Discussiones Mathematicae
Graph Theory 40 (2020) 107–125
doi:10.7151/dmgt.2115

THE PROPER DIAMETER OF A GRAPH

Vincent Coll

Lehigh University, Department of Mathematics
Bethlehem, PA 18015, USA

e-mail: vec208@lehigh.edu

Jonelle Hook

Mount St. Mary's University
Department of Mathematics and Computer Science
Emmitsburg, MD 21727, USA

e-mail: jhook@msmary.edu

Colton Magnant

Georgia Southern University
Department of Mathematical Sciences
Statesboro, GA 30460, USA

e-mail: cmagnant@georgiasouthern.edu

Karen McCready

King's College
Department of Mathematics and Computer Science
Wilkes-Barre, PA 18711, USA

e-mail: karenmccready@kings.edu

and

Kathleen Ryan

DeSales University
Department of Mathematics and Computer Science
Center Valley, PA 18034, USA

e-mail: kathleen.ryan@desales.edu

Abstract

A proper edge-coloring of a graph is a coloring in which adjacent edges receive distinct colors. A path is properly colored if consecutive edges have distinct colors, and an edge-colored graph is properly connected if there exists a properly colored path between every pair of vertices. In such a graph, we introduce the notion of the graph's proper diameter—which is a function of both the graph and the coloring— and define it to be the maximum length of a shortest properly colored path between any two vertices in the graph. We consider various families of graphs to find bounds on the gap between the diameter and possible proper diameters, paying singular attention to 2-colorings.

Keywords: diameter, properly connected, proper diameter.

2010 Mathematics Subject Classification: 05C12, 05C15.

1. Introduction

In a long distance wireless communications network, each node in the network simultaneously sends and receives signals. To avoid interference, the carrier waves for each of these incoming and outgoing signals must be operating at different frequencies. By representing each node by a vertex, each connection by an edge, and each frequency assignment by a color, the problem of finding a path over which a signal can travel without interference is translated to a question about the edge-coloring of graphs. More specifically, it is required that no two consecutive edges in the transmission path have the same color in order to maintain the integrity of a signal from start to finish. The design of such networks is therefore reliant on the ability to color a graph "properly". More formally, an edge-colored graph is said to be *properly colored* if no two adjacent edges share a color. An edge-colored connected graph G is called *properly connected* if between every pair of distinct vertices in G, there exists a path that is properly colored. For simplicity, an edge-coloring c of G is called *properly connected* if it makes G properly connected. Note that while a properly colored graph is necessarily properly connected, the converse is not true. See Figure 1.

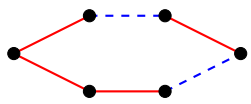

Figure 1. A properly connected but not properly colored graph.

The *proper connection number* of a connected graph G, $\mathrm{pc}(G)$, defined in [2] and also studied in [1, 5, 6], is the minimum number of colors needed to color

the edges of G to make it properly connected. One finds that if an edge-colored graph is sufficiently dense, then the existence of a properly colored path is assured. For example, using the connectivity $\kappa(G)$ of G, Borozan *et al.* [2] show that if $\kappa(G) \geq 3$, then $\mathrm{pc}(G) = 2$ and if $\kappa(G) \geq 2$, then $\mathrm{pc}(G) \leq 3$. We refer the reader to [8] for a dynamic survey on the topic of proper connection.

The notion of a properly colored graph and concepts such as proper connection number are obviously related to rainbow colored graphs and analogous concepts introduced by Chartrand *et al.* in [3]. In fact, the communications application noted at the outset is a generalization of a security protocol devised by Chartrand *et al.* to address communications between disparate governmental and legal agencies in the wake of the terrorist attack of Sept. 11, 2001 [4].

However, despite a spate of recent papers on the proper and rainbow connection numbers of a graph (inclusive of the basic work in [3] and [2]) the study of the spectrum of possible lengths of certain properly (respectively, rainbow) colored paths in a properly (respectively, rainbow) connected graph has not yet been investigated. In fact, the only known results relating properly colored paths to distance in the graph has been forcing the properly colored paths to be geodesics, shortest paths, as studied in [1, 6, 7]. Here we begin to address this omission by introducing the notion of the *proper diameter* of a properly connected graph.

Throughout this paper, we assume that all graphs are properly connected. The *proper distance* between two vertices u and v is the minimum length of a properly colored path from u to v. Observe that such a distance function does not define a metric. See Figure 2 for an example of a properly connected graph where the triangle inequality fails.

Figure 2. The proper distance between u and v is greater than the sum of the proper distances between u and x and between x and v.

Generalizing the notion of the diameter of a graph, $\mathrm{diam}(G)$, we formally define proper diameter.

Definition. The *proper diameter* of a graph G with a properly connected k-coloring c is defined to be the maximum proper distance between any two vertices in G and will be denoted as $\mathrm{pdiam}_k(G, c)$.

We will consider all colorings c where G is properly connected and discuss $\mathrm{pdiam}_k(G)$ which is the maximum of $\mathrm{pdiam}_k(G, c)$ where c makes G properly connected. Since coloring the edges of a graph cannot shorten the lengths of paths, it follows that the diameter of a graph is a trivial lower bound for the proper diameter. Note that the proper diameter of a graph is a function of

both the graph and the coloring, but for some classes of graphs, such as trees and complete graphs, the diameter and proper diameter are necessarily equal – regardless of the coloring.

We now ask two questions which drive our current investigation.

- How large can the gap be between $\mathrm{diam}(G)$ and $\mathrm{pdiam}_2(G)$?
- What is the spectrum of possible values for $\mathrm{pdiam}_2(G, c)$?

As a first step in answering these questions, we address—and completely resolve—the two driving questions for the following graph classes: cycles (Section 2.1), fans (Section 2.2), and complete multipartite graphs (Section 2.3). After the fashion of the result of Borozan *et al.* [2] noted above, the connectivity of a graph can also be used to give a tight upper bound of $n - \kappa(G) + 1$ for $\mathrm{pdiam}_2(G)$. We prove this result in Section 3. We conclude with two lemmas which provide additional bounds on the proper diameter. The first provides a general lower bound for graphs with odd girth, and the second gives an upper bound on the proper diameter of a graph formed by adding a vertex of degree 2 to a properly connected graph under certain conditions. Note that throughout the proofs of this paper we use $c(uv) = 1$ or $c(uv) = 2$ to denote an edge uv of color 1 or color 2, respectively. In the figures, color 1 is represented by a solid red line and color 2 is represented by a dashed blue line.

2. General Classes of Graphs

2.1. Cycles

We begin by considering the straightforward case of even and odd cycles. This class of graphs illustrates that using only two colors is sufficient to produce a colored graph whose proper diameter is quite large—nearly double the diameter.

In order for an odd cycle C_{2m+1} to be properly connected with two colors, there must be exactly one pair of consecutive edges with the same color. The path between the vertices at opposite ends of these two edges yields proper diameter $2m - 1$.

There are two possible properly connected 2-colorings of an even cycle C_{2m}. If the cycle is properly colored, the proper diameter is certainly equal to the diameter m. Otherwise, exactly one set of three consecutive edges must have the same color while all other consecutive pairs are different. Suppose v_1v_2, v_2v_3 and v_3v_4 all have color 1. Then the path between vertices v_1 and v_3 gives a proper diameter of $2m - 2$.

Observation 1. *Given any properly connected 2-coloring c of a cycle graph, $\mathrm{pdiam}_2(C_{2m+1}, c)$ is $2m-1$ and $\mathrm{pdiam}_2(C_{2m}, c)$ is either m or $2m-2$ where both values are attainable.*

2.2. Fans

Next, we consider fan graphs $F_{1,n} \cong K_1 \vee P_n$, where $\vee$ denotes graph join, to find tight bounds on the attainable values of $\mathrm{pdiam}_2(F_{1,n}, c)$. This class of graphs illustrates that two colors are sufficient to produce a very large difference between the diameter and the proper diameter of a graph. In particular, $F_{1,2k+1}$ can be 2-colored so that its proper diameter is k times its diameter. See Example 2.

Example 2. A fan graph $F_{1,n}$ has diameter 2 and can be 2-colored so that the proper diameter is $n-1$. Color all edges incident to the vertex u with degree n using color 1. Color the edges of the path that remain by alternating colors. The proper diameter will be the length of this path, $n-1$. See Figure 3 for this coloring on $F_{1,8}$.

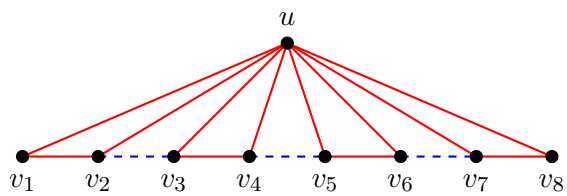

Figure 3. A properly connected 2-coloring c of $F_{1,8}$ with $\mathrm{diam}(F_{1,8}) = 2$ and $\mathrm{pdiam}_2(F_{1,8}, c) = 7$.

Example 3. A fan graph $F_{1,n}$ can be given a properly connected 2-coloring c so that the proper diameter is 3. Alternate colors of edges incident to the vertex u with degree n beginning with $c(uv_1) = 2$. Color the edges of the path by alternating colors beginning with $c(v_1v_2) = 2$. The proper diameter is 3. If i and j have opposite parity, there is properly colored path v_iuv_j of length 2. Otherwise, there is a properly colored path $v_iv_{i+1}uv_j$ of length 3. See Figure 4 for this coloring on $F_{1,8}$.

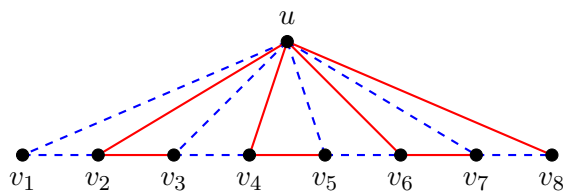

Figure 4. A properly connected 2-coloring c of $F_{1,8}$ with $\mathrm{diam}(F_{1,8}) = 2$ and $\mathrm{pdiam}_2(F_{1,8}, c) = 3$.

Theorem 4. *For a properly connected 2-coloring c of $F_{1,n}$, if $n \geq 7$, then* $\mathrm{pdiam}_2(F_{1,n}, c) \in [3, n-1]$. *These bounds are tight. If $3 \leq n \leq 6$, then a lower bound of 2 is attainable.*

Proof. First suppose $n \geq 7$. Let u be the vertex of degree n and label the path vertices as $\{v_1, v_2, \ldots, v_n\}$. Suppose that there is a properly connected 2-coloring c of $F_{1,n}$ with proper diameter equal to the diameter, 2. Without loss of generality, suppose $c(uv_1) = 1$. Regardless of the coloring of the edges of the path, $c(uv_4) = 2$ since the only path of length 2 from v_1 to v_4 passes through u. Similarly, all edges from u to $\{v_5, v_6, v_7\}$ must have color 2. Since the only path of length 2 from v_4 to v_7 passes through u, there is no properly colored path of length 2 from v_4 to v_7. As a result, every properly connected 2-coloring of $F_{1,n}$ with $n \geq 7$ has a proper diameter at least 3. Example 3 shows that 3 is a tight lower bound for $F_{1,n}$ when $n \geq 7$.

If $3 \leq n \leq 6$, then a proper diameter of 2 can be achieved by coloring the first $\lceil \frac{n}{2} \rceil$ edges from u to the path with color 1 and the remaining edges from u to the path with color 2. Alternate coloring edges of the path with colors 1 and 2.

For the upper bound, suppose there is some properly connected 2-coloring c of $F_{1,n}$ with proper diameter n. This implies that there is a properly colored path of length n, say H, between some pair of vertices and this is a shortest properly colored path. Note that H is a Hamiltonian path. The endpoints of H cannot be nonadjacent vertices v_i and v_j with $1 < i < j < n$ since any such path would miss at least one vertex and thus would not be Hamiltonian. Also, the vertex u is adjacent to every vertex on the path and so it cannot be an endpoint of H. The observations in the previous two sentences imply that either v_1 or v_n must be an endpoint of H.

Suppose first that the endpoints of H are v_1 and v_j for some $2 < j < n$. The only Hamiltonian path from v_1 to v_j is $v_1 v_2 \cdots v_{j-1} u v_n v_{n-1} \cdots v_j$ which must alternate between color 1 and color 2. This implies that uv_{j-1} and uv_n must be opposite in color, say color 1 and color 2, respectively. In order to avoid a shorter properly colored path, the edge uv_j must also have color 1. If $c(uv_1) = 1$, then we have a properly colored path $v_1 u v_n v_{n-1} \cdots v_j$ which has length at most $n-1$. If $c(uv_1) = 2$, then $v_1 u v_j$ is a properly colored path of length 2. A similar argument can be made if we assume the endpoints of H are v_n and v_i for some $1 < i < n-1$.

Thus, we have eliminated all possible endpoints of H except for a path beginning at v_1 and ending at v_n. In order for H to contain u, the path must use edges of the form $v_i u$ and uv_{i+1} of opposite color.

Case 1. Let $1 < i < n-1$. Without loss of generality, assume that $c(uv_i) = 1$, $c(uv_{i+1}) = 2$, and the remaining edges on the path alternate color. This implies that uv_n must be colored 1 or else there is a shorter properly colored path

$v_1v_2\cdots v_iuv_n$. If $c(uv_1)=2$, then v_1uv_n is a properly colored path of length 2. If $c(uv_1)=1$, then $v_1uv_{i+1}v_{i+2}\cdots v_n$ is a properly colored path of length at most $n-1$, which is a contradiction.

Case 2. Let $v_1=v_i$ with $c(uv_1)=1$ and $c(uv_2)=2$. As in the previous case, uv_n must be colored 1. Since $c(v_2v_3)=1$, the edge v_1v_2 must also be colored 1 to avoid a properly colored path without u. But, now $v_1v_2uv_n$ is a properly colored path of length 3, which is again a contradiction.

Case 3. Let $v_{n-1}=v_i$ with $c(uv_{n-1})=1$ and $c(uv_n)=2$. If uv_1 is colored 1, then v_1uv_n is a properly colored path of length 2. In a fashion similar to the second case, $c(v_{n-2}v_{n-1})=c(v_{n-1}v_n)=2$ to avoid a properly colored path without u. If $c(uv_1)$ is color 2, then $v_1uv_{n-1}v_n$ is a properly colored path of length 3, giving a contradiction.

Example 2 demonstrates the sharpness of the upper bound. ■

Theorem 4 establishes tight bounds on the range of values of $\mathrm{pdiam}_2(F_{1,n},c)$. The next theorem establishes that all values in this range are attainable.

Theorem 5. *If $n\geq 7$, then every value in the interval $[3,n-1]$ is an attainable value of* $\mathrm{pdiam}_2(F_{1,n},c)$.

Proof. By Theorem 4, we know $\mathrm{pdiam}_2(F_{1,n},c)$ is within the range of values $[3,n-1]$. Also, as described in the proof of Theorem 4, the endpoints of this range are achieved as in Examples 3 and 2, respectively. We will use these colorings to create a properly connected 2-coloring of $F_{1,n}$ where $\mathrm{pdiam}_2(F_{1,n},c)=n-k$ for $1\leq k\leq n-3$ and $n\geq 7$. Thus, we provide a properly connected 2-coloring c for every value within the given range.

As before, let u be the vertex of degree n and label the path vertices as v_1 through v_n. Let L consist of the set of vertices v_i where $1\leq i\leq n-(k-1)$ and let R consist of the set of vertices v_i where $n-(k-1)+1\leq i\leq n$. Note that when $k=1$, the set R is empty and the coloring we describe is as in Example 2.

Color the edges from u to the vertices of L with color 1. If $|R|$ is odd, then color the edges from u to the vertices of R by alternating in color starting with color 2. If $|R|$ is even, then color the edges from u to the vertices of R by alternating in color starting with color 2 up to and including v_{n-1}. For reasons evident in the following paragraph, let $c(uv_n)=2$.

Now, we color the edges of the path vertices v_1 through $v_{n-(k-1)}$ by alternating in color starting with color 1. Color the edge from $v_{n-(k-1)}$ to $v_{n-(k-1)+1}$ with color 2. Then continue coloring the path vertices $v_{n-(k-1)+1}$ through v_n by alternating in color starting with color 2 to eliminate shorter properly colored paths. This coloring is depicted in Figure 5. Clearly, u is adjacent to v_i for all i and thus has a path of length 1 to these vertices. Since the subgraph induced on $L\cup\{u\}$ is colored as in Example 2, $\mathrm{pdiam}_2\left(F_{1,n}\left[L\cup\{u\}\right],c\right)=n-(k-1)-1=n-k$. The

subgraph induced on $R \cup \{u\}$ is colored as in Example 3, but with $c(uv_n) = 2$ in order to have a properly colored path from v_1 to v_n. If $1 \leq k \leq 4$, then $\text{pdiam}_2\left(F_{1,n}\left[R \cup \{u\}\right], c\right) \leq 2$. If $k \geq 5$, then $\text{pdiam}_2\left(F_{1,n}\left[R \cup \{u\}\right], c\right) = 3$. We will now show that the path length between any vertex in L and any vertex in R is at most 3. For every vertex v_i in L, $c(uv_i) = 1$, and so we can show this for a single vertex in L, say v_1, without loss of generality. Since $c(uv_1) = 1$, v_1 can reach any vertex v_j in R with $c(uv_j) = 2$ via the path v_1uv_j of length 2. Otherwise, if v_j in R has $c(uv_j) = 1$ with $j < n$, then $v_1uv_{j+1}v_j$ is a path of length 3 from v_1 to v_j. Since the proper distance between v_1 and $v_{n-(k-1)}$ is $n - k \geq 3$ and since the proper distance between any other pair of vertices is either 2 or 3, we see that $\text{pdiam}(G, c) = n - k$, as desired. ■

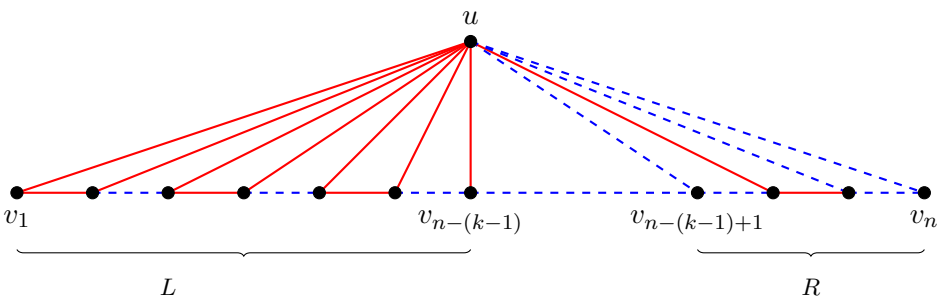

Figure 5. A properly connected fan with $\text{pdiam}_2(F_{1,n}, c) = n - k$ for $1 \leq k \leq n - 3$ and $n \geq 7$.

2.3. Complete multipartite graphs

Finally, we consider properly connected 2-colorings c of complete multipartite graphs $K_{n_1,n_2,\ldots,n_\ell}$ with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$. If $\ell \geq 3$, then $\text{pdiam}_2(K_{n_1,n_2,\ldots,n_\ell}, c) \in [2, 4]$. These bounds are tight and all values are attainable. If $\ell = 2$, then the graph is a complete bipartite graph $K_{n,m}$. Since no pair of vertices in the same partite set can have a proper distance of 3, the only possible values of $\text{pdiam}_2(K_{n,m}, c)$ are 2 and 4. Both values are attainable.

Theorem 6. *Let $K_{n_1,n_2,\ldots,n_\ell}$ be a properly connected 2-colored, complete ℓ-partite graph with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, where $\ell \geq 2$. The proper diameter of $K_{n_1,n_2,\ldots,n_\ell}$ falls in the interval $[2, 4]$.*

Proof. Consider any properly connected 2-coloring c of a complete ℓ-partite graph with $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, where $\ell \geq 2$. Since there is a path of length 1 between vertices in distinct partite sets, we only need to consider the lengths of properly colored paths between vertices in the same partite set. Let

a_1 and a_2 be any two distinct vertices in a partite set A. Let $P = a_1x_1x_2\cdots x_ja_2$ be a shortest properly colored path between a_1 and a_2. Assume for the moment that the proper distance between a_1 and a_2 is at least 5 so that $j \geq 4$. For some h between 2 and $j-1$, there is an x_h on P such that x_h is not in A. Since P is properly colored, we may assume that $c(x_{h-1}x_h) = 1$ and $c(x_hx_{h+1}) = 2$. Then $c(a_1x_h) = 2$ as otherwise $a_1x_hx_{h+1}\cdots x_ja_2$ is a shorter properly colored path than P. Similarly, $c(x_ha_2) = 1$ as otherwise $a_1\cdots x_{h-1}x_ha_2$ is a shorter properly colored path than P. However, since $c(a_1x_h) = 2$ and $c(x_ha_2) = 1$, the path $a_1x_ha_2$ is a properly colored path of length 2 between a_1 and a_2, which is a contradiction. ■

In Theorems 7, 8 and 9, we respectively classify when a proper diameter of 2, 3, or 4 is attainable for complete multipartite graphs. Focusing on complete bipartite graphs, a proper diameter of 2 is attainable except for those $K_{n,m}$ where one partite set is much larger than the other $\big(\max\{n,m\} > 2^{\min\{n,m\}}\big)$. On the other hand, besides those $K_{n,m}$ with very small partite sets ($\min\{n,m\} = 1$ or $\max\{n,m\} \leq 2$) a proper diameter of 4 can always be achieved.

Theorem 7. *Let $K_{n_1,n_2,\ldots,n_\ell}$ be an ℓ-partite graph of order n where $n_\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$. There exists a properly connected 2-coloring of $K_{n_1,n_2,\ldots,n_\ell}$ with proper diameter 2 if and only if $n_\ell \leq 2^{n-n_\ell}$.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_\ell}$. Label the partite sets of G as $A_1, A_2, \ldots, A_\ell$ where $A_i = \{a_1^i, a_2^i, \ldots, a_{n_i}^i\}$, so that partite set A_i has size n_i. We let B refer to the $n-n_\ell$ vertices not in A_ℓ, and for clarity, we label these vertices as $b_1, b_2, \ldots, b_{n-n_\ell}$ when otherwise distinguishing between the distinct partite sets of the vertices of B would overcomplicate the proof.

In any properly connected 2-coloring c of G, each vertex $a \in A_\ell$ has a *coloring tuple* with B, say $(c_1, c_2, \ldots, c_{n-n_\ell})$ which indicates that the edge ab_i has color $c_i \in \{1,2\}$ where $1 \leq i \leq n-n_\ell$. Thus, there are 2^{n-n_ℓ} possible coloring tuples and so 2^{n-n_ℓ} ways to color the edges between any vertex $a \in A_\ell$ and the $n-n_\ell$ vertices of B.

We prove the forward direction by contrapositive. Let $n_\ell > 2^{n-n_\ell}$ and consider any properly connected 2-coloring c of G. We argue that G does not have proper diameter 2. Since $n_\ell > 2^{n-n_\ell}$, the number of vertices in A_ℓ is greater than the number of possible coloring tuples, and thus, at least two vertices in A_ℓ, say a_i^ℓ and a_j^ℓ, have the same coloring tuple with B. This means that for every vertex $b \in B$, $c\big(a_i^\ell b\big) = c\big(a_j^\ell b\big)$ for all $b \in B$. As a result, there does not exist a properly colored path of length 2 between a_i^ℓ and a_j^ℓ and so the proper diameter of G is greater than 2.

For the backwards direction, we construct a coloring with proper diameter 2. The first step of this coloring process is described in this paragraph. For

each A_i where $1 \leq i \leq \ell - 1$, do the following. Consider the edges in the set $\{a_1^i a_1^\ell, a_2^i a_2^\ell, \ldots, a_{n_i}^i a_{n_i}^\ell\}$. This set of edges is a matching between A_i and the first n_i vertices of A_ℓ. Color these edges with color 1. Then color all remaining edges between A_i and the first $n_{\ell-1}$ vertices of A_ℓ, namely, $a_1^\ell, a_2^\ell, \ldots, a_{n_{\ell-1}}^\ell$, with color 2. This portion of the coloring ensures that there is a properly colored path of length 2 between any pair of vertices in A_i, as desired. To see this, for any 2 distinct vertices a_j^i and a_l^i in A_i, note that the path $a_j^i a_j^\ell a_l^i$ is a properly colored path of length 2.

Upon completing the first step of our coloring construction, described in the previous paragraph, observe that a color has been assigned to each of the edges incident to the first $n_{\ell-1}$ vertices of A_ℓ and to none of the edges incident to the final $n_\ell - n_{\ell-1}$ vertices of A_ℓ. Furthermore, by choice of the coloring in step 1, each of the initial $n_{\ell-1}$ vertices of A_ℓ has a distinct coloring tuple $(c_1, c_2, \ldots, c_{n-n_\ell})$ with the $n - n_\ell$ vertices of B. This is due to how we colored the edges incident to vertices in $A_{\ell-1}$, since for all vertices $a_i^\ell \in A_\ell$ where $1 \leq i \leq n_{\ell-1}$, a_i^ℓ is the only vertex connected to $a_i^{\ell-1}$ by an edge colored by c with color 1. Therefore, when done step 1, we used $n_{\ell-1}$ of the 2^{n-n_ℓ} possible coloring tuples with B and so there are $2^{n-n_\ell} - n_{\ell-1}$ that remain.

Finally, we discuss how to color the edges incident to the remaining $n_\ell - n_{\ell-1}$ vertices in A_ℓ. Since $n_\ell \leq 2^{n-n_\ell}$, we see that $n_\ell - n_{\ell-1} \leq 2^{n-n_\ell} - n_{\ell-1}$ which means that the number of vertices connected to A_ℓ with uncolored edges is no larger than the number of unused coloring tuples. Thus, there are enough coloring tuples to ensure that we can color the edges between the remaining vertices of A_ℓ and B so that each remaining vertex $a_j^\ell \in A_\ell$ has a distinct coloring tuple with B. After doing so, all vertices in A_ℓ have a distinct coloring tuple with B.

It remains to show that there is a path of length 2 between any pair of vertices in A_ℓ. Let a_i^ℓ and a_j^ℓ be distinct vertices in A_ℓ with coloring tuples $(c_1, c_2, \ldots, c_{n-n_\ell})$ and $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{n-n_\ell})$, respectively. By construction, these tuples are distinct and so $c_l \neq \hat{c}_l$ for some l. Then $a_i^\ell b_l a_j^\ell$ is a properly colored path of length 2.

Finally, color the edges between the partite sets $A_1, A_2, \ldots, A_{\ell-1}$ in any fashion. The proper diameter of the constructed 2-coloring of G is 2, as desired. ■

Theorem 8. *For $\ell \geq 3$, there exists a properly connected 2-coloring with proper diameter 3 of any complete ℓ-partite graph $K_{n_1, n_2, \ldots, n_\ell}$ where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$ and $n_\ell \geq 2$.*

Proof. Let A, B and C be three partite sets, where A is a largest partite set. First, let us suppose that there are at least four vertices in A. Let A_1 be all but any two vertices of A with all edges of color 1 to B and C. Let A_2 be a single vertex of A with all edges of color 1 to B and all edges of color 2 to C. Let A_3 be a single vertex of A with all edges of color 2 to B and all edges of color 1 to C.

Color the edges between B and C with color 2. Then there is a properly colored path containing a vertex in B and a vertex in C between any pair of vertices in A_1, giving a shortest properly colored path of length 3. Similarly, there is a properly colored path of length 3 between any existing pairs of vertices in B or in C using the vertex in A_3 or A_2, respectively. Since vertices in different partite sets are adjacent, and the vertices in A_1, A_2 and A_3 that are not both in A_1 are connected by a properly colored path of length 2, the proper diameter of this coloring is 3. See Figure 6.

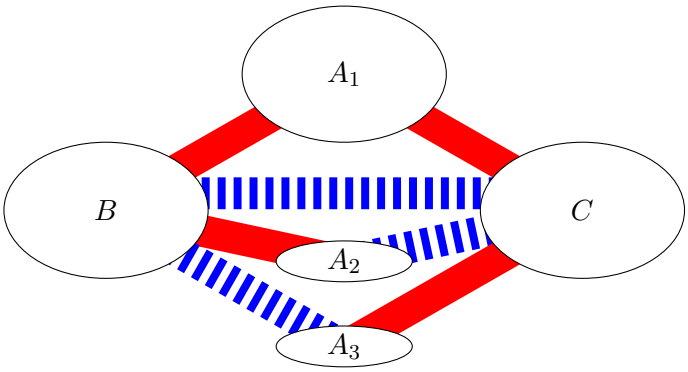

Figure 6. This figure depicts a properly connected 2-coloring of a complete 3-partite graph with proper diameter 3. A thick edge indicates that all edges between the sets are the same color.

To generalize the above coloring for any number of partite sets, color the edges from any additional partite set D to every existing vertex in the graph using color 1. Between each pair of vertices in D, there is a properly colored path of length 3 using an edge of color 1 from D to B, an edge of color 2 from B to C, and an edge of color 1 from C to D.

If A has exactly 3 vertices and at least one of B or C has 2 or 3 vertices, then each set A_1, A_2, and A_3 consists of a single vertex with the same coloring described above. Since either B or C has 2 vertices, the proper diameter of 3 is achieved by the properly colored path between the pair of vertices in that set. If either A has 3 vertices and both B and C have a single vertex or if each partite set has at most 2 vertices, color a single edge between B and C with color 2, and color all remaining edges in the graph with color 1. Then any existing pairs of vertices in B or in C have proper distance 2, and the shortest properly colored path between each pair of vertices in A requires an edge of color 1 from A to B, the edge of color 2 from B to C, and an edge of color 1 from C to A. Since A must have at least 2 vertices, this gives a proper diameter of 3. In any of these cases, the same generalization as above also holds for any additional partite set D by coloring the edges between D and all other partite sets with color 1. ■

Theorem 9. *There exists a properly connected 2-coloring of $K_{n,m}$ with proper diameter 4 if and only if $\min\{n,m\} \geq 2$ and $\max\{n,m\} \geq 3$. Furthermore, for $\ell \geq 3$, there exists a properly connected 2-coloring of any complete ℓ-partite graph $K_{n_1,n_2,\ldots,n_\ell}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, with proper diameter 4 if and only if the largest partite set contains at least three vertices.*

Proof. Let $G = K_{n,m}$ be a complete bipartite graph with a properly connected 2-coloring c. If $\min\{n,m\} = 1$ or $\max\{n,m\} \leq 2$, then $G \cong K_{1,\max\{n,m\}}$ or $G \cong C_4$, none of which can be colored so as to have proper diameter 4. Thus, if $\mathrm{pdiam}_2(K_{n,m}, c) = 4$, then $\min\{n,m\} \geq 2$ and $\max\{n,m\} \geq 3$.

Now, consider $K_{n,m}$ where the smaller partite set has at least 2 vertices and the larger partite set has at least 3. Let A be a set of any two vertices from the larger partite set and let A' be a set containing the remaining vertices of A. Also, let B be a set containing any one vertex from the smaller partite set and let B' contain the remaining vertices of B. Color all edges between A and B and between A' and B' with color 1. Color all other edges with color 2. The proper distance between the two vertices in A is 4 and so the proper diameter of this coloring is 4 as well. See Figure 7 for an example involving $K_{3,5}$. A similar coloring to Figure 7 appears in [2].

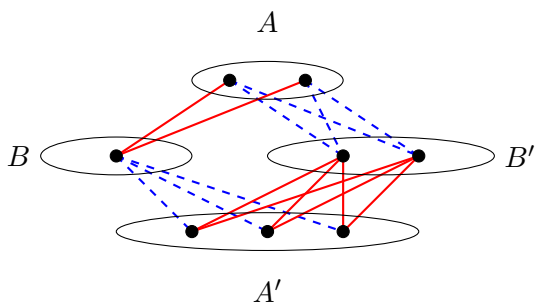

Figure 7. A properly connected 2-coloring of $K_{3,5}$ with proper diameter 4.

For $\ell \geq 3$, suppose that a largest partite set of a complete ℓ-partite graph $K_{n_1,n_2,\ldots,n_\ell}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, contains at least 3 vertices. As in the bipartite case, let A be a set of any two vertices from a largest partite set and let A' contain the remaining vertices. Let B and C denote additional partite sets. Color all edges between A and B, between A' and C, and between B and C with color 1. Color all edges between A and C and between A' and B with color 2. Then the proper distance between the pair of vertices in A is 4, and the proper distance between any other pair of vertices is at most 4. Hence, the proper diameter is 4. For any additional partite set D, color all edges between D and the other partite sets with color 1. Then the proper distance is 3 for any existing pairs of vertices in D, and so the proper diameter remains 4.

Finally, assume a complete ℓ-partite graph $K_{n_1,n_2,\ldots,n_\ell}$, where $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$, has proper diameter 4 and each partite set contains at most 2 vertices. Then there must be a partite set A with exactly 2 vertices, say a_1 and a_2. Suppose $a_1x_1x_2x_3a_2$ is a shortest properly colored path from a_1 to a_2 of length 4 with $c(x_1x_2) = 1$ and $c(x_2x_3) = 2$. Since x_2 is not in A, there is an edge a_1x_2, which must have color 2 otherwise $a_1x_2x_3a_2$ is a shorter properly colored path. Similarly, edge x_2a_2 must have color 1. This leads to an alternating path $a_1x_2a_2$ of length 2, giving a contradiction. Therefore, if no partite set contains at least 3 vertices, then the proper diameter must be less than 4. ■

We conclude with Corollary 10, which follows from Theorem 6, Theorem 7, and Theorem 9, and summarizes the facts about complete bipartite graphs.

Corollary 10. *Let $m \geq n$ and $m \geq 2$. The proper diameter of any properly connected 2-coloring of $K_{n,m}$ is 2 or 4. Moreover,*

(i) *A proper diameter of 2 is attainable if and only if $m \leq 2^n$.*

(ii) *A proper diameter of 4 is attainable if and only if $n \geq 2$ and $m \geq 3$.*

3. General Bounds

Whereas the results of previous sections focus on specific graph families, in this section we provide preliminary bounds on the maximum value of $\mathrm{pdiam}_2(G, c)$ which is $\mathrm{pdiam}_2(G)$ for general graphs G. Analyzing the connectivity of G is a natural starting point, since a large value of $\kappa(G)$ provides density and flexibility in routing from a given vertex to another. Theorem 12 provides a worst-case bound on the proper diameter of a 2-coloring of G and shows that as the connectivity increases for a fixed value of n, the maximum proper diameter for 2-colorings decreases. The proof is by contradiction and relies on Lemma 11. To guide the reader through the proof of Lemma 11, we first assume that a 3-connected graph has a shortest properly colored Hamiltonian path between v_1 and v_t. Then we find a shorter properly colored path by starting at v_1 and iterating through edges of the form $v_{i_j}v_{i_j-1}\cdots v_{i'_{j+1}}v_{i_{j+1}}$ for all j needed to reach v_t. For example, in Figure 8, a shorter properly colored path is $v_1v_{i_1}v_{i_1-1}v_{i'_2}v_{i_2}v_{i_2-1}v_{i'_3}v_{i_3}v_{i_3-1}v_{i'_4}v_{i_4}$.

Lemma 11. *In any properly connected graph G on t vertices with $\kappa(G) \geq 3$,* $\mathrm{pdiam}_2(G) < t - 1$.

Proof. Assume G is a 3-connected graph on t vertices with a properly connected 2-coloring c, where $\mathrm{pdiam}_2(G, c) = t - 1$. Then there exists a shortest properly colored path $P = v_1v_2\cdots v_t$ between some vertices v_1 and v_t. The edges of P

alternate between 2 colors, so suppose $c(v_{2i-1}v_{2i}) = 1$ and $c(v_{2i}v_{2i+1}) = 2$ for each integer i with $1 \leq i \leq \left\lfloor \frac{t-1}{2} \right\rfloor$. Note that $c(v_{t-1}, v_t) = 1$ if t is even.

We now construct a specific sequence of vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_m}$ in order to find a shorter properly colored path from v_1 to v_t than P. Since G is 3-connected, v_1 must have 2 neighbors besides v_2. This corresponds to two edges of the form v_1v_i for $i > 2$. Let i_1 be the largest index such that $v_1v_{i_1} \in E(G)$. In particular, $3 < i_1 < t$. Following similar logic as that just used to define v_{i_1}, define v_{i_j} for $j \geq 2$ reiteratively as below. See Figure 8 for an illustration.

• If v_t is adjacent to a vertex of $\{v_1, v_3, v_4, \ldots, v_{i_{j-1}-1}\}$, then let $i_j = t$ and $m = j$.

• Otherwise, there must be an edge from $L_{i_j} = \{v_1, v_3, v_4, \ldots, v_{i_{j-1}-1}\}$ to $R_{i_j} = \{v_{i_{j-1}+1}, \ldots, v_t\}$. If not, then $\{v_2, v_{i_{j-1}}\}$ is a cut set. From all the edges that exist between L_{i_j} and R_{i_j}, let v_{i_j} be the vertex with the largest index from R_{i_j} that is incident to such an edge. Define e_j as an edge from L_{i_j} to R_{i_j} with endpoints $v_{i'_j}$ and v_{i_j}, respectively. Note that e_j cannot be incident to vertices v_2 or $v_{i_{j-1}}$ because they are excluded from $L_{i_j} \cup R_{i_j}$. We exclude v_2 from L_{i_j} in order to build paths from v_1 to v_t that skip v_2. Note that $i'_j < i'_{j+1}$ by construction.

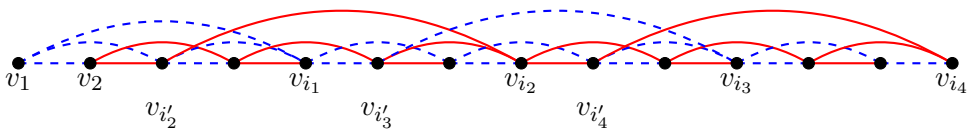

Figure 8. A strategic sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_m}$ along P where $m = 4$.

We proceed by induction on j to show that for $1 \leq j \leq m-1$, we have $c(e_{j+1}) = c\left(v_{i'_{j+1}}v_{i_{j+1}}\right) = c\left(v_{i_{j+1}}v_{i_{j+1}+1}\right)$, and additionally, there is a properly colored path $P_{i_{j+1}}$ from v_1 to $v_{i_{j+1}}$ of length at most $i_j - 1$ with terminal edge $v_{i'_{j+1}}v_{i_{j+1}}$. When $j = 1$, we have $v_{i_{j+1}} = v_{i_2}$. If $i_2 = t$, then depending on whether the color of edge $e_2 = v_{i'_2}v_{i_2} = v_{i'_2}v_t$ differs from $c\left(v_{i'_2-1}v_{i'_2}\right)$ or from $c\left(v_{i'_2}v_{i'_2+1}\right)$, either $v_1v_2\cdots v_{i'_2}v_t$ or $v_1v_{i_1}v_{i_1-1}\cdots v_{i'_2}v_t$, respectively, would be a properly colored path from v_1 to v_t that is shorter than P, as these paths miss v_{i_1} and v_2, respectively. Thus, we may assume that $i_2 < t$. In this case, if $c(e_2) = c\left(v_{i'_2}v_{i_2}\right) = c(v_{i_2-1}v_{i_2})$, then one of $v_1v_2\cdots v_{i'_2}v_{i_2}v_{i_2+1}\cdots v_t$ or $v_1v_{i_1}v_{i_1-1}\cdots v_{i'_2}v_{i_2}v_{i_2+1}\cdots v_t$ would be a shorter properly colored path from v_1 to v_t. Therefore, we must have $c(e_2) = c\left(v_{i'_2}v_{i_2}\right) = c(v_{i_2}v_{i_2+1})$. Additionally, if $c\left(v_{i'_2-1}v_{i'_2}\right) = c\left(v_{i'_2}v_{i_2}\right)$, there is a properly colored path P_{i_2} of length at most $i_1 - 1$, given by $v_1v_{i_1}v_{i_1-1}\cdots v_{i'_2}v_{i_2}$. Note that this path is properly colored only if $c(v_1v_{i_1}) \neq c(v_{i_1}v_{i_1-1})$, which we may assume is true since otherwise

$c(v_1v_{i_1}) \neq c(v_{i_1}v_{i_1+1})$ and then $v_1v_{i_1}v_{i_1+1}\cdots v_t$ is a properly colored path from v_1 to v_t that is shorter than P. If $c\left(v_{i'_2-1}v_{i'_2}\right) \neq c\left(v_{i'_2}v_{i_2}\right)$, then $v_1v_2\cdots v_{i'_2-1}v_{i'_2}v_{i_2}$ is a properly colored path of length at most i_1-1, giving the desired P_{i_2}.

Assume $c\left(v_{i'_{j+1}}v_{i_{j+1}}\right) = c\left(v_{i_{j+1}}v_{i_{j+1}+1}\right)$ for all j such that $1 \leq j < \ell$ for some $\ell \leq m-1$. Additionally, assume that there is a properly colored path $P_{i_{j+1}}$ from v_1 to $v_{i_{j+1}}$ of length at most $i_j - 1$ with terminal edge $v_{i'_{j+1}}v_{i_{j+1}}$. Suppose that $c\left(v_{i'_{\ell+1}}v_{i_{\ell+1}}\right) \neq c\left(v_{i_{\ell+1}}v_{i_{\ell+1}+1}\right)$. Now if $c\left(v_{i'_{\ell+1}-1}v_{i'_{\ell+1}}\right) \neq c\left(v_{i'_{\ell+1}}v_{i_{\ell+1}}\right)$, then $P' = v_1v_2\cdots v_{i'_{\ell+1}}v_{i_{\ell+1}}v_{i_{\ell+1}+1}\cdots v_t$ is a shorter properly colored path from v_1 to v_t that skips v_{i_ℓ}, which is a contradiction. Otherwise, we know that $c\left(v_{i'_{\ell+1}-1}v_{i'_{\ell+1}}\right) = c\left(v_{i'_{\ell+1}}v_{i_{\ell+1}}\right)$.

By induction, there is a properly colored path P_{i_ℓ} from v_1 to v_{i_ℓ} of length at most $i_{\ell-1}-1$ with terminal edge $v_{i'_\ell}v_{i_\ell}$ such that $c\left(v_{i'_\ell}v_{i_\ell}\right) = c(v_{i_\ell}v_{i_\ell+1})$. Thus, $P_{i_\ell}v_{i_\ell-1}\cdots v_{i'_{\ell+1}+1}v_{i'_{\ell+1}}v_{i_{\ell+1}}v_{i_{\ell+1}+1}\cdots v_t$ is a shorter properly colored path from v_1 to v_t, which is a contradiction. This means that our supposition is incorrect and thus $c\left(v_{i'_{\ell+1}}v_{i_{\ell+1}}\right) = c(v_{i_{\ell+1}}v_{i_{\ell+1}+1})$. Moreover, the path $P_{i_\ell}v_{i_\ell-1}\cdots v_{i'_{\ell+1}}v_{i_{\ell+1}}$ is the desired proper path of length at most $i_\ell - 1$ with terminal edge $v_{i'_{\ell+1}}v_{i_{\ell+1}}$. When $\ell = m-1$ this gives a properly colored path from v_1 to $v_{i_m} = v_t$ of length at most $i_{m-1} - 1 < t-1$, yielding a contradiction. ■

Theorem 12. *For any properly connected* 2*-colored graph* G *of order* $n \geq 2$*,*

$$\text{pdiam}_2(G) \leq n - \kappa(G) + 1. \tag{1}$$

The bound in (1) *is tight.*

Proof. Let c be a properly connected 2-coloring of a graph G of order $n \geq 2$. First we will show that the bound is tight by the following construction. Let ℓ be a positive integer and start with an alternating path $P_{2\ell+1} = v_1v_2\cdots v_{2\ell+1}$ in which $c(v_{2i-1}v_{2i}) = 1$ and $c(v_{2i}v_{2i+1}) = 2$ for all i. To this, we add the edges $v_1v_{2\ell}$ and $v_2v_{2\ell+1}$ coloring the former with color 2 and the latter with color 1. This graph is properly connected, the underlying uncolored graph is 2-connected, and the proper diameter is 2ℓ. Finally, we add $k-2$ vertices, coloring all possible edges between each vertex and the vertices of $P_{2\ell+1}$ with color 1, thus inducing a complete graph in color 1. See Figure 9. The resulting graph is still properly connected, the underlying uncolored graph is k-connected, and the proper diameter is still $2\ell = n-k+1$. Note that we achieve the desired bound if $k = \kappa(G)$ and also if $n-k$ is even using a similar construction.

Suppose that the proper diameter exceeds $n - \kappa(G) + 1$. Then there exists a pair of vertices in G, say u and v, such that the proper distance between u and v

is at least $n - \kappa(G) + 2$. Let P be a shortest properly colored path on t vertices from u to v. Note that t is one more than the proper distance between u and v. Let R be the remaining vertices and let $G' = G[P]$. Since $|R| \leq \kappa(G) - 3$, we see that $\kappa(G') \geq 3$. By Lemma 11, $\text{pdiam}_2(G', c) < t - 1$, and so there exists a properly colored path between u and v in G' with length less than P, which is a contradiction. ■

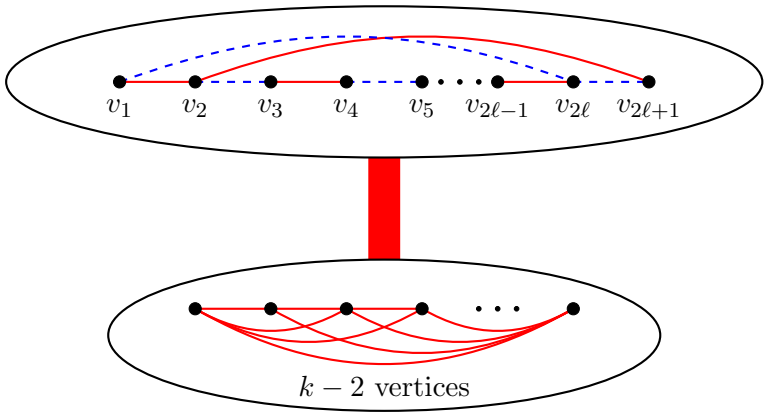

Figure 9. This figure illustrates a properly connected graph G with proper diameter $n-k+1$. The thick solid red edge depicts that all edges between the two sets of vertices are color 1.

Next, we look at Lemma 13, which gives a general lower bound using girth. In Section 2.1 we argue that any properly colored odd cycle on $2m+1$ vertices has proper diameter $2m-1$. This observation generalizes to any graph whose girth is given by an odd cycle and also demonstrates that the bound in Lemma 13 is tight.

Lemma 13. *If G is a graph with properly connected 2-coloring c and girth $2m+1$ for $m \geq 2$, then* $\text{pdiam}_2(G, c) \geq 2m - 1$.

Proof. Let $v_1, v_2, \ldots, v_{2m+1}$ denote the vertices of a $(2m+1)$-cycle in G, with $m \geq 2$. However the edges of the cycle are colored, since there are two colors and an odd number of edges, some pair of adjacent edges $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ must share a color. Thus, the shortest properly colored path from v_i to v_{i+2} on this cycle must have length $2m-1$. Suppose there is a properly colored path of length less than $2m-1$ from v_i to v_{i+2}. Let $v_i w_1 w_2 \cdots w_l v_{i+2}$ denote such a path of minimum length in G. Then $v_i w_1 w_2 \cdots w_l v_{i+2} v_{i+1} v_i$ is a cycle of length less than $2m+1$ since there is no shorter path between any two vertices on this path. Therefore, $\text{pdiam}_2(G, c) \geq 2m - 1$ if G contains an odd cycle with length $2m+1$ and no smaller cycle. ■

We conclude this section by considering how a small modification to a properly connected graph can affect the proper diameter. In general, removing a vertex from a properly connected graph does not necessarily result in a new graph that is still properly connected since the deleted vertex may have been required to obtain a properly colored path between some pair of vertices. Additionally, adding vertices and edges to a graph does not always yield a properly connected graph, depending on how the additional edges are colored. In the following lemma, given a non-complete graph G with a properly connected 2-coloring c, we look at one way to form a new graph G' with a properly connected 2-coloring c' by adding a degree-2 vertex and two edges to G, and we consider how the proper diameter of G' under c' is related to that of G under c. By Lemma 14, under certain conditions, the addition of a single vertex of degree 2 to G at worst doubles the proper diameter of the original edge-colored graph.

Lemma 14. *Let G be a graph with a properly connected 2-coloring c such that* $\mathrm{pdiam}_2(G, c) = d > 1$. *Let $u_1, u_{d+1} \in V(G)$ such that the proper distance between u_1 and u_{d+1} is d, and let $u_1u_2 \cdots u_du_{d+1}$ be a shortest properly colored path from u_1 to u_{d+1}. Let G' be a graph formed by adding a vertex v to G and edges vu_1 and vu_{d+1} with edge-coloring $c' : E(G') \to \{1, 2\}$ such that*

$$c'(e) = \begin{cases} c(e) & \text{if } e \in E(G), \\ 3 - c(u_1u_2) & \text{if } e = u_1v, \\ 3 - c(u_du_{d+1}) & \text{if } e = u_{d+1}v. \end{cases}$$

Then $\mathrm{pdiam}_2(G', c') \leq 2d$.

Figure 10. A properly connected P_5 and P_6 and a degree-2 vertex v in the cases where $d = 4$ and $d = 5$.

Proof. The only edges that do not exist in both G and G' are those incident with v. Since c and c' assign the same colors to all other edges, it suffices to show that for any vertex $w \in V(G')$, there exists a properly colored path of length of at most $2d$ between w and v.

Observe that by definition, $c'(u_1v) \neq c'(u_1u_2)$ and $c'(u_{d+1}v) \neq c'(u_du_{d+1})$. Hence, paths $P_1 = vu_1u_2 \cdots u_du_{d+1}$ and $P_2 = u_1u_2 \cdots u_du_{d+1}v$ are both properly colored in G'. Since P_1 and P_2 each has length $d+1$, the proper distance between v and u_i for $1 \leq i \leq d+1$ is therefore at most $d+1$, as desired.

Let us now consider any vertex $w_1 \neq v$ where w_1 is not on the path $P = u_1u_2\cdots u_{d+1}$. Let $Q = w_1w_2\cdots w_n$ be a shortest properly colored path in G between w_1 and any vertex on P. Then the endpoint of Q is a vertex on P, so $w_n = u_j$ for some $1 \leq j \leq d+1$. Also, the vertices $w_1, w_2, \ldots, w_{n-1}$ are not on P as otherwise there exists a properly colored path that is shorter than Q between w_1 and some vertex on P. Finally, the path Q has length at most d since $\mathrm{pdiam}_2(G,c) = d$.

The path Q exists in G' and its edges are colored the same by c' as c. Since the color of the final edge of Q must be distinct from either $c'(u_{j-1}, u_j)$ or $c'(u_j, u_{j+1})$, either the path QP_1' formed by concatenating Q with $P_1' = u_ju_{j-1}\cdots u_1v$ or the path QP_2' formed by concatenating Q with $P_2' = u_ju_{j+1}\cdots u_{d+1}v$ is a properly colored path from w_1 to v. Note that P_1' and P_2' have length at most d if Q ends at an internal vertex of P but one of P_1' or P_2' has length exactly $d+1$ if Q ends at an endpoint of P. Hence, if Q ends at an internal vertex of P or if Q has length strictly less than d, then both QP_1' and QP_2' have length at most $2d$, and so no matter which of QP_1' and QP_2' is properly colored, the vertices w_1 and v have a proper distance of at most $2d$, as desired.

Otherwise the path Q has length exactly d and ends at an endpoint of P. Since Q by definition is a shortest properly colored path in G between w_1 and any vertex on P and since $\mathrm{pdiam}_2(G,c) = d$, this means that a shortest properly colored path Q' between w_1 and u_2 exists and is also of length exactly d. Say Q' has an internal vertex $u_k \in P = u_1u_2\cdots u_{d+1}$. Then the subpath on Q' from w_1 to u_k yields a properly colored path shorter than d from w_1 to P, giving a contradiction. Hence, only the endpoint of Q' is on P. As before, either concatenating the path Q' with $u_2u_3\cdots u_{d+1}v$ or u_2u_1v yields a properly colored path from w_1 to v. Also, both of these concatenated paths have length at most $2d$ since u_2 is an internal vertex of P. Thus, the claim holds. ■

G	diam(**G**)	$\mathrm{pdiam}_2(\mathbf{G},\mathbf{c})$	**Notes**
path, P	diam(P)	diam(P)	If the path has n vertices, then $\mathrm{diam}(P) = \mathrm{pdiam}_2(P,c) = n-1$ holds for all colorings c.
K_n	1	1	This is the only possibility.
C_{2m}	m	$\{m, 2m-2\}$	These are the only possibilities. See Observation 1.
C_{2m+1}	m	$2m-1$	This is the only possibility. See Observation 1.
$F_{1,n}$, $n \geq 7$	2	$[3, n-1]$	Bounds are tight and all values are attainable. See Theorems 4 and 5.
$K_{n,m}$	2	$\{2,4\}$	See Corollary 10.
$K_{n_1,n_2,\ldots,n_\ell}$	2	$[2,4]$	Bounds are tight and all values are attainable. See Theorems 6 through 9.

Table 1. Diameter vs. proper diameter.

4. Conclusion

Table 1 compares the proper diameter for the graph families discussed in this paper to their diameter. For odd cycles and sufficiently large fan graphs, the proper diameter cannot equal the diameter of the graph using two colors. We also see that fan graphs and complete multipartite graphs (with at least three partite sets) are two families of graphs that can attain any proper diameter value between the upper and lower bounds with two colors.

Acknowledgments

The authors are grateful to the referees for their valuable comments and suggestions. The authors would like to thank an anonymous referee for the suggestion of improving the notation for proper diameter to emphasize the coloring c.

References

[1] E. Andrews, E. Laforge, C. Lumduanhom and P. Zhang, *On proper-path colorings in graphs*, J. Combin. Math. Combin. Comput. **97** (2016) 189–207.

[2] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero and Zs. Tuza, *Proper connection of graphs*, Discrete Math. **312** (2012) 2550–2560. doi:10.1016/j.disc.2011.09.003

[3] G. Chartrand, G. L. Johns, A. K. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[4] G. Chartrand, G. Johns, K. McKeon and P. Zhang, *The rainbow connectivity of a graph*, Networks **54** (2009) 75–81. doi:10.1002/net.20296

[5] S. Fujita, A. Gerek and C. Magnant, *Proper connection with many colors*, J. Comb. **3** (2012) 683–693. doi:10.4310/JOC.2012.v3.n4.a6

[6] E. Laforge, C. Lumduanhom and P. Zhang, *Characterizations of graphs having large proper connection numbers*, Discuss. Math. Graph Theory **36** (2016) 439–453. doi:10.7151/dmgt.1867

[7] E. Laforge, C. Lumduanhom and P. Zhang, *Chromatic-connection in graphs*, Congr. Numer. **225** (2015) 37–54.

[8] X. Li and C. Magnant, *Properly colored notions of connectivity—a dynamic survey*, Theory Appl. Graphs **0** (2015), Iss. 1, Article 2. doi:10.20429/tag.2015.000102

Received 23 January 2017
Revised 15 February 2018
Accepted 15 February 2018