Discussiones Mathematicae
Graph Theory 40 (2020) 51–65
doi:10.7151/dmgt.2116

CONFLICT-FREE VERTEX-CONNECTIONS OF GRAPHS

Xueliang Li[1], Yingying Zhang, Xiaoyu Zhu

Center for Combinatorics and LPMC
Nankai University
Tianjin 300071, China

e-mail: lxl@nankai.edu.cn
zyydlwyx@163.com
zhuxy@mail.nankai.edu.cn

Yaping Mao, Haixing Zhao

School of Mathematics and Statistics
Qinghai Normal University
Xining, Qinghai 810008, China

e-mail: maoyaping@ymail.com
h.x.zhao@163.com

and

Stanislav Jendrol'

Institute of Mathematics
P.J. Šafárik University
Jesenná 5, 04001 Košice, Slovakia

e-mail: stanislav.jendrol@upjs.sk

Abstract

A path in a vertex-colored graph is called *conflict-free* if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be *conflict-free vertex-connected* if any two vertices of the graph are connected by a conflict-free path. This paper investigates the question: for a connected graph G, what is the smallest number of colors needed in a vertex-coloring of G in order to make G conflict-free vertex-connected. As a result, we get that the answer is easy for 2-connected graphs, and very difficult for connected graphs with more cut-vertices, including trees.

Keywords: vertex-coloring, conflict-free vertex-connection, 2-connected graph, tree.

2010 Mathematics Subject Classification: 05C15, 05C40, 05C75.

[1]Corresponding author.

1. Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to a book [1] for undefined notation and terminology in graph theory. A path in an edge-colored graph is a *rainbow path* if its edges have different colors. An edge-colored graph is *rainbow connected* if any two vertices of the graph are connected by a rainbow path of the graph. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colors required to make G rainbow connected. This concept was first introduced by Chartrand *et al.* in [4,5]. Since then, a lot of results on the rainbow connection have been obtained; see [14, 15].

As a natural counterpart of the concept of rainbow connection, the concept of rainbow vertex connection was first introduced by Krivelevich and Yuster in [10]. A path in a vertex-colored graph is a *vertex-rainbow path* if its internal vertices have different colors. A vertex-colored graph is *rainbow vertex-connected* if any two vertices of the graph are connected by a vertex-rainbow path of the graph. For a connected graph G, the *rainbow vertex-connection number* of G, denoted by $rvc(G)$, is defined as the smallest number of colors required to make G rainbow vertex-connected. There are many results on this topic, we refer to [6, 11–13, 17].

In [7], Czap *et al.* introduced the concept of conflict-free connection. A path in an edge-colored graph is called *conflict-free* if there is a color used on exactly one of its edges. An edge-colored graph is said to be *conflict-free connected* if any two vertices of the graph are connected by a conflict-free path. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is defined as the smallest number of colors required to make G conflict-free connected. Note that for a nontrivial connected graph G with order n, we have

$$1 \leq cfc(G) \leq rc(G) \leq n - 1.$$

Moreover, $cfc(G) = 1$ if and only if G is a complete graph, and $cfc(G) = n - 1$ if and only if G is a star. For more results, we refer to [2, 3, 7, 8].

Motivated by the above mentioned concepts, as a natural counterpart of conflict-free connection number, in this paper we introduce the concept of conflict-free vertex-connection number. A path in a vertex-colored graph is called *conflict-free* if there is a color used on exactly one of its vertices. A vertex-colored graph is said to be *conflict-free vertex-connected* if any two vertices of the graph are connected by a conflict-free path. The *conflict-free vertex-connection number* of a connected graph G, denoted by $vcfc(G)$, is defined as the smallest number of colors required to make G conflict-free vertex-connected. Note that for a nontrivial connected graph G with order n, we can easily observe that

$$2 \leq vcfc(G) \leq n.$$

The lower bound is trivial since there is a path of order at least two between any two vertices in G, while the upper bound is also trivial since one may color all the vertices of G with distinct colors. The main problem studied in this paper is the following.

Problem 1.1. For a given graph G, determine its conflict-free vertex-connection number.

The rest of this paper is organized as follows. In Section 2, we prove some preliminary results. In Section 3, we study the structure of graphs having conflict-free vertex-connection number two and three respectively. In Section 4, we obtain some sharp bounds of the conflict-free vertex-connection number for trees.

2. PRELIMINARIES

The following observation is immediate.

Observation 1. *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $vcfc(G) \leq vcfc(H)$. In particular, $vcfc(G) \leq vcfc(T)$ for every spanning tree T of G.*

Lemma 2.1. *Let G be a 2-connected graph and w be a vertex of G. Then for any two vertices u and v in G, there is a u-v path containing the vertex w.*

Proof. It is clearly true for the case that $w \in \{u, v\}$ since G is 2-connected. Now suppose that $w \in V(G)\backslash\{u, v\}$. Let P_1 and P_2 be two internally vertex disjoint paths from u to w in G. If there is a v-w path P such that P and P_1 are vertex-disjoint except for the vertex w, then the path uP_1wPv is the desired path. Otherwise, let x be the first common vertex of P and P_1 when going along P from v. Then the path uP_2wP_1xPv is the desired path. ■

For a path, we have the following result.

Theorem 2.1. *Let P_n be a path of order n. Then $vcfc(P_n) = \lceil \log_2(n + 1)\rceil$.*

Proof. The proof goes similarly to that of Theorem 3 in [7]. Let $P_n = v_1v_2 \cdots v_n$. First we show that $vcfc(P_n) \leq \lceil \log_2(n + 1)\rceil$. Define a vertex-coloring of P_n by coloring the vertex v_i with color $x + 1$, where $i \in [n]$ and 2^x is the largest power of 2 that divides i. Clearly, the largest number in such a coloring is $\lceil \log_2(n + 1)\rceil$. Moreover, it is easy to check that the maximum color of the vertices on each subpath Q of P_n appears only once on Q. Then P_n is conflict-free vertex-connected, and so $vcfc(P_n) \leq \lceil \log_2(n + 1)\rceil$.

Next we just need to prove that $vcfc(P_n) \geq \lceil \log_2(n + 1)\rceil$. To show it, it suffices to show that any path with conflict-free vertex-connection number

k has at most $2^k - 1$ vertices. We apply induction on k. The statement is evidently true for $k = 2$. Give the path P_n with $vcfc(P_n) = k$ a conflict-free vertex-connection k-coloring. Then there is a vertex, say v_i, in P_n with a unique color. Delete the vertex v_i from P_n. The resulting paths are $P_{i-1} = v_1v_2 \cdots v_{i-1}$ and $P_{n-i} = v_{i+1}v_{i+2} \cdots v_n$ with $vcfc(P_{i-1}) \leq k - 1$ and $vcfc(P_{n-i}) \leq k - 1$. By the induction hypothesis, P_{i-1} and P_{n-i} have at most $2^{k-1} - 1$ vertices, respectively. Thus P_n has at most $2\left(2^{k-1} - 1\right) + 1 = 2^k - 1$ vertices, and so $vcfc(P_n) \geq \lceil \log_2(n+1) \rceil$.

Therefore, $vcfc(P_n) = \lceil \log_2(n+1) \rceil$. ■

Remark 1. From Theorem 2.1 and [7, Theorem 3], we have that $vcfc(P_n) \geq cfc(P_n)$. However, $vcfc(G) \leq cfc(G)$ if G is a star of order at least 3. Thus, one of $vcfc(G)$ and $cfc(G)$ cannot be bounded in terms of the other.

3. Graphs with Conflict-Free Vertex-Connection Number Two or Three

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Then the block is either a cut-edge, say *trivial block*, or a maximal 2-connected subgraph, say *nontrivial block*. Let $B_1, B_2, \ldots, B_k$ be the blocks of G. The *block graph* of G, denoted by $B(G)$, has vertex-set $\{B_1, B_2, \ldots, B_k\}$ and B_iB_j is an edge if and only if the blocks B_i and B_j have a cut-vertex in common, where $1 \leq i, j \leq k$.

The following lemma is a preparation of Theorem 3.1.

Lemma 3.1. *Let G be a 2-connected graph. Then $vcfc(G) = 2$.*

Proof. Since $vcfc(G) \geq 2$, we just need to show that $vcfc(G) \leq 2$. Let w be a vertex of G. Define a 2-coloring c of the vertices of G by coloring the vertex w with color 2 and all the other vertices of G with color 1. By Lemma 2.1, for any two vertices u and v in G, there is a u-v path containing the vertex w. According to the coloring c of G, this u-v path is a conflict-free path. Thus $vcfc(G) \leq 2$, and the proof is complete. ■

From Theorem 2.1 and Lemma 3.1, we have the following result.

Corollary 3.1. *For the complete graph K_n with $n \geq 2$, $vcfc(K_n) = 2$.*

After the above preparation, graphs with $vcfc(G) = 2$ can be characterized.

Theorem 3.1. *Let G be a connected graph of order at least 3. Then $vcfc(G) = 2$ if and only if G is 2-connected or G has only one cut-vertex.*

Proof. Firstly, we prove its sufficiency. If G is 2-connected, then it follows from Lemma 3.1 that $vcfc(G) = 2$. Now suppose that G has exactly one cut-vertex, say w. Since $vcfc(G) \geq 2$, we just need to show that $vcfc(G) \leq 2$. Define a 2-coloring c of the vertices of G by coloring the vertex w with color 2 and all the other vertices with color 1. Since G has only one cut-vertex, it follows that G consists of some blocks which have the common vertex w. Next it remains to check that for any two vertices u and v in G, there is a conflict-free path between them. It is clearly true for the case that $w \in \{u, v\}$. Thus we may assume that $w \in V(G) \backslash \{u, v\}$. If u and v are in the same block, then the block must be nontrivial. From Lemma 2.1 and the coloring c of G, we get that there is a conflict-free path from u to v in the block. If u and v are in two different blocks, then there is a u-w path P_1 and a v-w path P_2 in the two blocks, respectively. Clearly, the path uP_1wP_2v is the desired path.

Now, we show its necessity. Let $vcfc(G) = 2$. By Lemma 3.1, it remains to show that if G is not 2-connected, then G has only one cut-vertex. Suppose that G has at least two cut-vertices. Let B_1 and B_2 be two blocks in G each of which contains only one cut-vertex, respectively. Moreover, denote by v_1 and v_2 the cut-vertices in B_1 and B_2, respectively. Note that for any two vertices in the same block, all paths connecting them are in the block. Thus, each block needs two colors. Let u_1 be the vertex whose color is different from v_1 in B_1 and u_2 be the vertex whose color is different from v_2 in B_2. Clearly, all paths from u_1 to u_2 in G must pass through the vertices v_1 and v_2. However, the four vertices u_1, v_1, u_2, v_2 use each of two colors twice. Thus there does not exist a conflict-free path between u_1 and u_2 in G, a contradiction. Hence G has only one cut-vertex. ∎

The following corollary is immediate from Theorem 3.1.

Corollary 3.2. *Let G be a connected graph. Then $vcfc(G) \geq 3$ if and only if G has at least two cut-vertices.*

Next we give two sufficient conditions for a graph G to have $vcfc(G) = 3$.

Theorem 3.2. *Let G be a connected graph of order n with maximum degree $\Delta(G)$. If G has at least two cut-vertices and $n - 4 \leq \Delta(G) \leq n - 2$, then $vcfc(G) = 3$.*

Proof. Since G has at least two cut-vertices, it follows that $vcfc(G) \geq 3$ by Corollary 3.2, and so we only need to show that $vcfc(G) \leq 3$. We distinguish the following three cases to show this theorem.

Case 1. $\Delta(G) = n - 2$. In this case, G must have a spanning tree T_1 shown in Figure 1. Moreover, a 3-coloring of the vertices of T_1 is shown in Figure 1 to

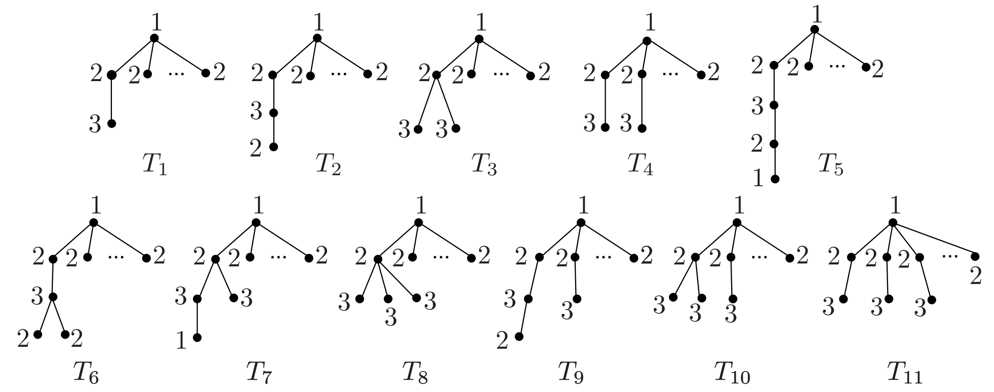

Figure 1. The eleven graphs in Theorem 3.

make T_1 conflict-free vertex-connected. Thus $vcfc(T_1) \leq 3$. From Observation 1, we have $vcfc(G) \leq vcfc(T_1)$, and hence $vcfc(G) \leq 3$.

Case 2. $\Delta(G) = n - 3$. Since $\Delta(G) = n - 3$, it follows that G must have a spanning tree depicted as one of $T_i s$ $(2 \leq i \leq 4)$ shown in Figure 1. For $2 \leq i \leq 4$, a 3-coloring of the vertices of T_i is shown in Figure 1 to make T_i conflict-free vertex-connected. From Observation 1, we have $vcfc(G) \leq vcfc(T_i) \leq 3$.

Case 3. $\Delta(G) = n - 4$. Since $\Delta(G) = n - 4$, it follows that G must have a spanning tree depicted as one of $T_i s$ $(5 \leq i \leq 11)$ shown in Figure 1. For $5 \leq i \leq 11$, a 3-coloring of the vertices of T_i is shown in Figure 1 to make T_i conflict-free vertex-connected. From Observation 1, we have $vcfc(G) \leq vcfc(T_i) \leq 3$.

From the above argument, we conclude that $vcfc(G) = 3$. ■

Remark 2. The condition on the maximum degree above cannot be improved, since if G is T' shown in Figure 2, then $\Delta(G) = n-5$ and $vcfc(G) = 4$. Note that there is only one path between any two vertices in a tree. Then any two adjacent vertices in T' need two different colors. Considering this, we can check that three colors cannot make T' conflict-free vertex-connected and so $vcfc(T') \geq 4$. Moreover, a 4-coloring of the vertices of T' is shown in Figure 2 to make T' conflict-free vertex-connected. Hence $vcfc(T') = 4$.

Let $C(G)$ denote the subgraph of G induced by the set of cut-edges of G.

Theorem 3.3. *Let G be a connected graph with at least two cut-vertices. If $C(G)$ is a star and each nontrivial block has a common vertex with $C(G)$, then $vcfc(G) = 3$.*

Proof. By Corollary 3.2, it suffices to show that $vcfc(G) \leq 3$, since G has at least two cut-vertices. Let $V(C(G)) = \{v_0, v_1, \ldots, v_t\}$, where $t \geq 1$ and v_0 is the

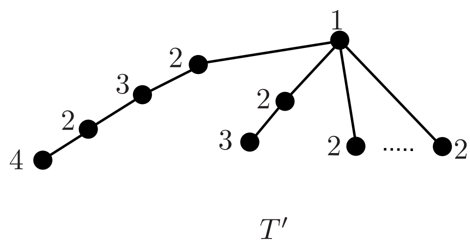

Figure 2. A 4-coloring of the vertices of T'.

center of the star $C(G)$. Define a 3-coloring c of the vertices of G by coloring the vertex v_0 with color 1, the pendant vertices $\{v_1, \dots, v_t\}$ of $C(G)$ with color 2 and all the other vertices with color 3. Next, it remains to check that for any two vertices u and v in G, there is a conflict-free path between them. If $u, v \in V(C(G))$, then the desired path is the unique path from u to v in $C(G)$. If u and v belong to the same nontrivial block, then by Lemma 2.1, there is a u-v path in the block containing the vertex which is also in $C(G)$. Clearly, this path is the desired path. Now we may assume that u and v are in two different nontrivial blocks B and B'. If B and B' do not have a common vertex, then a shortest u-v path in G must go through the center v_0 which has the unique color 1 and so it is the desired path. Otherwise, B and B' have a unique common vertex v_i $(0 \le i \le t)$ which has the unique color $c(v_i)$ on the u-v path. Thus, $vcfc(G) \le 3$, and the proof is complete. ■

The *t-corona* of a graph H, denoted by $Cor_t(H)$, is a graph obtained from H by adding t pendant edges to each vertex of H.

Proposition 3.4. *Let C_n be a cycle and G be its t-corona, where $t \ge 1$. Then $vcfc(G) = 3$.*

Proof. Since G has at least three cut-vertices, we have $vcfc(G) \ge 3$ by Theorem 3.1, and so it remains to show that $vcfc(G) \le 3$. Define a 3-coloring c of the vertices of G by coloring all the pendant vertices with color 1, one of the vertices of C_n with color 2 and the other vertices with color 3. It is easy to check that for any two vertices of G, there is a conflict-free path between them. Then $vcfc(G) \le 3$, and we complete the proof. ■

It seems that it is not easy to characterize graphs G with $vcfc(G) = 3$. But, below we study a family of graphs with conflict-free vertex-connection number three. Before it, we provide the concept of a segment: Let G be a connected graph whose block graph is a path. Let $B_1, B_2, \dots, B_k$ be the blocks of G such that $|V(B_i) \cap V(B_{i+1})| = 1$ and $E(B_i) \cap E(B_{i+1}) = \emptyset$ $(1 \le i \le k-1)$. We call $F_{p,q}$ $(1 \le p \le q \le k)$ a *segment* of G if $F_{p,q} = \bigcup_{i=p}^{q} B_i$.

Theorem 3.5. *Let G be a connected graph with at least two cut-vertices, and its block graph $B(G)$ is a path. Then $vcfc(G) = 3$ if and only if G is a segment of one type of the thirteen graphs listed below.*

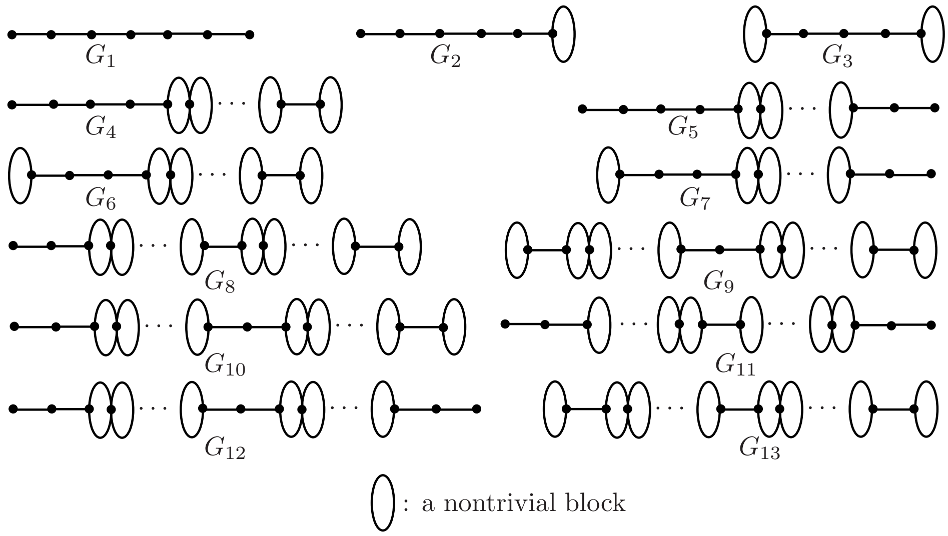

Figure 3. The thirteen types of graphs.

Proof. First we show its sufficiency. Let G be a segment of one type of the thirteen graphs listed in Figure 3. Since G has at least two cut-vertices, $vcfc(G) \geq 3$ according to Corollary 3.2 and so we just need to show that $vcfc(G) \leq 3$, Namely, we should give a 3-coloring of the vertices in G such that G is conflict-free vertex-connected. Since G is a segment of one type of the thirteen graphs listed in Figure 3, we only consider the cases when $G = G_i$ ($1 \leq i \leq 13$).

If $G = G_1$, we assign the color sequence $1, 2, 1, 3, 1, 2, 1$ to the vertices from left to right. If $G = G_2$, we assign the color sequence $1, 2, 1, 3, 1, 2$ to the six cut-vertices from left to right and the remaining vertices are assigned to the color 1. If $G = G_3$, we assign the color sequence $2, 1, 3, 1, 2$ to the five cut-vertices from left to right and the remaining vertices are assigned to the color 1.

Assume $G = G_6$. Let $B, B_1, B_2, \ldots, B_t, B'$ be the nontrivial blocks from left to right. Assign the color sequence $2, 1, 3$ to the left three cut-vertices. If t is odd, assign the rightmost cut-vertex with color 3; otherwise, assign it with color 2. Pick a vertex avoiding the cut-vertex from each B_i ($1 \leq i \leq t$). Assign the color 2 to all these vertices if i is odd and assign the color 3 otherwise. All remaining vertices are assigned to the color 1. For the case $G = G_4$, we only need to view the leftmost vertex in G_4 as the vertices colored 1 of B in G_6 and the other vertices in G_4 are colored as those in G_6. Similarly we can give a 3-coloring for $G = G_5$ and $G = G_7$. Namely view the leftmost vertex and rightmost vertex in G_5 as the vertices colored 1 of B and B' in G_6 respectively, and view the rightmost vertex in G_7 as the vertices colored 1 of B' in G_6. Following this structural law, we only give the 3-coloring of G when $G = G_9$ for $G = G_9, G_{10}$ or G_{12}, and give the 3-coloring of G when $G = G_{13}$ for $G = G_8, G_{11}$ or G_{13} in the following.

Assume $G = G_9$. Denote by $B_0, B_{11}, B_{12}, \ldots, B_{1s}$ $(s \geq 1)$, $B_{21}, B_{22}, \ldots, B_{2t}$ $(t \geq 1), B'_0$ the nontrivial blocks and $v_0, v_1, v_2, \ldots, v_{s+1}, v, v'_1, v'_2, \ldots, v'_{t+1}, v'_0$ the cut-vertices from left to right, respectively. First consider the case that s is odd. Pick a vertex avoiding the cut-vertex from each of B_{1i} and B_{2j} when i and j are even. Assign the color 2 to all these vertices and v_0, v. Moreover, pick a vertex avoiding the cut-vertex from each of B_{1i} and B_{2j} when i and j are odd. Give all these vertices the color 3. For v'_0, if t is odd, color it with color 2; otherwise, color it with color 3. All remaining vertices are assigned to the color 1. Then consider the case that s is even. Pick a vertex avoiding the cut-vertex from each B_{1i} when i is even and a vertex from each B_{2j} when j is odd. Assign the color 2 to all these vertices and v_0. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i is odd and a vertex from each B_{2j} when j is even. Give all these vertices and v the color 3. For v'_0, if t is odd, color it with color 3; otherwise, color it with color 2. All remaining vertices are assigned to the color 1.

Assume $G = G_{13}$. Denote by $B_0, B_{11}, B_{12}, \ldots, B_{1s}$ $(s \geq 1)$, $B_{21}, B_{22}, \ldots, B_{2t}$ $(t \geq 1)$, B'_0 the nontrivial blocks and $v_0, v_1, v_2, \ldots, v_{s+1}, v'_1, v'_2, \ldots, v'_{t+1}, v'_0$ the cut-vertices from left to right, respectively. First consider the case that s is odd. Pick a vertex avoiding the cut-vertex from each B_{1i} when i is even and a vertex from each B_{2j} when j is odd and $3 \leq j \leq t$. Assign the color 2 to all these vertices and v_0, v'_1. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} when i is odd and a vertex from each B_{2j} when j is even. Give all these vertices the color 3. For v'_0, if t is odd, color it with color 3; otherwise, color it with color 2. All remaining vertices are assigned to the color 1. Then consider the case that s is even. Pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i and j are even. Assign the color 2 to all these vertices and v_0. Moreover, pick a vertex avoiding the cut-vertex from each B_{1i} and B_{2j} when i and j are odd and $3 \leq j \leq t$. Give all these vertices and v'_1 the color 3. For v'_0, if t is odd, color it with color 2; otherwise, color it with color 3. All remaining vertices are assigned to the color 1.

Using Lemma 2.1, it can be easily checked that the conflict-free paths can be found between the vertices of the same block. For other pairs of vertices, we can also find conflict-free paths between them under the above colorings for $G = G_i$ $(1 \leq i \leq 13)$. Thus $vcfc(G) = 3$.

Next we show its necessity. Let $B_1, \ldots, B_n$ be blocks of G such that $|V(B_i) \cap V(B_{i+1})| = 1$ and $v_i \in V(B_i) \cap V(B_{i+1})$ be the cut-vertex of G, where $1 \leq i \leq n-1$. Let $\mathcal{F}$ be the family of graphs listed in Theorem 3.5. From the sufficient part of the proof we know that for $G \in \mathcal{F}$ there is $vcfc(G) = 3$. Suppose there is another graph G with $vcfc(G) = 3$ but $G \notin \mathcal{F}$. Let ϕ be a conflict-free vertex-connection coloring of G with colors from $\{a, b, c\}$ and c be a color with the least number of appearances on cut-vertices of G. Then the following two claims are evident.

Claim 1. *Any three consecutive blocks B_{i-1}, B_i, B_{i+1} contain on their vertices all three colors.*

Claim 2. *Color c appears at most once on the cut-vertices of G.*

Claim 3. *G does not contain a segment of seven consecutive blocks $B_{i-1}, \ldots, B_{i+5}$ such that all three blocks $B_{i+1}, B_{i+2}, B_{i+3}$ are trivial.*

Proof. If such a segment exists, without loss of generality let $\phi(v_{i+1}) = \phi(v_{i+3}) = a, \phi(v_i) = b$ and $\phi(v_{i+2}) = c$. From Claims 1 and 2, there are vertices $x \in V(B_{i-1}) \cup V(B_i)$ with $\phi(x) = c$ and $y \in V(B_{i+4}) \cup V(B_{i+5})$ with $\phi(y) = b$, and any x-y path is a conflict one. □

Consider the following segment $\mathbf{S}$ of blocks of the graph G:

$$\mathbf{S} = B_{i-1}, B_i, B_{i+1}, \ldots, B_j, B_{j+1}, \ldots, B_k, B_{k+1}, B_{k+2}, B_{k+3},$$

where B_{i+1} and B_{k+1} are trivial, B_{j+1} is trivial only if $C(G)$ has at least three components, and the three blocks B_{i-1}, B_j, B_k are always nontrivial; the remaining blocks may be, in dependence on the components of $C(G)$, either trivial or nontrivial. We suppose, without loss of generality, that blocks B_{i+1}, B_{j+1} and B_{k+1} belong to three different consecutive components of $C(G)$ (if exist).

Claim 4. *Both B_{i-1} and B_{k+3} are not present simultaneously in G.*

Proof. If both are present, then, because of Claims 1 and 2, there exist two vertices $x \in V(B_{i-1}) \cup V(B_i)$ with $\{\phi(x), \phi(v_i), \phi(v_{i+1})\} = \{a, b, c\}$ and $y \in V(B_{k+2}) \cup V(B_{k+3})$ with $\{\phi(y), \phi(v_k), \phi(v_{k+1})\} = \{a, b, c\}$, such that every x-y path is conflict. □

Let $C(G)$ contain a path on at least four vertices as a component. Then, by Claim 4, we can put $i = 1$ and suppose that the block B_0 is not present in $\mathbf{S}$. If $C(G)$ contains P_7 or P_6 as a component, then either G contains a graph from $\{G_1, G_2\}$ as a proper segment and so we have a contradiction with Claim 3 or G is contained by G_1 or G_2 which is also a contradiction. If $C(G)$ contains P_5 as a component and G contains G_3 as a proper segment, we have a contradiction with Claim 3. Thus we can suppose that either $C(G)$ contains P_5 as a component and G does not contain G_3 as a proper segment or $C(G)$ contains P_4 as a component. If, in $\mathbf{S}$, the blocks B_2, B_3, B_4 are all trivial (B_1 is also trivial if G contains P_5), and no other block is trivial in it, then $\mathbf{S}$ and hence G, is in $\mathcal{F}$. If there is another trivial block B_{j+1} in G, let v_j be its first cut-vertex incident with this block. Because of Claim 2, $\{\phi(v_j), \phi(v_{j+1})\} = \{a, b\}$, then $j = k$. Otherwise there are two more blocks B_{j+2} and B_{j+3} containing a vertex y with $\phi(y) = c$, and there is a vertex $x \in V(B_1) \setminus \{v_1\}$ such that every x-y path is conflict. The existence of the block B_{j+3} in the case $j = k$ yields a contradiction analogously as above. If the block B_{j+3} is not in $\mathbf{S}$, then G is in $\mathcal{F}$, again a contradiction.

Let $C(G)$ do not contain a path on at least four vertices. By the set $\mathcal{F}$ we can suppose that $C(G)$ has at least two components. Observe that $\mathbf{S}$ contains at least two components of $C(G)$ and, by the definition of $\mathbf{S}$, at most three components of $C(G)$. By Claim 4, at most one of B_{i-1} and B_{k+3} is present in G.

Then, without loss of generality we can suppose that the last block of $\mathbf{S}$ is B_{k+2} and that there is the block B_{i-1}. If there are only two components for $C(G)$, then surely $\mathbf{S}$ and hence G, is in $\mathcal{F}$, a contradiction. When $C(G)$ has three components, which allows $i + 2 < k - 1$. We discuss this case as follows.

Case 1. There is $m \in \{i + 2, \ldots, k - 1\}$ such that $\phi(v_m) = c$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ with $\phi(x) = c$ and every x-v_{k+1} path is conflict, a contradiction.

Case 2. There is no $m \in \{i + 2, \ldots, k - 1\}$ such that $\phi(v_m) = c$.

Case 2.1. Let $c \in \{\phi(v_i), \phi(v_{i+1})\}$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ such that $\{\phi(x), \phi(v_i), \phi(v_{i+1})\} = \{a, b, c\}$ and a vertex $y \in V(B_{k-1}) \cup V(B_k) \cup V(B_{k+1})$ with $\phi(y) = c$. Then every x-y path is conflict, a contradiction.

Case 2.2. Let $c \notin \{\phi(v_i), \phi(v_{i+1})\}$. Then there is a vertex $x \in V(B_{i-1}) \cup V(B_i)$ such that $\phi(x) = c$ and a vertex $y \in V(B_{j+2}) \cup V(B_{j+3})$ such that $\{\phi(v_j), \phi(v_{j+1}), \phi(y)\} = \{a, b, c\}$. As a result, every x-y path is conflict, a contradiction.

If both B_{i-1} and B_{k+3} are not present, then $\mathbf{S}$ and hence G, is in $\mathcal{F}$, a contradiction except for the case that B_{i+2} is trivial, B_i is nontrivial and there exists a trivial block B_{j+1}. But in this case, we can find a vertex $x \in B_i$ with $\{\phi(x), \phi(v_{i+1}), \phi(v_{i+2})\} = \{a, b, c\}$ and a vertex $y \in B_{j+2} \cup B_{j+3}$ with $\{\phi(y), \phi(v_j), \phi(v_{j+1})\} = \{a, b, c\}$, which also leads to the contradiction that there is no conflict-free path between x and y.

This finishes the proof of the necessary part of Theorem 5.

The proof is complete. ■

At the end of this section, we pose the following problem.

Problem 3.1. Characterize all the graphs G with $vcfc(G) = 3$.

4. Trees

A *k-ranking* of a connected graph G is a labeling of its vertices with labels $1, 2, 3, \ldots, k$ such that every path between any two vertices with the same label i in G contains at least one vertex with label $j > i$. A graph G is said to be *k-rankable* if it has a k-ranking. The minimum k for which G is k-rankable is denoted by $r(G)$.

Iyer [9] obtained the following result.

Lemma 4.1 [9]**.** *Let T be a tree of order $n \geq 3$. Then $r(T) \leq \log_{\frac{3}{2}} n$.*

The next two lemmas are preparations for Theorem 4.1.

Lemma 4.2. *Let G be a connected graph. Then $vcfc(G) \leq r(G)$.*

Proof. Consider a ranking of the vertices of G. For any two vertices u and v of G, let P be a path between them and k be the maximum label of the vertices of P. If there is only one vertex with label k in P, then the proof is done. So we assume that P contains at least two vertices with label k. According to the definition of ranking, there must exist one vertex with label $j > k$ on P, which is a contradiction. Hence P contains only one vertex with label k. View the $r(G)$-ranking of G as its vertex-coloring with $r(G)$ colors. Then the path P is a conflict-free path between u and v in G. Thus $vcfc(G) \leq r(G)$. ■

Lemma 4.3. *Let T be a nontrivial tree. Then $vcfc(T) \geq \chi(T)$, where $\chi(T)$ is the chromatic number of T and the bound is sharp.*

Proof. Define a vertex-coloring of T with $vcfc(T)$ colors such that T is conflict-free vertex-connected. Since there is only one path between any two vertices in T, it follows that any two adjacent vertices must have different colors, and hence $vcfc(T) \geq \chi(T)$. To show the sharpness of the bound, we let T be a star of order at least two. Clearly, $\chi(T) = 2$. By Theorem 3.1, we have $vcfc(T) = 2$ $(= \chi(T))$. ■

Combining the lemmas above, we can have the following bounds for $vcfc(T)$ of a tree T.

Theorem 4.1. *Let T be a tree of order $n \geq 3$ and $d(T)$ be its diameter. Then*

$$\max\left\{\chi(T), \lceil \log_2(d(T)+2) \rceil\right\} \leq vcfc(T) \leq \log_{\frac{3}{2}} n.$$

Proof. The lower bound is an immediate result from Lemma 4.3 and Theorem 2.1, while the upper bound can be deduced from Lemmas 4.1 and 4.2. ■

Let G be a connected graph. The *eccentricity* $\epsilon_G(v)$ of a vertex v in G is the maximum value among the distances between v and the other vertices in G. The *radius* $rad(G)$ of G is the minimum eccentricity among all the vertices of G. A *central vertex* of radius $rad(G)$ is one whose eccentricity is $rad(G)$. Remind that $d_G(u, v)$ is the shortest distance between the two vertices u and v in G.

Theorem 4.2. *Let T be a tree with radius $rad(T)$. Then $vcfc(T) \leq rad(T) + 1$. Moreover, the bound is sharp.*

Proof. Let v be a central vertex of radius $rad(T)$ in T. Let $V_i = \{u \in V(T) : d_T(u, v) = i\}$, where $0 \leq i \leq rad(T)$. Hence $V_0 = \{v\}$. Define a vertex-coloring c of T with $rad(T) + 1$ colors by coloring the vertices of V_i with color $i + 1$, where $0 \leq i \leq rad(T)$. It is easy to check that for any two vertices of T, there is a conflict-free path between them, and hence $vcfc(T) \leq rad(T) + 1$. To show the sharpness of the bound, we let T be a star of order at least two. Clearly, $rad(T) = 1$. By Theorem 3.1, we have $vcfc(T) = 2$ $(= rad(T) + 1)$. ∎

For each connected graph G, we can always find a spanning tree T of G such that $rad(T) = rad(G)$. From Observation 1 and Theorem 4.2, we can get the following result.

Corollary 4.1. *Let G be a connected graph. Then $vcfc(G) \leq rad(G) + 1$.*

For trees, we can give an upper bound of its conflict-free vertex-connection number in term of its order.

Proposition 4.3. *Let T be a tree with order $n \geq 5$. Then $vcfc(T) \leq \lceil \frac{n}{2} \rceil$. Moreover, the bound is sharp.*

Proof. If T is a path, then it follows from Theorem 2.1 that $vcfc(T) = \lceil \log_2 (n+1) \rceil \leq \lceil \frac{n}{2} \rceil$. From now on, we suppose that T is not a path. Then the longest path in T has at most $n - 1$ vertices. So we have $rad(T) \leq \frac{n-1}{2}$ if n is odd and $rad(T) \leq \frac{n-2}{2}$ if n is even. By Theorem 4.2, we have $vcfc(T) \leq rad(T) + 1$, and hence $vcfc(T) \leq \lceil \frac{n}{2} \rceil$. To show the sharpness of the bound, we set $T = P_5$. Then $vcfc(T) = 3$ by Theorem 2.1 and $\lceil \frac{n}{2} \rceil = 3$. ∎

Let G be a nontrivial connected graph of order n. For $vcfc(G)$, it can be easily seen that the trivial lower bound is 2. Based on Observation 1, the upper bound can be attained when G is a tree. Note that the path P_n is a tree of order n. After experiments on the graphs with small order, we believe that P_n might be the one attaining the upper bound among trees. Recently, Li and Wu [16] have confirmed this as $vcfc(G) \leq vcfc(P_n)$.

Acknowledgments

The authors are very grateful to the referees for their helpful comments and suggestions. The project was supported by National Science Foundation of China (Nos. 11531011, 11601254, 11551001), the Science Found of Qinghai Province (Nos. 2016-ZJ-948Q, 2014-ZJ-907), and the project on the key lab of IOT of Qinghai Province (No. 2017-Z-Y21). The sixth author was supported by the Slovak Research and Development Agency under the contract No. APPVV-15-0116 and by the Slovak VEGA Grant 1/0368/16.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer, 2008).

[2] H. Chang, Z. Huang, X. Li, Y. Mao and H. Zhao, *Nordhaus-Gaddum-type theorem for conflict-free connection number of graphs*. arXiv:1705.08316 [math.CO].

[3] H. Chang, M. Ji, X. Li and J. Zhang, *Conflict-free connection of trees*. arXiv:1712.10010 [math.CO]

[4] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2018) 85–98.

[5] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *The rainbow connectivity of a graph*, Networks **54** (2009) 75–81. doi:10.1002/net.20296

[6] L. Chen, X. Li and Y. Shi, *The complexity of determining the rainbow vertex-connection of a graph*, Theoret. Comput. Sci. **412** (2011) 4531–4535. doi:10.1016/j.tcs.2011.04.032

[7] J. Czap, S. Jendrol' and J. Valiska, *Conflict-free connections of graphs*, Discuss. Math. Graph Theory **38** (2018) 911–920. doi:10.7151/dmgt.2036

[8] B. Deng, W. Li, X. Li, Y. Mao and H. Zhao, *Conflict-free connection numbers of line graphs*, Lecture Notes in Comput. Sci. **10627** (2017) 141–151. doi:10.1007/978-3-319-71150-8_14

[9] A.V. Iyer, H.D. Ratliff and G. Vijayan, *Optimal node ranking of trees*, Inform. Process. Lett. **28** (1988) 225–229. doi:10.1016/0020-0190(88)90194-9

[10] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is* (*at most*) *reciprocal to its minimum degree*, J. Graph Theory **63** (2010) 185–191.

[11] X. Li and S. Liu, *Tight upper bound of the rainbow vertex-connection number for* 2-*connected graphs*, Discrete Appl. Math. **173** (2014) 62–69. doi:10.1016/j.dam.2014.04.002

[12] X. Li, Y. Mao and Y. Shi, *The strong rainbow vertex-connection of graphs*, Util. Math. **93** (2014) 213–223.

[13] X. Li and Y. Shi, *On the rainbow vertex-connection*, Discuss. Math. Graph Theory **33** (2013) 307–313.
doi:10.7151/dmgt.1664

[14] X. Li, Y. Shi and Y. Sun, *Rainbow connections of graphs: A survey*, Graphs Combin. **29** (2013) 1–38.
doi:10.1007/s00373-012-1243-2

[15] X. Li and Y. Sun, Rainbow Connections of Graphs (Springer, New York, 2012).
doi:10.1007/978-1-4614-3119-0

[16] Z. Li and B. Wu, *On the maximum value of conflict-free vertex-connection number of graphs*.
arxiv: 1709.01225 [math.CO]

[17] H. Liu, Â. Mestre and T. Sousa, *Rainbow vertex k-connection in graphs*, Discrete Appl. Math. **161** (2013) 2549–2555.
doi:10.1016/j.dam.2013.04.025

Received 22 May 2017
Revised 18 January 2018
Accepted 22 January 2018