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Note

A SHORT PROOF FOR A LOWER BOUND ON THE ZERO FORCING NUMBER

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Abstract

We provide a short proof of a conjecture of Davila and Kenter concerning a lower bound on the zero forcing number $Z(G)$ of a graph G. More specifically, we show that $Z(G) \geq (g-2)(\delta-2)+2$ for every graph G of girth g at least 3 and minimum degree δ at least 2.

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1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology.

For an integer n, let $[n]$ denote the set of positive integers at most n. For a graph G, a set Z of vertices of G is a *zero forcing set* of G if the elements of $V(G) \setminus Z$ have a linear order $u_1, \ldots, u_k$ such that, for every i in $[k]$, there is some vertex v_i in $Z \cup \{u_j : j \in [i-1]\}$ such that u_i is the only neighbor of v_i outside of $Z \cup \{u_j : j \in [i-1]\}$; in particular, $N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\}$ for $i \in [k]$. The *zero forcing number* $Z(G)$ of G, defined as the minimum order of a zero forcing set of G, was proposed by the AIM Minimum Rank - Special Graphs Work Group [1] as an upper bound on the nullity of matrices associated with a given graph. The same parameter was also considered in connection with quantum physics [5, 7, 14] and logic circuits [6].

In [11] Davila and Kenter conjectured that

$$Z(G) \geq (g-2)(\delta-2)+2 \tag{1}$$

for every graph G of girth g at least 3 and minimum degree δ at least 2. They observe that, for $g > 6$ and sufficiently large δ in terms of g, the conjectured bound follows by combining results from [3] and [8]. For $g \leq 6$, it was shown in [12, 13], Davila and Henning [9] showed it for $7 \leq g \leq 10$, and, eventually, Davila, Kalinowski, and Stephen [10] completed the proof. The proof in [10] is rather short itself but relies on [12, 13, 9]. While the cases $g \leq 6$ have rather short proofs, the proof in [9] for $7 \leq g \leq 10$ extends over more than eleven pages and requires a detailed case analysis. Therefore, the complete proof of (1) obtained by combining [9, 10, 12, 13] is rather long.

In the present note we propose a considerably shorter and simpler proof. Our approach only requires a special treatment for the triangle-free case $g = 4$ [12], involves a new lower bound on the zero forcing number, and an application of the Moore bound [2].

2. Proof of (1)

Our first result is a natural generalization of the well known fact $Z(G) \geq \delta(G)$ [4], where $\delta(G)$ is the minimum degree of a graph G. For a set X of vertices of a graph G of order n, let $N_G(X) = \left(\bigcup_{u \in X} N_G(u)\right) \setminus X$, $N_G[X] = X \cup N_G(X)$, and $\delta_p(G) = \min\{|N_G(X)| : X \subseteq V(G) \text{ and } |X| = p\}$ for $p \in [n]$. Note that $\delta_1(G)$ equals $\delta(G)$.

Lemma 1. *If G is a graph of order n, then $Z(G) \geq \delta_p(G)$ for every $p \in [n]$.*

Proof. Let Z be a zero forcing set of minimum order. Let $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ be as in the introduction. Since, by definition, $\delta_p(G) \leq n - p$, the result is trivial for $p \geq k = n - |Z|$, and we may assume that $p < k$. As noted above, we have $N_G[v_i] \setminus (Z \cup N_G[v_1] \cup \cdots \cup N_G[v_{i-1}]) = \{u_i\}$ for $i \in [k]$, which implies that $X = \{v_1, \ldots, v_p\}$ is a set of p distinct vertices of G. Furthermore, it implies that $|N_G[X]| \leq |Z| + p$, and, hence, $\delta_p(G) \leq |N_G(X)| = |N_G[X]| - p \leq |Z|$ as required. ■

For later reference, we recall the Moore bound for irregular graphs.

Theorem 2 (Alon, Hoory and Linial [2])**.** *If G is a graph of order n, girth at least $2r$ for some integer r, and average degree d at least 2, then $n \geq 2\sum\limits_{i=0}^{r-1}(d-1)^i$.*

We also need the following numerical fact.

Lemma 3. *For positive integers p and q with $p \geq 5$ and $2p-1 \leq q \leq \binom{p}{2}$,*

$$\left(1+\frac{2(q-p)}{q+p}\right)^{\left\lceil \frac{p}{2}\right\rceil+1} > q-p+1.$$

Proof. For $p \geq 17$, it follows from $q \geq 2p-1$ that $1+\frac{2(q-p)}{q+p} \geq 1.64$, and, since $1.64^{\left\lceil \frac{p}{2}\right\rceil+1} > \binom{p}{2}-p+1$, the desired inequality follows for these values of p. For the finitely many pairs (p,q) with $5 \leq p \leq 16$ and $2p-1 \leq q \leq \binom{p}{2}$, we verified it using a computer. ■

We proceed to the proof of (1).

Theorem 4. *If G is a graph of girth g at least 3 and minimum degree δ at least 2, then $Z(G) \geq (g-2)(\delta-2)+2$.*

Proof. For $g=3$, the inequality simplifies to the known fact $Z(G) \geq \delta(G)$, and, for $g=4$, it has been shown in [12]. Now, let $g \geq 5$. Let X be a set of $g-2$ vertices of G with $|N_G(X)| = \delta_{g-2}(G)$, and, let $N = N_G(X)$. By the girth condition, the components of $G[X]$ are trees, and no vertex in N has more than one neighbor in any component of $G[X]$.

Let $K_1, \ldots, K_p$ be the vertex sets of the components of $G[X]$.

If $p \geq 3$ and there are two vertices in N that both have neighbors in the same two distinct components of $G[X]$, then G contains a cycle of order at most $2+|K_i|+|K_j| \leq 2+(g-2)-(p-2) < g$ which is a contradiction. Thus, $0 \leq |N_G(K_i) \cap N_G(K_j)| \leq 1$ for $1 \leq i < j \leq n$. Similarly, if $p=2$, and there are three vertices u, v, and w in N that all three have neighbors in K_1 and K_2, then let u_i, v_i, and w_i denote the corresponding neighbors in K_i for $i \in \{1,2\}$, respectively. If any of u_1, v_1, and w_1 are distinct, then $G[K_1]$ contains a path between two of the vertices u_1, v_1, and w_1 avoiding the third, and G contains a cycle of order at most $2+(|K_1|-1)+|K_2| = g-1$, which is a contradiction. By symmetry, this implies $u_1=v_1=w_1$ and $u_2=v_2=w_2$, and G contains the cycle $u_1uu_2vu_1$ of order 4, which is a contradiction. Thus, $0 \leq |N_G(K_1) \cap N_G(K_2)| \leq 2$.

Combining these observations, we obtain

$$\sum_{1\leq i<j\leq p} |N_G(K_i) \cap N_G(K_j)| \leq \begin{cases} \binom{p}{2}, & \text{for } p \geq 3, \text{ and} \\ 2p-2, & \text{for } p \in \{1,2\}. \end{cases} \tag{2}$$

Let the bipartite graph H arise from $G[X \cup N]$ by contracting the component K_i of $G[X]$ to a single vertex u_i for every $i \in [p]$, and removing all edges of $G[N]$. Note that $\sum_{i\in[p]} d_H(u_i) - \sum_{v\in N} d_H(v) = 0$ in the bipartite graph H with partite sets $\{u_1, \ldots, u_p\}$ and N. By the girth condition, no vertex in N has two neighbors in K_i, and K_i induces a tree, which implies $d_H(u_i) = \sum_{v\in K_i} d_G(v) - 2(|K_i|-1) \geq$

$\delta|K_i| - 2(|K_i| - 1)$ for every $i \in [p]$. Let $q = \sum_{v\in N}(d_H(v) - 1)$. Now, Lemma 1 implies

$$\begin{aligned}
Z(G) \geq \delta_{g-2}(G) = |N| &= \sum_{v\in N} 1 + \left(\sum_{i\in[p]} d_H(u_i) - \sum_{v\in N} d_H(v)\right)\\
&= \sum_{i\in[p]} d_H(u_i) - q \geq \sum_{i=1}^{p}\Big(\delta|K_i| - 2(|K_i| - 1)\Big) - q\\
&= (g-2)(\delta-2) + 2 + ((2p-2) - q).
\end{aligned}$$

If $q \leq 2p-2$, then this implies (1). Hence, we may assume $q \geq 2p-1$.

Note that

$$2p - 1 \leq q = \sum_{v\in N}(d_H(v) - 1) \leq \sum_{v\in N}\binom{d_H(v)}{2} = \sum_{1\leq i<j\leq p}|N_G(K_i)\cap N_G(K_j)|,$$

where the last equality follows, because every vertex v in N contributes exactly $\binom{d_H(v)}{2}$ to the right hand side. Now, (2) implies $p \geq 5$.

Let H' arise by removing all vertices of degree 1 from H. Since, for every $i \in [p]$, we have $d_H(u_i) \geq \delta|K_i| - 2(|K_i| - 1) \geq 2$, the graph H' contains all p vertices $u_1,\dots,u_p$. Let H' contain r vertices of N. Since H' has order $p+r$ and size

$$\sum_{v\in N\cap V(H')} d_H(v) = r + \sum_{v\in N}(d_H(v) - 1) = r + q,$$

its average degree is at least $\frac{2(r+q)}{p+r}$, which is at least 2, because $q \geq 2p-1 \geq p$.

If H' contains a cycle of order 2ℓ, then G contains a cycle that alternates between X and N, contains ℓ vertices from N, and avoids $p-\ell$ of the components of $G[X]$, which implies that this cycle has order at most $\ell + (|X| - (p-\ell)) = \ell + (g-2) - (p-\ell)$. By the girth condition, this implies that the bipartite graph H' has girth at least $p+2$, if p is even, and $p+3$, if p is odd.

Using Theorem 2 and $q \geq r$, we obtain

$$\begin{aligned}
p + r &\geq 2\sum_{i=0}^{\lceil\frac{p}{2}\rceil}\left(\frac{2(r+q)}{p+r} - 1\right)^i = 2\frac{p+r}{2(q-p)}\left(\left(1 + \frac{2(q-p)}{p+r}\right)^{\lceil\frac{p}{2}\rceil+1} - 1\right)\\
&\geq 2\frac{p+r}{2(q-p)}\left(\left(1 + \frac{2(q-p)}{p+q}\right)^{\lceil\frac{p}{2}\rceil+1} - 1\right),
\end{aligned}$$

which implies $\left(1 + \frac{2(q-p)}{q+p}\right)^{\lceil\frac{p}{2}\rceil+1} \leq q - p + 1$. Since $q \geq 2p-1$, and, by (2), $q \leq \binom{p}{2}$, this contradicts Lemma 3, which completes the proof. ■

References

[1] AIM Minimum Rank - Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. Cioabă, D. Cvetković, S. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K.V. Meulen and A.W. Wehe), *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl. **428** (2008) 1628–1648. doi:10.1016/j.laa.2007.10.009

[2] N. Alon, S. Hoory and N. Linial, *The Moore bound for irregular graphs*, Graphs Combin. **18** (2002) 53–57. doi:10.1007/s003730200002

[3] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, *Parameters related to tree-width, zero forcing, and maximum nullity of a graph*, J. Graph Theory **72** (2013) 146–177. doi:10.1002/jgt.21637

[4] A. Berman, S. Friedland, L. Hogben, U.G. Rothblum and B. Shader, *An upper bound for the minimum rank of a graph*, Linear Algebra Appl. **429** (2008) 1629–1638. doi:10.1016/j.laa.2008.04.038

[5] D. Burgarth and V. Giovannetti, *Full control by locally induced relaxation*, Phys. Rev. Lett. **99** (2007) 100501. doi:10.1103/PhysRevLett.99.100501

[6] D. Burgarth, V. Giovannetti, L. Hogben, S. Severini and M. Young, *Logic circuits from zero forcing*, Nat. Comput. **14** (2015) 485–490. doi:10.1007/s11047-014-9438-5

[7] D. Burgarth and K. Maruyama, *Indirect Hamiltonian identification through a small gateway*, New J. Phys. **11** (2009) 103019. doi:10.1088/1367-2630/11/10/103019

[8] L.S. Chandran and C.R. Subramanian, *Girth and treewidth*, J. Combin. Theory Ser. B **93** (2005) 23–32. doi:10.1016/j.jctb.2004.05.004

[9] R. Davila and M. Henning, *The forcing number of graphs with given girth*, arXiv:1610.08435v1.

[10] R. Davila, T. Kalinowski and S. Stephen, *Proof of a conjecture of Davila and Kenter regarding a lower bound for the forcing number in terms of girth and minimum degree*. arXiv:1611.06557v2

[11] R. Davila and F. Kenter, *Bounds for the zero forcing number of graphs with large girth*, Theory Appl. Graphs **2(2)** (2015) Art. 1. doi:10.20429/tag.2015.020201

[12] M. Gentner, L.D. Penso, D. Rautenbach and U.S. Souza, *Extremal values and bounds for the zero forcing number*, Discrete Appl. Math. **214** (2016) 196–200. doi:10.1016/j.dam.2016.06.004

[13] M. Gentner and D. Rautenbach, *Some bounds on the zero forcing number of a graph*, Discrete Appl. Math. **236** (2018) 203–213. doi:10.1016/j.dam.2017.11.015

[14] S. Severini, *Nondiscriminatory propagation on trees*, J. Phys. A **41** (2008) 48. doi:10.1088/1751-8113/41/48/482002

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