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Note

A SHORT PROOF FOR A LOWER BOUND ON THE ZERO FORCING NUMBER

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Abstract

We provide a short proof of a conjecture of Davila and Kenter concerning a lower bound on the zero forcing number $Z(G)$ of a graph G. More specifically, we show that $Z(G) \geq (g-2)(\delta-2)+2$ for every graph G of girth g at least 3 and minimum degree δ at least 2.

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1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology.

For an integer n, let $[n]$ denote the set of positive integers at most n. For a graph G, a set Z of vertices of G is a *zero forcing set* of G if the elements of $V(G) \setminus Z$ have a linear order $u_1, \ldots, u_k$ such that, for every i in $[k]$, there is some vertex v_i in $Z \cup \{u_j : j \in [i-1]\}$ such that u_i is the only neighbor of v_i outside of $Z \cup \{u_j : j \in [i-1]\}$; in particular, $N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\}$ for $i \in [k]$. The *zero forcing number* $Z(G)$ of G, defined as the minimum order of a zero forcing set of G, was proposed by the AIM Minimum Rank - Special Graphs Work Group [1] as an upper bound on the nullity of matrices associated with a given graph. The same parameter was also considered in connection with quantum physics [5, 7, 14] and logic circuits [6].

In [11] Davila and Kenter conjectured that

$$Z(G) \geq (g-2)(\delta-2)+2 \tag{1}$$

for every graph G of girth g at least 3 and minimum degree δ at least 2. They observe that, for $g > 6$ and sufficiently large δ in terms of g, the conjectured bound follows by combining results from [3] and [8]. For $g \leq 6$, it was shown in [12, 13], Davila and Henning [9] showed it for $7 \leq g \leq 10$, and, eventually, Davila, Kalinowski, and Stephen [10] completed the proof. The proof in [10] is rather short itself but relies on [12, 13, 9]. While the cases $g \leq 6$ have rather short proofs, the proof in [9] for $7 \leq g \leq 10$ extends over more than eleven pages and requires a detailed case analysis. Therefore, the complete proof of (1) obtained by combining [9, 10, 12, 13] is rather long.

In the present note we propose a considerably shorter and simpler proof. Our approach only requires a special treatment for the triangle-free case $g = 4$ [12], involves a new lower bound on the zero forcing number, and an application of the Moore bound [2].

2. Proof of (1)

Our first result is a natural generalization of the well known fact $Z(G) \geq \delta(G)$ [4], where $\delta(G)$ is the minimum degree of a graph G. For a set X of vertices of a graph G of order n, let $N_G(X) = \left(\bigcup_{u\in X} N_G(u)\right) \setminus X$, $N_G[X] = X \cup N_G(X)$, and $\delta_p(G) = \min\{|N_G(X)| : X \subseteq V(G) \text{ and } |X| = p\}$ for $p \in [n]$. Note that $\delta_1(G)$ equals $\delta(G)$.

Lemma 1. *If G is a graph of order n, then $Z(G) \geq \delta_p(G)$ for every $p \in [n]$.*

Proof. Let Z be a zero forcing set of minimum order. Let $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ be as in the introduction. Since, by definition, $\delta_p(G) \leq n - p$, the result is trivial for $p \geq k = n - |Z|$, and we may assume that $p < k$. As noted above, we have $N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\}$ for $i \in [k]$, which implies that $X = \{v_1, \ldots, v_p\}$ is a set of p distinct vertices of G. Furthermore, it implies that $|N_G[X]| \leq |Z| + p$, and, hence, $\delta_p(G) \leq |N_G(X)| = |N_G[X]| - p \leq |Z|$ as required. ■

For later reference, we recall the Moore bound for irregular graphs.

Theorem 2 (Alon, Hoory and Linial [2]). *If G is a graph of order n, girth at least $2r$ for some integer r, and average degree d at least 2, then $n \geq 2\sum\limits_{i=0}^{r-1}(d-1)^i$.*

We also need the following numerical fact.

Lemma 3. *For positive integers p and q with $p \geq 5$ and $2p-1 \leq q \leq \binom{p}{2}$,*

$$\left(1+\frac{2(q-p)}{q+p}\right)^{\left\lceil \frac{p}{2}\right\rceil+1} > q-p+1.$$

Proof. For $p \geq 17$, it follows from $q \geq 2p-1$ that $1+\frac{2(q-p)}{q+p} \geq 1.64$, and, since $1.64^{\left\lceil \frac{p}{2}\right\rceil+1} > \binom{p}{2} - p + 1$, the desired inequality follows for these values of p. For the finitely many pairs (p, q) with $5 \leq p \leq 16$ and $2p-1 \leq q \leq \binom{p}{2}$, we verified it using a computer. ■

We proceed to the proof of (1).

Theorem 4. *If G is a graph of girth g at least 3 and minimum degree δ at least 2, then $Z(G) \geq (g-2)(\delta-2)+2$.*

Proof. For $g = 3$, the inequality simplifies to the known fact $Z(G) \geq \delta(G)$, and, for $g = 4$, it has been shown in [12]. Now, let $g \geq 5$. Let X be a set of $g-2$ vertices of G with $|N_G(X)| = \delta_{g-2}(G)$, and, let $N = N_G(X)$. By the girth condition, the components of $G[X]$ are trees, and no vertex in N has more than one neighbor in any component of $G[X]$.

Let $K_1, \ldots, K_p$ be the vertex sets of the components of $G[X]$.

If $p \geq 3$ and there are two vertices in N that both have neighbors in the same two distinct components of $G[X]$, then G contains a cycle of order at most $2 + |K_i| + |K_j| \leq 2 + (g-2) - (p-2) < g$ which is a contradiction. Thus, $0 \leq |N_G(K_i) \cap N_G(K_j)| \leq 1$ for $1 \leq i < j \leq n$. Similarly, if $p = 2$, and there are three vertices u, v, and w in N that all three have neighbors in K_1 and K_2, then let u_i, v_i, and w_i denote the corresponding neighbors in K_i for $i \in \{1, 2\}$, respectively. If any of u_1, v_1, and w_1 are distinct, then $G[K_1]$ contains a path between two of the vertices u_1, v_1, and w_1 avoiding the third, and G contains a cycle of order at most $2 + (|K_1| - 1) + |K_2| = g - 1$, which is a contradiction. By symmetry, this implies $u_1 = v_1 = w_1$ and $u_2 = v_2 = w_2$, and G contains the cycle $u_1 u u_2 v u_1$ of order 4, which is a contradiction. Thus, $0 \leq |N_G(K_1) \cap N_G(K_2)| \leq 2$.

Combining these observations, we obtain

$$(2) \qquad \sum_{1 \leq i < j \leq p} |N_G(K_i) \cap N_G(K_j)| \leq \begin{cases} \binom{p}{2}, & \text{for } p \geq 3, \text{ and} \\ 2p-2, & \text{for } p \in \{1, 2\}. \end{cases}$$

Let the bipartite graph H arise from $G[X \cup N]$ by contracting the component K_i of $G[X]$ to a single vertex u_i for every $i \in [p]$, and removing all edges of $G[N]$. Note that $\sum_{i \in [p]} d_H(u_i) - \sum_{v \in N} d_H(v) = 0$ in the bipartite graph H with partite sets $\{u_1, \ldots, u_p\}$ and N. By the girth condition, no vertex in N has two neighbors in K_i, and K_i induces a tree, which implies $d_H(u_i) = \sum_{v \in K_i} d_G(v) - 2(|K_i| - 1) \geq$

$\delta|K_i| - 2(|K_i| - 1)$ for every $i \in [p]$. Let $q = \sum_{v\in N}(d_H(v) - 1)$. Now, Lemma 1 implies

$$\begin{aligned} Z(G) \geq \delta_{g-2}(G) = |N| &= \sum_{v\in N} 1 + \left(\sum_{i\in[p]} d_H(u_i) - \sum_{v\in N} d_H(v)\right) \\ &= \sum_{i\in[p]} d_H(u_i) - q \geq \sum_{i=1}^{p} \Big(\delta|K_i| - 2(|K_i| - 1)\Big) - q \\ &= (g-2)(\delta-2) + 2 + ((2p-2) - q). \end{aligned}$$

If $q \leq 2p - 2$, then this implies (1). Hence, we may assume $q \geq 2p - 1$.

Note that

$$2p - 1 \leq q = \sum_{v\in N}(d_H(v) - 1) \leq \sum_{v\in N}\binom{d_H(v)}{2} = \sum_{1\leq i<j\leq p}|N_G(K_i) \cap N_G(K_j)|,$$

where the last equality follows, because every vertex v in N contributes exactly $\binom{d_H(v)}{2}$ to the right hand side. Now, (2) implies $p \geq 5$.

Let H' arise by removing all vertices of degree 1 from H. Since, for every $i \in [p]$, we have $d_H(u_i) \geq \delta|K_i| - 2(|K_i| - 1) \geq 2$, the graph H' contains all p vertices $u_1, \ldots, u_p$. Let H' contain r vertices of N. Since H' has order $p + r$ and size

$$\sum_{v\in N\cap V(H')} d_H(v) = r + \sum_{v\in N}(d_H(v) - 1) = r + q,$$

its average degree is at least $\frac{2(r+q)}{p+r}$, which is at least 2, because $q \geq 2p - 1 \geq p$.

If H' contains a cycle of order 2ℓ, then G contains a cycle that alternates between X and N, contains ℓ vertices from N, and avoids $p-\ell$ of the components of $G[X]$, which implies that this cycle has order at most $\ell + (|X| - (p - \ell)) = \ell + (g-2) - (p-\ell)$. By the girth condition, this implies that the bipartite graph H' has girth at least $p + 2$, if p is even, and $p + 3$, if p is odd.

Using Theorem 2 and $q \geq r$, we obtain

$$\begin{aligned} p + r &\geq 2\sum_{i=0}^{\lceil \frac{p}{2}\rceil}\left(\frac{2(r+q)}{p+r} - 1\right)^i = 2\frac{p+r}{2(q-p)}\left(\left(1 + \frac{2(q-p)}{p+r}\right)^{\lceil \frac{p}{2}\rceil+1} - 1\right) \\ &\geq 2\frac{p+r}{2(q-p)}\left(\left(1 + \frac{2(q-p)}{p+q}\right)^{\lceil \frac{p}{2}\rceil+1} - 1\right), \end{aligned}$$

which implies $\left(1 + \frac{2(q-p)}{q+p}\right)^{\lceil \frac{p}{2}\rceil+1} \leq q - p + 1$. Since $q \geq 2p - 1$, and, by (2), $q \leq \binom{p}{2}$, this contradicts Lemma 3, which completes the proof. ■

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