined in the proof of Theorem 4 constitute the set

$$\{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + mt - 1\}.$$

In particular, this implies that the upper bound for $\text{ths}(mG, H)$ given in Theorem 4 is tight if G is a graph that satisfies the conditions mentioned above.

Theorem 6. *Let G be a graph admitting an H -covering given by t subgraphs isomorphic to H . Let f be an H -irregular total $\text{ths}(G, H)$ -labeling of G such that*

$$\{wt_f(H^j) : j = 1, 2, \dots, t\} = \{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \dots, wt_f^{\min}(H) + t - 1\}.$$

If the fraction $\frac{t}{|V(H)|+|E(H)|}$ is an integer then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + \frac{(m-1)t}{|V(H)|+|E(H)|}.$$

Moreover, if $\text{ths}(G, H) = \left\lceil 1 + \frac{t}{|V(H)|+|E(H)|} \right\rceil = 1 + \frac{t}{|V(H)|+|E(H)|}$ then

$$\text{ths}(mG, H) = \text{ths}(G, H) + \frac{(m-1)t}{|V(H)|+|E(H)|} = 1 + \frac{mt}{|V(H)|+|E(H)|}.$$

Theorem 2 gives the exact value for the total P_s -irregularity strength for a path P_r . Moreover, the P_s -irregular total $(\lceil (s+r-1)/(2s-1) \rceil)$ -labeling of P_r described in the proof of Theorem 2 in [7] has the property that the set of P_s -weights consists of t consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in P_r isomorphic to P_s . As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that

$$\begin{aligned} \text{ths}(mP_r, P_s) &= \text{ths}(P_r, P_s) + (m-1)\frac{r-s+1}{2s-1} = \left\lceil \frac{s+r-1}{2s-1} \right\rceil + (m-1)\frac{r-s+1}{2s-1} \\ &= \left\lceil \frac{r-s+1+2s-1-1}{2s-1} \right\rceil + (m-1)\frac{r-s+1}{2s-1} \\ &= \left\lceil \frac{r-s+1}{2s-1} + 1 - \frac{1}{2s-1} \right\rceil + (m-1)\frac{r-s+1}{2s-1} \\ &= \frac{r-s+1}{2s-1} + 1 + (m-1)\frac{r-s+1}{2s-1} = m\frac{r-s+1}{2s-1} + 1. \end{aligned}$$

Thus we obtain the following result.

Corollary 7. *Let $m, r, s, m \geq 1, 2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then*

$$\text{ths}(mP_r, P_s) = \frac{m(r-s+1)}{(2s-1)} + 1.$$

If H is isomorphic to K_2 then $\text{ths}(G, K_2) = \text{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

Corollary 8. *Let m be a positive integer. Then*

$$\left\lceil \frac{m|E(G)|+2}{3} \right\rceil \leq \text{ths}(mG, K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m-1) \left\lceil \frac{wt_f^{\max} - wt_f^{\min} + 1}{3} \right\rceil,$$

where $wt_f^{\max}$ and $wt_f^{\min}$ are the largest and smallest edge weights under a total $\text{tes}(G)$ -labeling f of G .

3. DISJOINT UNION OF TWO NON-ISOMORPHIC GRAPHS

In this section we will deal with the total H -irregularity strength of two graphs G_1 and G_2 admitting an H -covering. From Theorem 1 we immediately obtain

Corollary 9. *Let G_i , $i = 1, 2$, be a graph admitting an H -covering given by t_i subgraphs isomorphic to H . Then*

$$\text{ths}(G_1 \cup G_2, H) \geq \left\lceil 1 + \frac{t_1+t_2-1}{|V(H)|+|E(H)|} \right\rceil.$$

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

Theorem 10. *Let G_i , $i = 1, 2$, be a graph having an H -irregular total $\text{ths}(G_i, H)$ -labeling f_i . Then*

$$\begin{aligned} & \text{ths}(G_1 \cup G_2, H) \\ & \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}, \right. \\ & \quad \left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\} \right\}, \end{aligned}$$

where $wt_{f_i}^{\max}(H)$ and $wt_{f_i}^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\text{ths}(G, H)$ -labeling f_i of G_i .

Proof. Let G_i , $i = 1, 2$, be a graph that admits an H -covering given by t_i subgraphs isomorphic to H . We denote these subgraphs as $H_i^1, H_i^2, \dots, H_i^{t_i}$. Assume that f_i is an H -irregular total k_i -labeling of a graph G_i with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph H under the total k_i -labeling f_i is denoted by the symbol $wt_{f_i}^{\min}(H)$. Evidently

$$(10) \quad wt_{f_i}^{\min}(H) \geq |V(H)| + |E(H)|.$$

Analogously, the largest weight of a subgraph H under the total k_i -labeling f_i is denoted by the symbol $wt_{f_i}^{\max}(H)$. It holds that

$$(11) \quad wt_{f_i}^{\max}(H) \geq wt_{f_i}^{\min}(H) + t_i - 1$$

and

$$(12) \quad wt_{f_i}^{\max}(H) \leq (|V(H)| + |E(H)|)k_i.$$

Thus $f_i : V(G_i) \cup E(G_i) \rightarrow \{1, 2, \dots, k_i\}$ and

$$(13) \quad \{wt_{f_i}(H_i^j) : j = 1, 2, \dots, t_i\} \subset \{wt_{f_i}^{\min}(H), wt_{f_i}^{\min}(H) + 1, \dots, wt_{f_i}^{\max}(H)\}.$$

Let us define the total labeling g of $G_1 \cup G_2$ in the following way.

$$\begin{aligned} g(x) &= f_1(x) && \text{if } x \in V(G_1) \cup E(G_1), \\ g(x) &= f_2(x) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil && \text{if } x \in V(G_2) \cup E(G_2). \end{aligned}$$

Evidently, all the labels are not greater than

$$\max \left\{ k_1, k_2 + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}.$$

For the weight of the subgraph H_1^j , $j = 1, 2, \dots, t_1$, isomorphic to the graph H under the labeling g we get

$$wt_g(H_1^j) = \sum_{v \in V(H_1^j)} g(v) + \sum_{e \in E(H_1^j)} g(e) = \sum_{v \in V(H_1^j)} f_1(v) + \sum_{e \in E(H_1^j)} f_1(e) = wt_{f_1}(H_1^j).$$

For the weight of the subgraph H_2^j , $j = 1, 2, \dots, t_2$, isomorphic to the graph H under the labeling g we get

$$\begin{aligned} wt_g(H_2^j) &= \sum_{v \in V(H_2^j)} g(v) + \sum_{e \in E(H_2^j)} g(e) \\ &= \sum_{v \in V(H_2^j)} \left(f_2(v) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\ &\quad + \sum_{e \in E(H_2^j)} \left(f_2(e) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right) \\ &= \sum_{v \in V(H_2^j)} f_2(v) + \sum_{e \in E(H_2^j)} f_2(e) + |V(H)| \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\ &\quad + |E(H)| \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \\ &= wt_{f_2}(H_2^j) + (|V(H)| + |E(H)|) \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil. \end{aligned}$$

According to (13) we get that the largest weight of a subgraph H under the total labeling g in G_1 , denoted by $wt_g^{\max}(H : H \subset G_1)$, is at most

$$wt_g^{\max}(H : H \subset G_1) = wt_{f_1}^{\max}(H)$$

and the smallest weight of a subgraph H under the total labeling g in G_2 , denoted by $wt_g^{\min}(H : H \subset G_2)$, is at least

$$wt_g^{\min}(H : H \subset G_2) \geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.$$

Note, that when writting H_i we only consider subgraphs of G_i isomorphic to H .
As

$$\left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \geq \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|}$$

we get

$$\begin{aligned} wt_g^{\min}(H : H \subset G_2) &\geq wt_{f_2}^{\min}(H) + (|V(H)| + |E(H)|) \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \\ &\geq wt_{f_2}^{\min}(H) + (wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1) = wt_{f_1}^{\max}(H) + 1 \\ &> wt_{f_1}^{\max}(H) = wt_g^{\max}(H : H \subset G_1). \end{aligned}$$

Thus all the H -weights under the labeling g in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling h of $G_1 \cup G_2$ such that

$$\begin{aligned} h(x) &= f_2(x) && \text{if } x \in V(G_2) \cup E(G_2), \\ h(x) &= f_1(x) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil && \text{if } x \in V(G_1) \cup E(G_1). \end{aligned}$$

Using similar arguments we can also show that under the total labeling h the H -weights in $G_1 \cup G_2$ are distinct.

Thus g and h are H -irregular total labelings of G . Immediately from this fact we get

$$\begin{aligned} &\text{ths}(G_1 \cup G_2, H) \\ &\leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\max}(H) - wt_{f_1}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}, \right. \\ &\quad \left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\max}(H) - wt_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\} \right\}. \end{aligned}$$

■

Ramdani, Salman, Assiyatum, Semaničová-Feňovčíková and Bača [18] gave an upper bound for the total edge irregularity strength of the disjoint union of graphs by the following form.

Theorem 11 [18]. *The total edge irregularity strength of the disjoint union of graphs $G_1, G_2, \dots, G_m$, $m \geq 2$, is*

$$\text{tes} \left(\bigcup_{i=1}^m G_i \right) \leq \sum_{i=1}^m \text{tes}(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor.$$

If H is isomorphic to K_2 then from Theorem 10 it follows that

$$\begin{aligned} \text{ths}(G_1 \cup G_2, K_2) &= \text{tes}(G_1 \cup G_2) \\ &\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lceil \frac{3\text{tes}(G_2)-2}{3} \right\rceil \right\}, \right. \\ &\quad \left. \max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lceil \frac{3\text{tes}(G_1)-2}{3} \right\rceil \right\} \right\} \\ &= \text{tes}(G_1) + \text{tes}(G_2) \end{aligned}$$

which is equal to the result from Theorem 11.

4. CONCLUSION

In this paper, we have estimated lower and upper bounds for the total H -irregularity strength for the disjoint union of m copies of a graph. We have proved that if a graph G admits an H -irregular total $\text{ths}(G, H)$ -labeling f and m is a positive integer then

$$\text{ths}(mG, H) \leq \text{ths}(G, H) + (m-1) \left\lceil \frac{wt_f^{\max}(H) - wt_f^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil,$$

where $wt_f^{\max}(H)$ and $wt_f^{\min}(H)$ are the largest and smallest weights of a subgraph H under a total $\text{ths}(G, H)$ -labeling f of G . This upper bound is tight.

We have also proved an upper bound for the total H -irregularity strength for the disjoint union of two non-isomorphic graphs.

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