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CONFLICT-FREE VERTEX-CONNECTIONS OF GRAPHS

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Abstract

A path in a vertex-colored graph is called *conflict-free* if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be *conflict-free vertex-connected* if any two vertices of the graph are connected by a conflict-free path. This paper investigates the question: for a connected graph G, what is the smallest number of colors needed in a vertex-coloring of G in order to make G conflict-free vertex-connected. As a result, we get that the answer is easy for 2-connected graphs, and very difficult for connected graphs with more cut-vertices, including trees.

Keywords: vertex-coloring, conflict-free vertex-connection, 2-connected graph, tree.

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to a book [1] for undefined notation and terminology in graph theory. A path in an edge-colored graph is a *rainbow path* if its edges have different colors. An edge-colored graph is *rainbow connected* if any two vertices of the graph are connected by a rainbow path of the graph. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colors required to make G rainbow connected. This concept was first introduced by Chartrand *et al.* in [4,5]. Since then, a lot of results on the rainbow connection have been obtained; see [14, 15].

As a natural counterpart of the concept of rainbow connection, the concept of rainbow vertex connection was first introduced by Krivelevich and Yuster in [10]. A path in a vertex-colored graph is a *vertex-rainbow path* if its internal vertices have different colors. A vertex-colored graph is *rainbow vertex-connected* if any two vertices of the graph are connected by a vertex-rainbow path of the graph. For a connected graph G, the *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is defined as the smallest number of colors required to make G rainbow vertex-connected. There are many results on this topic, we refer to [6, 11–13, 17].

In [7], Czap *et al.* introduced the concept of conflict-free connection. A path in an edge-colored graph is called *conflict-free* if there is a color used on exactly one of its edges. An edge-colored graph is said to be *conflict-free connected* if any two vertices of the graph are connected by a conflict-free path. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is defined as the smallest number of colors required to make G conflict-free connected. Note that for a nontrivial connected graph G with order n, we have

$$1 \leq cfc(G) \leq rc(G) \leq n - 1.$$

Moreover, $cfc(G) = 1$ if and only if G is a complete graph, and $cfc(G) = n - 1$ if and only if G is a star. For more results, we refer to [2, 3, 7, 8].

Motivated by the above mentioned concepts, as a natural counterpart of conflict-free connection number, in this paper we introduce the concept of conflict-free vertex-connection number. A path in a vertex-colored graph is called *conflict-free* if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be *conflict-free vertex-connected* if any two vertices of the graph are connected by a conflict-free path. The *conflict-free vertex-connection number* of a connected graph G, denoted by $vcfc(G)$, is defined as the smallest number of colors required to make G conflict-free vertex-connected. Note that for a nontrivial connected graph G with order n, we can easily observe that

$$2 \leq vcfc(G) \leq n.$$

The lower bound is trivial since there is a path of order at least two between any two vertices in G, while the upper bound is also trivial since one may color all the vertices of G with distinct colors. The main problem studied in this paper is the following.

Problem 1.1. For a given graph G, determine its conflict-free vertex-connection number.

The rest of this paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we study the structure of graphs having conflict-free vertex-connection number two and three respectively. In Section 4, we obtain some sharp bounds of the conflict-free vertex-connection number for trees.

2. Preliminaries

The following observation is immediate.

Observation 1. *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $vcfc(G) \leq vcfc(H)$. In particular, $vcfc(G) \leq vcfc(T)$ for every spanning tree T of G.*

Lemma 2.1. *Let G be a 2-connected graph and w be a vertex of G. Then for any two vertices u and v in G, there is a u-v path containing the vertex w.*

Proof. It is clearly true for the case that $w \in \{u, v\}$ since G is 2-connected. Now suppose that $w \in V(G)\backslash\{u, v\}$. Let P_1 and P_2 be two internally vertex disjoint paths from u to w in G. If there is a v-w path P such that P and P_1 are vertex-disjoint except for the vertex w, then the path uP_1wPv is the desired path. Otherwise, let x be the first common vertex of P and P_1 when going along P from v. Then the path uP_2wP_1xPv is the desired path. ■

For a path, we have the following result.

Theorem 2.1. *Let P_n be a path of order n. Then $vcfc(P_n) = \lceil \log_2(n+1) \rceil$.*

Proof. The proof goes similarly to that of Theorem 3 in [7]. Let $P_n = v_1v_2 \cdots v_n$. First we show that $vcfc(P_n) \leq \lceil \log_2(n+1) \rceil$. Define a vertex-coloring of P_n by coloring the vertex v_i with color $x+1$, where $i \in [n]$ and 2^x is the largest power of 2 that divides i. Clearly, the largest number in such a coloring is $\lceil \log_2(n+1) \rceil$. Moreover, it is easy to check that the maximum color of the vertices on each subpath Q of P_n appears only once on Q. Then P_n is conflict-free vertex-connected, and so $vcfc(P_n) \leq \lceil \log_2(n+1) \rceil$.

Next we just need to prove that $vcfc(P_n) \geq \lceil \log_2(n+1) \rceil$. To show it, it suffices to show that any path with conflict-free vertex-connection number

k has at most $2^k - 1$ vertices. We apply induction on k. The statement is evidently true for $k = 2$. Give the path P_n with $vcfc(P_n) = k$ a conflict-free vertex-connection k-coloring. Then there is a vertex, say v_i, in P_n with a unique color. Delete the vertex v_i from P_n. The resulting paths are $P_{i-1} = v_1 v_2 \cdots v_{i-1}$ and $P_{n-i} = v_{i+1} v_{i+2} \cdots v_n$ with $vcfc(P_{i-1}) \leq k - 1$ and $vcfc(P_{n-i}) \leq k - 1$. By the induction hypothesis, P_{i-1} and P_{n-i} have at most $2^{k-1} - 1$ vertices, respectively. Thus P_n has at most $2\left(2^{k-1} - 1\right) + 1 = 2^k - 1$ vertices, and so $vcfc(P_n) \geq \lceil \log_2(n+1) \rceil$.

Therefore, $vcfc(P_n) = \lceil \log_2(n+1) \rceil$. ∎

Remark 1. From Theorem 2.1 and [7, Theorem 3], we have that $vcfc(P_n) \geq cfc(P_n)$. However, $vcfc(G) \leq cfc(G)$ if G is a star of order at least 3. Thus, one of $vcfc(G)$ and $cfc(G)$ cannot be bounded in terms of the other.

3. Graphs with Conflict-Free Vertex-Connection Number Two or Three

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Then the block is either a cut-edge, say *trivial block*, or a maximal 2-connected subgraph, say *nontrivial block*. Let $B_1, B_2, \ldots, B_k$ be the blocks of G. The *block graph* of G, denoted by $B(G)$, has vertex-set $\{B_1, B_2, \ldots, B_k\}$ and B_iB_j is an edge if and only if the blocks B_i and B_j have a cut-vertex in common, where $1 \leq i, j \leq k$.

The following lemma is a preparation of Theorem 3.1.

Lemma 3.1. *Let G be a 2-connected graph. Then $vcfc(G) = 2$.*

Proof. Since $vcfc(G) \geq 2$, we just need to show that $vcfc(G) \leq 2$. Let w be a vertex of G. Define a 2-coloring c of the vertices of G by coloring the vertex w with color 2 and all the other vertices of G with color 1. By Lemma 2.1, for any two vertices u and v in G, there is a u-v path containing the vertex w. According to the coloring c of G, this u-v path is a conflict-free path. Thus $vcfc(G) \leq 2$, and the proof is complete. ∎

From Theorem 2.1 and Lemma 3.1, we have the following result.

Corollary 3.1. *For the complete graph K_n with $n \geq 2$, $vcfc(K_n) = 2$.*

After the above preparation, graphs with $vcfc(G) = 2$ can be characterized.

Theorem 3.1. *Let G be a connected graph of order at least 3. Then $vcfc(G) = 2$ if and only if G is 2-connected or G has only one cut-vertex.*

Proof. Firstly, we prove its sufficiency. If G is 2-connected, then it follows from Lemma 3.1 that $vcfc(G) = 2$. Now suppose that G has exactly one cut-vertex, say w. Since $vcfc(G) \geq 2$, we just need to show that $vcfc(G) \leq 2$. Define a 2-coloring c of the vertices of G by coloring the vertex w with color 2 and all the other vertices with color 1. Since G has only one cut-vertex, it follows that G consists of some blocks which have the common vertex w. Next it remains to check that for any two vertices u and v in G, there is a conflict-free path between them. It is clearly true for the case that $w \in \{u, v\}$. Thus we may assume that $w \in V(G)\backslash\{u, v\}$. If u and v are in the same block, then the block must be nontrivial. From Lemma 2.1 and the coloring c of G, we get that there is a conflict-free path from u to v in the block. If u and v are in two different blocks, then there is a u-w path P_1 and a v-w path P_2 in the two blocks, respectively. Clearly, the path uP_1wP_2v is the desired path.

Now, we show its necessity. Let $vcfc(G) = 2$. By Lemma 3.1, it remains to show that if G is not 2-connected, then G has only one cut-vertex. Suppose that G has at least two cut-vertices. Let B_1 and B_2 be two blocks in G each of which contains only one cut-vertex, respectively. Moreover, denote by v_1 and v_2 the cut-vertices in B_1 and B_2, respectively. Note that for any two vertices in the same block, all paths connecting them are in the block. Thus, each block needs two colors. Let u_1 be the vertex whose color is different from v_1 in B_1 and u_2 be the vertex whose color is different from v_2 in B_2. Clearly, all paths from u_1 to u_2 in G must pass through the vertices v_1 and v_2. However, the four vertices u_1, v_1, u_2, v_2 use each of two colors twice. Thus there does not exist a conflict-free path between u_1 and u_2 in G, a contradiction. Hence G has only one cut-vertex. ∎

The following corollary is immediate from Theorem 3.1.

Corollary 3.2. *Let G be a connected graph. Then $vcfc(G) \geq 3$ if and only if G has at least two cut-vertices.*

Next we give two sufficient conditions for a graph G to have $vcfc(G) = 3$.

Theorem 3.2. *Let G be a connected graph of order n with maximum degree $\Delta(G)$. If G has at least two cut-vertices and $n - 4 \leq \Delta(G) \leq n - 2$, then $vcfc(G) = 3$.*

Proof. Since G has at least two cut-vertices, it follows that $vcfc(G) \geq 3$ by Corollary 3.2, and so we only need to show that $vcfc(G) \leq 3$. We distinguish the following three cases to show this theorem.

Case 1. $\Delta(G) = n - 2$. In this case, G must have a spanning tree T_1 shown in Figure 1. Moreover, a 3-coloring of the vertices of T_1 is shown in Figure 1 to

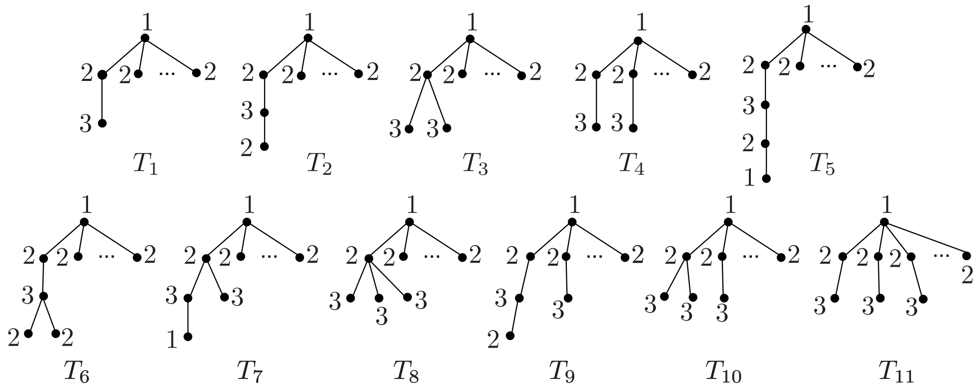

Figure 1. The eleven graphs in Theorem 3.

make T_1 conflict-free vertex-connected. Thus $vcfc(T_1) \leq 3$. From Observation 1, we have $vcfc(G) \leq vcfc(T_1)$, and hence $vcfc(G) \leq 3$.

Case 2. $\Delta(G) = n - 3$. Since $\Delta(G) = n - 3$, it follows that G must have a spanning tree depicted as one of T_is $(2 \leq i \leq 4)$ shown in Figure 1. For $2 \leq i \leq 4$, a 3-coloring of the vertices of T_i is shown in Figure 1 to make T_i conflict-free vertex-connected. From Observation 1, we have $vcfc(G) \leq vcfc(T_i) \leq 3$.

Case 3. $\Delta(G) = n - 4$. Since $\Delta(G) = n - 4$, it follows that G must have a spanning tree depicted as one of T_is $(5 \leq i \leq 11)$ shown in Figure 1. For $5 \leq i \leq 11$, a 3-coloring of the vertices of T_i is shown in Figure 1 to make T_i conflict-free vertex-connected. From Observation 1, we have $vcfc(G) \leq vcfc(T_i) \leq 3$.

From the above argument, we conclude that $vcfc(G) = 3$. ■

Remark 2. The condition on the maximum degree above cannot be improved, since if G is T' shown in Figure 2, then $\Delta(G) = n-5$ and $vcfc(G) = 4$. Note that there is only one path between any two vertices in a tree. Then any two adjacent vertices in T' need two different colors. Considering this, we can check that three colors cannot make T' conflict-free vertex-connected and so $vcfc(T') \geq 4$. Moreover, a 4-coloring of the vertices of T' is shown in Figure 2 to make T' conflict-free vertex-connected. Hence $vcfc(T') = 4$.

Let $C(G)$ denote the subgraph of G induced by the set of cut-edges of G.

Theorem 3.3. *Let G be a connected graph with at least two cut-vertices. If $C(G)$ is a star and each nontrivial block has a common vertex with $C(G)$, then $vcfc(G) = 3$.*

Proof. By Corollary 3.2, it suffices to show that $vcfc(G) \leq 3$, since G has at least two cut-vertices. Let $V(C(G)) = \{v_0, v_1, \ldots, v_t\}$, where $t \geq 1$ and v_0 is the

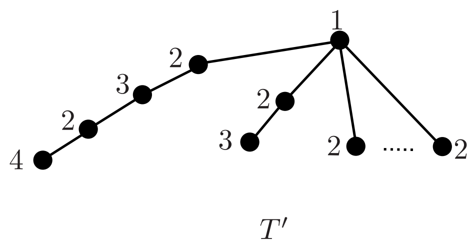

Figure 2. A 4-coloring of the vertices of T'.

center of the star $C(G)$. Define a 3-coloring c of the vertices of G by coloring the vertex v_0 with color 1, the pendant vertices $\{v_1, \ldots, v_t\}$ of $C(G)$ with color 2 and all the other vertices with color 3. Next, it remains to check that for any two vertices u and v in G, there is a conflict-free path between them. If $u, v \in V(C(G))$, then the desired path is the unique path from u to v in $C(G)$. If u and v belong to the same nontrivial block, then by Lemma 2.1, there is a u-v path in the block containing the vertex which is also in $C(G)$. Clearly, this path is the desired path. Now we may assume that u and v are in two different nontrivial blocks B and B'. If B and B' do not have a common vertex, then a shortest u-v path in G must go through the center v_0 which has the unique color 1 and so it is the desired path. Otherwise, B and B' have a unique common vertex v_i $(0 \leq i \leq t)$ which has the unique color $c(v_i)$ on the u-v path. Thus, $vcfc(G) \leq 3$, and the proof is complete. ■

The *t-corona* of a graph H, denoted by $Cor_t(H)$, is a graph obtained from H by adding t pendant edges to each vertex of H.

Proposition 3.4. *Let C_n be a cycle and G be its t-corona, where $t \geq 1$. Then $vcfc(G) = 3$.*

Proof. Since G has at least three cut-vertices, we have $vcfc(G) \geq 3$ by Theorem 3.1, and so it remains to show that $vcfc(G) \leq 3$. Define a 3-coloring c of the vertices of G by coloring all the pendant vertices with color 1, one of the vertices of C_n with color 2 and the other vertices with color 3. It is easy to check that for any two vertices of G, there is a conflict-free path between them. Then $vcfc(G) \leq 3$, and we complete the proof. ■

It seems that it is not easy to characterize graphs G with $vcfc(G) = 3$. But, below we study a family of graphs with conflict-free vertex-connection number three. Before it, we provide the concept of a segment: Let G be a connected graph whose block graph is a path. Let $B_1, B_2, \ldots, B_k$ be the blocks of G such that $|V(B_i) \cap V(B_{i+1})| = 1$ and $E(B_i) \cap E(B_{i+1}) = \emptyset$ $(1 \leq i \leq k-1)$. We call $F_{p,q}$ $(1 \leq p \leq q \leq k)$ a *segment* of G if $F_{p,q} = \bigcup_{i=p}^{q} B_i$.

Theorem 3.5. *Let G be a connected graph with at least two cut-vertices, and its block graph $B(G)$ is a path. Then $vcfc(G) = 3$ if and only if G is a segment of one type of the thirteen graphs listed below.*

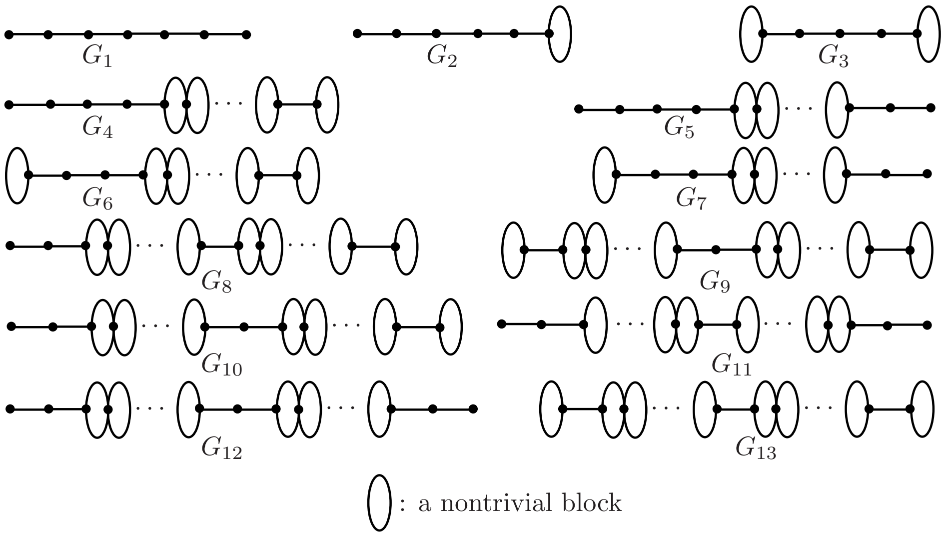

Figure 3. The thirteen types of graphs.

Proof. First we show its sufficiency. Let G be a segment of one type of the thirteen graphs listed in Figure 3. Since G has at least two cut-vertices, $vcfc(G) \geq 3$ according to Corollary 3.2 and so we just need to show that $vcfc(G) \leq 3$, Namely, we should give a 3-coloring of the vertices in G such that G is conflict-free vertex-connected. Since G is a segment of one type of the thirteen graphs listed in Figure 3, we only consider the cases when $G = G_i$ $(1 \leq i \leq 13)$.

If $G = G_1$, we assign the color sequence $1, 2, 1, 3, 1, 2, 1$ to the vertices from left to right. If $G = G_2$, we assign the color sequence $1, 2, 1, 3, 1, 2$ to the six cut-vertices from left to right and the remaining vertices are assigned to the color 1. If $G = G_3$, we assign the color sequence $2, 1, 3, 1, 2$ to the five cut-vertices from left to right and the remaining vertices are assigned to the color 1.

Assume $G = G_6$. Let $B, B_1, B_2, \ldots, B_t, B'$ be the nontrivial blocks from left to right. Assign the color sequence $2, 1, 3$ to the left three cut-vertices. If t is odd, assign the rightmost cut-vertex with color 3; otherwise, assign it with color 2. Pick a vertex avoiding the cut-vertex from each B_i $(1 \leq i \leq t)$. Assign the color 2 to all these vertices if i is odd and assign the color 3 otherwise. All remaining vertices are assigned to the color 1. For the case $G = G_4$, we only need to view the leftmost vertex in G_4 as the vertices colored 1 of B in G_6 and the other vertices in G_4 are colored as those in G_6. Similarly we can give a 3-coloring for $G = G_5$ and $G = G_7$. Namely view the leftmost vertex and rightmost vertex in G_5 as the vertices colored 1 of B and B' in G_6 respectively, and view the rightmost vertex in G_7 as the vertices colored 1 of B' in G_6. Following this structural law, we only give the 3-coloring of G when $G = G_9$ for $G = G_9, G_{10}$ or G_{12}, and give the 3-coloring of G when $G = G_{13}$ for $G = G_8, G_{11}$ or G_{13} in the following.

Assume $G = G_9$. Denote by $B_0, B_{11}, B_{12}, \ldots, B_{1s}$ $(s \geq 1)$, $B_{21}, B_{22}, \ldots, B_{2t}$ $(t \geq 1), B'_0$ the nontrivial blocks and $v_0, v_1, v_2, \ldots, v_{s+1}, v, v'_1, v'_2, \ldots, v'_{t+1}, v'_0$ the cut-vertices from left to right, respectively. First consider the case that s is odd. Pick a vertex avoiding the cut-vertex from each of B_{1i} and B_{2j} when i and j are even. Assign the color 2 to all these vertices and v_0, v. Moreover, pick a vertex avoiding the cut-vertex from each of B_{1i} and B_{2j} when i and j are odd. Give all these vertices the color 3. For v'_0, if t is odd, color it with color 2; otherwise, color it with color 3. All remaining vertices are assigned to the color 1. Then consider the case that s is even. Pick a vertex avoiding the cut-vertex from each B_{1i} when i is even and a vertex from each B_{2j} when j is odd. Assign the color 2 to all these vertices and v_0. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i is odd and a vertex from each B_{2j} when j is even. Give all these vertices and v the color 3. For v'_0, if t is odd, color it with color 3; otherwise, color it with color 2. All remaining vertices are assigned to the color 1.

Assume $G = G_{13}$. Denote by $B_0, B_{11}, B_{12}, \ldots, B_{1s}$ $(s \geq 1)$, $B_{21}, B_{22}, \ldots, B_{2t}$ $(t \geq 1)$, B'_0 the nontrivial blocks and $v_0, v_1, v_2, \ldots, v_{s+1}, v'_1, v'_2, \ldots, v'_{t+1}, v'_0$ the cut-vertices from left to right, respectively. First consider the case that s is odd. Pick a vertex avoiding the cut-vertex from each B_{1i} when i is even and a vertex from each B_{2j} when j is odd and $3 \leq j \leq t$. Assign the color 2 to all these vertices and v_0, v'_1. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} when i is odd and a vertex from each B_{2j} when j is even. Give all these vertices the color 3. For v'_0, if t is odd, color it with color 3; otherwise, color it with color 2. All remaining vertices are assigned to the color 1. Then consider the case that s is even. Pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i and j are even. Assign the color 2 to all these vertices and v_0. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i and j are odd and $3 \leq j \leq t$. Give all these vertices and v'_1 the color 3. For v'_0, if t is odd, color it with color 2; otherwise, color it with color 3. All remaining vertices are assigned to the color 1.

Using Lemma 2.1, it can be easily checked that the conflict-free paths can be found between the vertices of the same block. For other pairs of vertices, we can also find conflict-free paths between them under the above colorings for $G = G_i$ $(1 \leq i \leq 13)$. Thus $vcfc(G) = 3$.

Next we show its necessity. Let $B_1, \ldots, B_n$ be blocks of G such that $|V(B_i) \cap V(B_{i+1})| = 1$ and $v_i \in V(B_i) \cap V(B_{i+1})$ be the cut-vertex of G, where $1 \leq i \leq n-1$. Let $\mathcal{F}$ be the family of graphs listed in Theorem 3.5. From the sufficient part of the proof we know that for $G \in \mathcal{F}$ there is $vcfc(G) = 3$. Suppose there is another graph G with $vcfc(G) = 3$ but $G \notin \mathcal{F}$. Let ϕ be a conflict-free vertex-connection coloring of G with colors from $\{a, b, c\}$ and c be a color with the least number of appearances on cut-vertices of G. Then the following two claims are evident.

Claim 1. *Any three consecutive blocks B_{i-1}, B_i, B_{i+1} contain on their vertices all three colors.*

Claim 2. *Color c appears at most once on the cut-vertices of G.*

Claim 3. *G does not contain a segment of seven consecutive blocks $B_{i-1}, \ldots, B_{i+5}$ such that all three blocks $B_{i+1}, B_{i+2}, B_{i+3}$ are trivial.*

Proof. If such a segment exists, without loss of generality let $\phi(v_{i+1}) = \phi(v_{i+3}) = a, \phi(v_i) = b$ and $\phi(v_{i+2}) = c$. From Claims 1 and 2, there are vertices $x \in V(B_{i-1}) \cup V(B_i)$ with $\phi(x) = c$ and $y \in V(B_{i+4}) \cup V(B_{i+5})$ with $\phi(y) = b$, and any x-y path is a conflict one. □

Consider the following segment $\mathbf{S}$ of blocks of the graph G:

$$\mathbf{S} = B_{i-1}, B_i, B_{i+1}, \ldots, B_j, B_{j+1}, \ldots, B_k, B_{k+1}, B_{k+2}, B_{k+3},$$

where B_{i+1} and B_{k+1} are trivial, B_{j+1} is trivial only if $C(G)$ has at least three components, and the three blocks B_{i-1}, B_j, B_k are always nontrivial; the remaining blocks may be, in dependence on the components of $C(G)$, either trivial or nontrivial. We suppose, without loss of generality, that blocks B_{i+1}, B_{j+1} and B_{k+1} belong to three different consecutive components of $C(G)$ (if exist).

Claim 4. *Both B_{i-1} and B_{k+3} are not present simultaneously in G.*

Proof. If both are present, then, because of Claims 1 and 2, there exist two vertices $x \in V(B_{i-1}) \cup V(B_i)$ with $\{\phi(x), \phi(v_i), \phi(v_{i+1})\} = \{a, b, c\}$ and $y \in V(B_{k+2}) \cup V(B_{k+3})$ with $\{\phi(y), \phi(v_k), \phi(v_{k+1})\} = \{a, b, c\}$, such that every x-y path is conflict. □

Let $C(G)$ contain a path on at least four vertices as a component. Then, by Claim 4, we can put $i = 1$ and suppose that the block B_0 is not present in $\mathbf{S}$. If $C(G)$ contains P_7 or P_6 as a component, then either G contains a graph from $\{G_1, G_2\}$ as a proper segment and so we have a contradiction with Claim 3 or G is contained by G_1 or G_2 which is also a contradiction. If $C(G)$ contains P_5 as a component and G contains G_3 as a proper segment, we have a contradiction with Claim 3. Thus we can suppose that either $C(G)$ contains P_5 as a component and G does not contain G_3 as a proper segment or $C(G)$ contains P_4 as a component. If, in $\mathbf{S}$, the blocks B_2, B_3, B_4 are all trivial (B_1 is also trivial if G contains P_5), and no other block is trivial in it, then $\mathbf{S}$ and hence G, is in $\mathcal{F}$. If there is another trivial block B_{j+1} in G, let v_j be its first cut-vertex incident with this block. Because of Claim 2, $\{\phi(v_j), \phi(v_{j+1})\} = \{a, b\}$, then $j = k$. Otherwise there are two more blocks B_{j+2} and B_{j+3} containing a vertex y with $\phi(y) = c$, and there is a vertex $x \in V(B_1) \setminus \{v_1\}$ such that every x-y path is conflict. The existence of the block B_{j+3} in the case $j = k$ yields a contradiction analogously as above. If the block B_{j+3} is not in **S**, then G is in $\mathcal{F}$, again a contradiction.

Let $C(G)$ do not contain a path on at least four vertices. By the set $\mathcal{F}$ we can suppose that $C(G)$ has at least two components. Observe that **S** contains at least two components of $C(G)$ and, by the definition of **S**, at most three components of $C(G)$. By Claim 4, at most one of B_{i-1} and B_{k+3} is present in G.

Then, without loss of generality we can suppose that the last block of **S** is B_{k+2} and that there is the block B_{i-1}. If there are only two components for $C(G)$, then surely **S** and hence G, is in $\mathcal{F}$, a contradiction. When $C(G)$ has three components, which allows $i + 2 < k - 1$. We discuss this case as follows.

Case 1. There is $m \in \{i + 2, \ldots, k - 1\}$ such that $\phi(v_m) = c$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ with $\phi(x) = c$ and every x-v_{k+1} path is conflict, a contradiction.

Case 2. There is no $m \in \{i + 2, \ldots, k - 1\}$ such that $\phi(v_m) = c$.

Case 2.1. Let $c \in \{\phi(v_i), \phi(v_{i+1})\}$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ such that $\{\phi(x), \phi(v_i), \phi(v_{i+1})\} = \{a, b, c\}$ and a vertex $y \in V(B_{k-1}) \cup V(B_k) \cup V(B_{k+1})$ with $\phi(y) = c$. Then every x-y path is conflict, a contradiction.

Case 2.2. Let $c \notin \{\phi(v_i), \phi(v_{i+1})\}$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ such that $\phi(x) = c$ and a vertex $y \in V(B_{j+2}) \cup V(B_{j+3})$ such that $\{\phi(v_j), \phi(v_{j+1}), \phi(y)\} = \{a, b, c\}$. As a result, every x-y path is conflict, a contradiction.

If both B_{i-1} and B_{k+3} are not present, then **S** and hence G, is in $\mathcal{F}$, a contradiction except for the case that B_{i+2} is trivial, B_i is nontrivial and there exists a trivial block B_{j+1}. But in this case, we can find a vertex $x \in B_i$ with $\{\phi(x), \phi(v_{i+1}), \phi(v_{i+2})\} = \{a, b, c\}$ and a vertex $y \in B_{j+2} \cup B_{j+3}$ with $\{\phi(y), \phi(v_j), \phi(v_{j+1})\} = \{a, b, c\}$, which also leads to the contradiction that there is no conflict-free path between x and y.

This finishes the proof of the necessary part of Theorem 5.

The proof is complete. ■

At the end of this section, we pose the following problem.

Problem 3.1. Characterize all the graphs G with $vcfc(G) = 3$.

4. Trees

A *k-ranking* of a connected graph G is a labeling of its vertices with labels $1, 2, 3, \ldots, k$ such that every path between any two vertices with the same label i in G contains at least one vertex with label $j > i$. A graph G is said to be *k-rankable* if it has a k-ranking. The minimum k for which G is k-rankable is denoted by $r(G)$.

Iyer [9] obtained the following result.

Lemma 4.1 [9]**.** *Let T be a tree of order $n \geq 3$. Then $r(T) \leq \log_{\frac{3}{2}} n$.*

The next two lemmas are preparations for Theorem 4.1.

Lemma 4.2. *Let G be a connected graph. Then $vcfc(G) \leq r(G)$.*

Proof. Consider a ranking of the vertices of G. For any two vertices u and v of G, let P be a path between them and k be the maximum label of the vertices of P. If there is only one vertex with label k in P, then the proof is done. So we assume that P contains at least two vertices with label k. According to the definition of ranking, there must exist one vertex with label $j > k$ on P, which is a contradiction. Hence P contains only one vertex with label k. View the $r(G)$-ranking of G as its vertex-coloring with $r(G)$ colors. Then the path P is a conflict-free path between u and v in G. Thus $vcfc(G) \leq r(G)$. ∎

Lemma 4.3. *Let T be a nontrivial tree. Then $vcfc(T) \geq \chi(T)$, where $\chi(T)$ is the chromatic number of T and the bound is sharp.*

Proof. Define a vertex-coloring of T with $vcfc(T)$ colors such that T is conflict-free vertex-connected. Since there is only one path between any two vertices in T, it follows that any two adjacent vertices must have different colors, and hence $vcfc(T) \geq \chi(T)$. To show the sharpness of the bound, we let T be a star of order at least two. Clearly, $\chi(T) = 2$. By Theorem 3.1, we have $vcfc(T) = 2$ $(= \chi(T))$. ∎

Combining the lemmas above, we can have the following bounds for $vcfc(T)$ of a tree T.

Theorem 4.1. *Let T be a tree of order $n \geq 3$ and $d(T)$ be its diameter. Then*

$$\max\left\{\chi(T), \lceil \log_2(d(T)+2) \rceil\right\} \leq vcfc(T) \leq \log_{\frac{3}{2}} n.$$

Proof. The lower bound is an immediate result from Lemma 4.3 and Theorem 2.1, while the upper bound can be deduced from Lemmas 4.1 and 4.2. ∎

Let G be a connected graph. The *eccentricity* $\epsilon_G(v)$ of a vertex v in G is the maximum value among the distances between v and the other vertices in G. The *radius* $rad(G)$ of G is the minimum eccentricity among all the vertices of G. A *central vertex* of radius $rad(G)$ is one whose eccentricity is $rad(G)$. Remind that $d_G(u, v)$ is the shortest distance between the two vertices u and v in G.

Theorem 4.2. *Let T be a tree with radius $rad(T)$. Then $vcfc(T) \leq rad(T) + 1$. Moreover, the bound is sharp.*

Proof. Let v be a central vertex of radius $rad(T)$ in T. Let $V_i = \{u \in V(T) : d_T(u, v) = i\}$, where $0 \leq i \leq rad(T)$. Hence $V_0 = \{v\}$. Define a vertex-coloring c of T with $rad(T) + 1$ colors by coloring the vertices of V_i with color $i + 1$, where $0 \leq i \leq rad(T)$. It is easy to check that for any two vertices of T, there is a conflict-free path between them, and hence $vcfc(T) \leq rad(T) + 1$. To show the sharpness of the bound, we let T be a star of order at least two. Clearly, $rad(T) = 1$. By Theorem 3.1, we have $vcfc(T) = 2$ $(= rad(T) + 1)$. ∎

For each connected graph G, we can always find a spanning tree T of G such that $rad(T) = rad(G)$. From Observation 1 and Theorem 4.2, we can get the following result.

Corollary 4.1. *Let G be a connected graph. Then $vcfc(G) \leq rad(G) + 1$.*

For trees, we can give an upper bound of its conflict-free vertex-connection number in term of its order.

Proposition 4.3. *Let T be a tree with order $n \geq 5$. Then $vcfc(T) \leq \lceil \frac{n}{2} \rceil$. Moreover, the bound is sharp.*

Proof. If T is a path, then it follows from Theorem 2.1 that $vcfc(T) = \lceil \log_2 (n+1) \rceil \leq \lceil \frac{n}{2} \rceil$. From now on, we suppose that T is not a path. Then the longest path in T has at most $n - 1$ vertices. So we have $rad(T) \leq \frac{n-1}{2}$ if n is odd and $rad(T) \leq \frac{n-2}{2}$ if n is even. By Theorem 4.2, we have $vcfc(T) \leq rad(T) + 1$, and hence $vcfc(T) \leq \lceil \frac{n}{2} \rceil$. To show the sharpness of the bound, we set $T = P_5$. Then $vcfc(T) = 3$ by Theorem 2.1 and $\lceil \frac{n}{2} \rceil = 3$. ∎

Let G be a nontrivial connected graph of order n. For $vcfc(G)$, it can be easily seen that the trivial lower bound is 2. Based on Observation 1, the upper bound can be attained when G is a tree. Note that the path P_n is a tree of order n. After experiments on the graphs with small order, we believe that P_n might be the one attaining the upper bound among trees. Recently, Li and Wu [16] have confirmed this as $vcfc(G) \leq vcfc(P_n)$.

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