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THE PATH-PAIRABILITY NUMBER OF PRODUCT OF STARS

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Abstract

The study of a graph theory model of certain telecommunications network problems lead to the concept of path-pairability, a variation of weak linkedness of graphs. A graph G is *k-path-pairable* if for any set of $2k$ distinct vertices, s_i, t_i, $1 \leq i \leq k$, there exist pairwise edge-disjoint s_i, t_i-paths in G, for $1 \leq i \leq k$. The *path-pairability number* is the largest k such that G is k-path-pairable. Cliques, stars, the Cartesian product of two cliques (of order at least three) are 'fully pairable'; that is $\lfloor n/2 \rfloor$-pairable, where n is the order of the graph. Here we determine the path-pairability number of the Cartesian product of two stars.

Keywords: path-pairability, weak linkage, Cartesian product, star-like network, telecommunications network.

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1. Introduction

A telecommunications network (such as a data- or telephone network) is a collection of terminal nodes, links, and intermediate nodes which are assembled to enable simultaneous communication between the terminals. In typical applications pairs of communicating terminals are connected through transmission links. The basic characteristic of a given telecommunications network is, how many distinct communication lines can simultaneously function. In terms of graph theory terminals and intermediate nodes of a network are the vertices of a graph, the lines and the transmission links correspond to edges and paths in the graph. The graph theory model of telecommunications networks and the various practical connectedness requirements imposed on the real networks lead to the notion of various linkage or pairing properties of a graph. Here we focus on the 'k-path-pairability' of graphs, a concept introduced by Csaba *et al.* in [1] and has been investigated since then by several authors in [3, 4, 6, 7, 9].

For k fixed, a simple graph G with at least $2k$ vertices is *k-path-pairable* if for any list of $2k$ distinct vertices called *terminals*, $s_1, \ldots, s_k$, $t_1, \ldots, t_k$, there exist pairwise edge-disjoint s_i, t_i-paths in G, for $1 \leq i \leq k$. The concept of k-path-pairability is a variant of *weak k-linkedness*, a property close to edge connectivity, where on the list $s_1, \ldots, s_k$, $t_1, \ldots, t_k$ vertices may repeat. Indeed, a weakly k-linked graph must be k-edge connected (think of the choice of $s_i = s$, $t_i = t$, for $1 \leq i \leq k$, where $s \neq t$); on the other hand, k-edge connectivity of a graph is 'nearly' sufficient for weak k-linkedness (for instance, Huck proved in [5] that, for k odd, a $(k+1)$-edge-connected graph is weakly k-linked). In contrast, Faudree *et al.* showed in [2] that not even high degree is necessary for k-path-pairability: for every k, and sufficiently large n, there is a 3-regular and k-path-pairable graph of order n.

We define the *path-pairability number* of a graph G as $pp(G) = \max\{k \mid G$ is k-path-pairable$\}$. For instance, $pp(K_{2,2}) = 1$, $pp(K_{1,b}) = \lceil b/2 \rceil$, $pp(K_n) = \lfloor n/2 \rfloor$ (where $K_{a,b}$ is the complete $a \times b$ bipartite graph and K_n is the complete graph of order n). A k-path-pairable graph of order $2k$ is simply called *path-pairable.*

The most obvious path-pairable graphs are K_n, for n even, and $K_{1,a}$, for odd a. It is not quite obvious, but an easy exercise to verify that $K_{a,b}$ is path-pairable, for all $a, b \geq 3$ and ab even. The Petersen graph and the 3-cube $Q^{(3)}$ are path-pairable as well. A simple parity argument shows that the $2t$-dimensional hypercube, $Q^{(2t)}$, is not path-pairable. Csaba *et al.* made the following appealing conjecture in 1992 (see [1]) which is still open: the hypercube $Q^{(2t+1)}$ is path-pairable, for every integer $t \geq 1$.

To find path-pairable graphs with small maximum degree it suffices to consider the Cartesian product of two cliques (such that the product has even order at least 6), or the Cartesian product of two complete $n \times n$ bipartite graphs (for n even) (see [7, 9]). It was proved recently by Győri *et al.* in [4] that the Cartesian nth power of K_t is path-pairable, for $t \geq 18$.

Here we discuss the path-pairability number of the Cartesian product of stars, the only path-pairable trees. It is straightforward to show that $pp(K_{1,1} \square K_{1,1}) = pp(K_{1,1} \square K_{1,2}) = 1$ and $pp(K_{1,2} \square K_{1,2}) = pp(K_{1,2} \square K_{1,3}) = 2$ (see Proposition 11 in the Appendix). Our main result (proved in Section 2) answers the conjecture due to Mészáros in [8].

Theorem 1. $pp(K_{1,a} \square K_{1,b}) = \lceil (a+b)/2 \rceil$, *for every* $a, b \geq 3$.

According to Theorem 1 the path-pairability number of the Cartesian product of stars is not bounded. It is a natural question to ask whether the path-pairability number of the product of two non-path-pairable trees can be arbitrarily high. For bounded path-paribility, consider the example of a grid graph $P_n \square P_n$, where P_n is a path on n vertices. (In [6] the bound $pp(P_n \square P_n) \leq 5$ is obtained from a pairing where a cluster of six terminals in a 2×3 corner of the grid $P_n \square P_n$, $n \geq 4$, is cut off from their six terminal pairs by $3+2=5$ edges.) In Section 3 a somewhat unexpected answer is obtained by using a strategy similar to the one in Theorem 1. If $\widehat{K}_{1,m}$ denote a star $K_{1,m}$ with a subdivided edge, then $pp(\widehat{K}_{1,m}) = 1$; meanwhile, for $a, b \geq 3$, $pp(\widehat{K}_{1,a} \square \widehat{K}_{1,b}) \geq \lfloor \min\{a,b\}/2 \rfloor$ (Proposition 7). Determining the exact value of $pp(\widehat{K}_{1,a} \square \widehat{K}_{1,b})$ remains open.

2. THE PRODUCT OF TWO STARS

It was conjectured in [8] that $pp(K_{1,a} \square K_{1,b}) = \lceil (a+b)/2 \rceil$. We shall prove that the conjecture is true except for particular parameter values discussed in the Appendix. Let the vertices of $G(a,b) = K_{1,a} \square K_{1,b}$ be arranged in an $(a+1) \times (b+1)$ array, where the vertices are labeled with (i,j), $0 \leq i \leq a, 0 \leq j \leq b$, such that vertex $(0,0)$ has degree $a+b$, each vertex of the form $(0,j)$ has degree $a+1$, each vertex of the form $(i,0)$ has degree $b+1$, and all other vertices have degree 2. Then each row $A(i) = \{(i,j) \mid 0 \leq j \leq b\}$ induces a copy of $K_{1,b}$ with center $(i,0)$, and each column $B(j) = \{(i,j) \mid 0 \leq i \leq a\}$ induces a copy of $K_{1,a}$ with center $(0,j)$. The set $\Gamma = A(0) \cup B(0)$ induces a star $K_{1,a+b}$ with center at $(0,0)$; it is called the *boundary* star of $G(a,b)$.

Proposition 2. For $2 \leq a \leq b$ and $a+b \geq 6$, $pp(K_{1,a} \square K_{1,b}) \leq \lceil (a+b)/2 \rceil$.

Proof. For $a+b$ even, let $k = a+b+2$, and define the pairs s_ℓ, t_ℓ, $1 \leq \ell \leq k/2$ as follows:

	t_1								
s_1	s_2	s_3	s_4	$\cdots$	$s_{k/2}$	$t_{k/2}$	$t_{k/2-1}$	$\cdots$	t_{a+2}
	t_3	t_2							
	t_4								
	$\cdots$								
	t_a								
	t_{a+1}								

For $a+b$ odd, let $k = a+b+3$, and define the pairs similarly:

	t_1								
s_1	s_2	s_3	s_4	$\cdots$	$s_{k/2}$	$t_{k/2}$	$t_{k/2-1}$	$\cdots$	t_{a+3}
	t_4	t_2	t_3						
	t_5								
	$\cdots$								
	t_{a+1}								
	t_{a+2}								

Observe that $k/2 \leq b+1$; hence there is room for $s_{k/2}$ in $A(1)$. The collection of $k/2$ pairwise edge disjoint s_i, t_i-paths, $1 \leq i \leq k/2$, if it exists, called a *solution.* In each case either $s_2 s_1$ or $s_2 t_1$ is the first edge of the s_2, t_2-path P_2 of a solution. Since all vertices of degree two in $A(1) \cup B(1)$ are occupied by a terminal, P_2 must proceed using either the edge $s_1 - (0,0)$ or the edge $t_1 - (0,0)$. On the other hand, for the same reason, the s_1, t_1-path P_1 must use both edges $s_1 - (0,0)$ and $t_1 - (0,0)$. Thus P_1 and P_2 cannot be edge-disjoint. ■

2.1. Mating terminals

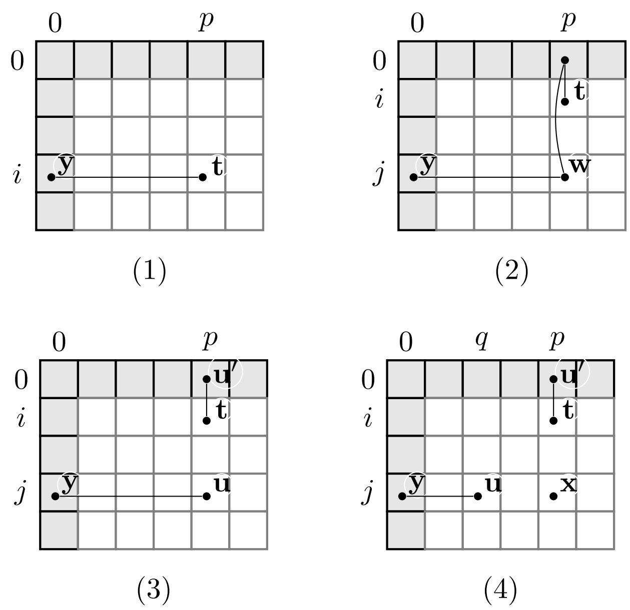
To verify that $G(a,b) = K_{1,a} \square K_{1,b}$ is k-path-pairable we must show that given any $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ of $G(a,b)$, we can find k pairwise edge-disjoint s_ℓ, t_ℓ-paths, for $1 \leq \ell \leq k$. Our procedure to obtain these s_ℓ, t_ℓ-paths will consist of two basic steps. First we assign distinct boundary vertices $\phi(s_\ell), \phi(t_\ell) \in \Gamma$, called the *mates* of s_ℓ and t_ℓ, such that they can be reached from s_ℓ and from t_ℓ via pairwise edge-disjoint *mating paths.* Second, we join each pair of mates using paths of length two through the center vertex in Γ. The concatenation of these three pieces, for each $1 \leq \ell \leq k$, yields a *solution*, the required k pairwise edge-disjoint paths.

For a terminal $u = (i,p) \notin \Gamma$, the vertices $(i,0)$ and $(0,p)$ are the possible 1-*step boundary mates* of u reached by the mating path $u - (i,0)$ or $u - (0,p)$. Vertices $(0,q)$ and $(j,0)$ such that $q \neq p$ and $j \neq i$ are the possible 3-*step boundary mates* of u. A 3-step boundary mate of u is reached by the mating path $u - (i,0) - (i,q) - (0,q)$ or $u - (0,p) - (j,p) - (j,0)$ through *transit vertex* (i,q) or (j,p).

Lemma 3. *For $n \le a+b$, and for any set X of n distinct terminals of $G(a,b)$, there is an injection $\phi : X \longmapsto \Gamma$ such that $\phi(x) = x$, for every $x \in X \cap \Gamma$, and $\phi(x)$ is a 1- or 3-step boundary mate of x, for every $x \in X \setminus \Gamma$, furthermore, the mating paths determined by ϕ are pairwise edge-disjoint.*

Proof. Let $L_0 = X \cap \Gamma$ and set $\phi(x) = x$, for each $x \in L_0$. The definition of ϕ will be successively extended for the terminals in $X \setminus L_0$ via an algorithm, which repeatedly updates an increasing set $L_1 \cup L_3$ of terminals already associated with 1- or 3-step boundary mates. At any given stage of the algorithm, L_i contains all terminals t associated with an i-step boundary mate $t' \in L_i'$. An auxiliary set T is also maintained and updated including the transit vertices belonging to the mating paths of length 3 between a terminal $t \in L_3$ and its actual mate $t' \in L_3'$. Set $L = L_0 \cup L_1 \cup L_3$, $L' = L_0 \cup L_1' \cup L_3'$. The algorithm terminates when $L = X$, and then $\phi(t) = t'$, for all $t \in L$.

If $L \ne X$, there is an unmated terminal $t \in X \setminus L$, and since $|L| < |X| \le a+b$, there is an unused boundary vertex $y \in \Gamma \setminus L'$. Let $t = (i,p)$ and $y = (j,0)$ (or $y = (0,q)$). Using the symmetry of the rows and columns in $G(a,b)$ we assume that $y = (j,0) \in \Gamma \setminus L'$. The mate of t is assigned as follows.



(1) If $j = i$, then we include t and $t' = y$ to L_1 and L_1', respectively. At each stage of the algorithm repeat this step as long as possible. If no such pair

t, y exists, then for each unmated terminal $t = (i, p) \in X \setminus L$ and for each unused boundary vertex $y = (j, 0)$ and $y = (0, q)$, we have $j \neq i$ and $q \neq p$.

(2) If $w = (j, p) \notin X \cup T$, then the path $t - (0, p) - w - y$ does not interfere with the mating paths used between the terminals in L and their mate in L'. Now we redefine the sets L_3, L'_3 and T by including t into L_3, w into T, and $t' = y$ into L'_3.

Assume now that $t = (i, p) \in X \setminus L$, $y = (j, 0) \in \Gamma \setminus L'$, and neither (1) nor (2) applies, in particular, $(j, p) \in L \cup T$.

(3) If $u = (j, p) \in L$, and since y is unused, we have $u \in L_1$ and $u' = (0, p) \in L'_1$. Now we redefine the 1-step mate of u by letting $u' = y$ and reassigning $(0, p)$ as the 1-step mate of t. The sets L_1 and L'_1 are updated according to the changes made in the mapping ϕ.

(4) If $x = (j, p) \in T$, then x is supposed to be used as a transit vertex in a 3-step mating path between some $u = (j, q) \in L_3$ and $u' = (0, p) \in L'_3$. Since y is unused, it is not possible due to our assumption.

The algorithm terminates when $L = X$, and then $\phi(t) = t'$, for all $t \in X$. ■

2.2. Mating lemma extended

In case of $a+b+1$ terminals there is not enough room for the mates in $\Gamma \setminus \{(0,0)\}$ and our procedure in Section 2.1 must be modified. The strategy is simple, before we start injecting the terminals into Γ we 'reserve' an s_ℓ, t_ℓ-path of length at most 6, for some ℓ, such that this reserved path does not use edges in Γ. Reserving a path will be done by 'blocking' the participating edges from being used later in the mating procedure. Lemma 4 extends Lemma 3 just for this purpose.

Lemma 4. *Let X be a set of $a + b$ distinct terminals in $G(a, b)$ and let Z be a set of at most three vertices of $V(G(a, b)) \setminus X$. Assume that for a set $X_0 \subseteq X$ there is an injective pre-map $\phi_0 : X_0 \longmapsto \Gamma$ such that*

– *$\phi_0(x) = x$, for every $x \in X_0 \cap \Gamma$,*
– *$\phi_0(x)$ is a 1- or 3-step boundary mate of x, for every $x \in X_0 \setminus \Gamma$,*
– *$(0, j), (i, 0) \in \phi_0(X_0)$, for every $(i, j) \in Z$,*
– *the mating paths determined by ϕ_0 are edge-disjoint and disjoint from Z.*

Then ϕ_0 has an extension to an injection $\phi : X \longmapsto \Gamma$ such that $\phi(x) = x$, for $x \in X \cap \Gamma$, $\phi(x)$ is a 1- or 3-step boundary mate of x, for every $x \in X \setminus \Gamma$, and the mating paths determined by ϕ are edge-disjoint and disjoint from Z.

Proof. We use the notation of Lemma 3. In particular, $L_0 = X \cap \Gamma$. Observe that the case $Z = \emptyset$ with $X_0 = L_0$ and the identity map $\phi_0 : X_0 \longmapsto X_0$ is the claim we proved by using the algorithm in Lemma 3.

For $|Z| \geq 1$, we repeat the algorithm in Lemma 3 by setting the initial values implied by the partial injection ϕ_0 as follows. Let $L = X_0$ and $L' = \phi(X_0)$ partitioned into $L_0 \cup L_1 \cup L_3$ and $L_0 \cup L'_1 \cup L'_3$, respectively, where L'_h is the set of all h-step mates assigned by ϕ_0 to the terminals in L_h, for $h = 1, 3$. Furthermore, let T_0 be the set of all transit vertices used in the mating paths of length 3 between $x \in X_0$ and its mate $\phi_0(x)$, and set $T = T_0 \cup Z$.

If $L \neq X$, there is an unmated terminal $t = (i, j) \in X \setminus L$, and there is an unused boundary vertex $y = (p, 0) \in \Gamma \setminus L'$ (or $y = (0, q) \in \Gamma \setminus L'$). Observe that $(p, j) \notin Z$ (or $(i, q) \notin Z$), since otherwise, $(p, 0) \in L'$ (or $(0, q) \in L'$), by our assumption on Z. As a consequence, a terminal in Z is never considered in steps (1)–(4) of the algorithm in Lemma 3; in fact, $t, u, x, z \notin Z$ in the corresponding steps (1)–(4). Thus Lemma 3 yields the required extension ϕ. ■

When we apply the mating lemma an s_ℓ, t_ℓ-path will be reserved by specifying a set Z of at most 3 vertices containing t_ℓ, together with X_0 and $\phi_0(X_0)$ such that the assumptions of Lemma 4 are satisfied. We will say that the mates in $\phi_0(X_0)$ are 'blocking' the terminals and the transit vertices in Z.

2.3. The row-column bipartite graph

Let H be the $a \times b$ bipartite graph of the rows and columns of $G(a, b)$ with vertex set $V(H) = \{A(1), \ldots, A(a)\} \cup \{B(1), \ldots, B(b)\}$, and there is an edge between $A(i)$ and $B(j)$, $1 \leq i \leq a, 1 \leq j \leq b$, if and only if $(i, j) \in X$. (Here we assume that $A(0) \cup B(0)$ contains no terminal; this restriction will be reconsidered later.) We will need an elementary lemma in Section 2.4.

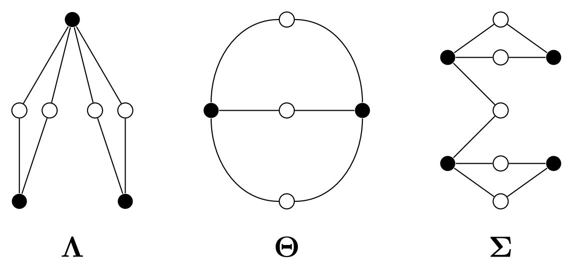


Figure 1. The base graphs in Lemma 5.

Lemma 5. *Let $\mathcal{H}$ be the family of all connected bipartite graphs such that each $H \in \mathcal{H}$ has $|V(H)| + 1$ edges and the degree of each vertex in one of the partition classes of H is equal to 2. Then every member of $\mathcal{H}$ can be obtained starting with one of the base graphs Λ, Θ, Σ in Figure 1 and by repeating even extensions: subdividing an edge with an even number of vertices or suspending a path of even length at any vertex of the smaller partition class.*

Proof. Let $H \in \mathcal{H}$ be a bipartite graph, with partition classes A and B, $|A| = a$, $|B| = b$, and assume that the vertices in B have degree 2. Because H has $2b = a + b + 1$ vertices we obtain $b = a + 1$. First observe that starting with H any of the described even extensions produce members in $\mathcal{H}$. Assume that $H \in \mathcal{H}$ is minimal with respect to the (inverse of the) two even extensions; in particular,

(i) $d_H(x) \geq 2$, for every $x \in A$, and

(ii) for a 4-path (x_0, y, x, y_0), if $x_0 \in A$ and $d_H(x) = 2$, then $x_0y_0 \in E(H)$.

We show that H is one of the graphs Λ, Θ, Σ. Since H has $2a + 2$ edges incident with a vertices of degree at least 2, by (i), the possible degree sequence of the vertices in A are $(4, \overbrace{2, \ldots, 2}^{a-1})$ and $(3, 3, \overbrace{2, \ldots, 2}^{a-2})$.

If $x \in A$ has degree 4, then by (ii), $H - x$ is the union of two paths of length 2, and $H \cong \Lambda$ follows. If $x_1, x_2 \in A$ are the two vertices of degree 3, then connectivity of H and (ii) imply that x_1, x_2 have a common neighbor y. Then $H - \{x_1, x_2, y\}$ induces two paths of length 0 or 2. Using property (ii) again, we obtain $H \cong \Theta$ or $H \cong \Sigma$. ■

Observe that with the only exception of Θ, every graph in $\mathcal{H}$ has a path of length 4 with end vertices lying in the smaller partition class. Lemma 5 has no natural extension for non-connected graphs, we will use the following observation, instead.

Proposition 6. *If a non-connected bipartite graph H contains $|V(H)| + 1$ edges and the degree of each vertex in one of the partition classes is equal to 2, then H has a connected component which is either a cycle or an even path with end vertices in the smaller partition class.*

Proof. Let H be an $a \times b$ bipartite graph, and let A and B be the partition classes with $|A| = a$, $|B| = b$. Let H_i, $1 \leq i \leq c$, with $c \geq 2$, be the connected components of H, where each H_i is an $a_i \times b_i$ bipartite graph and $\sum a_i = a$ and $\sum b_i = b$. If the vertices in B have degree 2, then H has $2b = a + b + 1$ edges.

By the connectivity of the components, we have $2b_i \geq a_i + b_i - 1$, for every $i = 1, \ldots, c$, with equality if H_i is an even path with end vertices in A. If it does not happen, then $2b_i \geq a_i + b_i$, for every $i = 1, \ldots, c$, with equality provided H_i is an even cycle. Assuming that this is not the case either, we have $2b_i \geq a_i + b_i + 1$, for every $i = 1, \ldots, c$. Adding up these inequalities results in $a + b + 1 = 2b \geq a + b + c$. Thus we obtain $c = 1$, a contradiction. ■

2.4. Proof of the main theorem

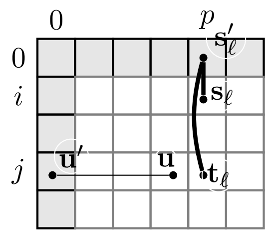
We prove that for every $a, b \geq 2$, except the case $\{a, b\} = \{2, 3\}$,

$$pp(K_{1,a} \square K_{1,b}) = \lceil (a + b)/2 \rceil.$$

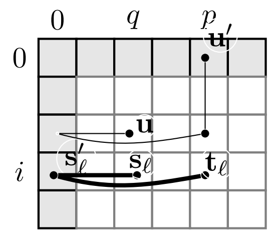
Proof. Let $a \le b$ and $k = \lceil (a+b)/2 \rceil$. By Proposition 2, it is enough to show that $G(a,b)$ is k-path-pairable. Let $X = \{s_1, t_1, \dots, s_k, t_k\}$ be a set of $2k$ distinct terminals.

For $a+b$ even, $2k = a+b$, and by Lemma 3, it follows that the terminals have an injection into 1- or 3-step mates on the boundary Γ of $G(a,b)$ along with edge-disjoint paths. Then for every pair s_ℓ, t_ℓ, $1 \le \ell \le k$, the mates $\phi(s_\ell)$ and $\phi(t_\ell)$ can be joined in Γ. The concatenation of these three pieces produces the required s_ℓ, t_ℓ-paths.

From now on we assume that $a+b$ is odd, in particular, $b \ge a+1$ and $2k = a+b+1$. In this case $\Gamma - (0,0)$ cannot receive the mates of all terminals, one pair s_ℓ, t_ℓ must be joined with a path P_ℓ not using edges of Γ. To make sure that the remaining pairs have an injection into Γ we need to 'reserve' P_ℓ by 'blocking' the edges of P_ℓ from being used in the mating procedure. In a case-by-case analysis we apply Lemma 4 that sets up a blocking by specifying an appropriate pre-map ϕ_0. The first two types of blocking are used when members of some pair of terminals lie in the same row or column.



(α)



(β)

(α) Let $s_\ell = (i,p), t_\ell = (j,p)$, where $0 \le i, j \le a$, $1 \le p \le b$. We reserve the path $s_\ell - (0,p) - t_\ell$ (or $s_\ell - t_\ell$). Let $u \in X \setminus B(p)$ be a terminal closest to $(j,0)$, and apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell\}$, $\phi_0(s_\ell) = (0,p)$, and $\phi_0(u) = (j,0)$.

(β) Let $s_\ell = (i,q), t_\ell = (i,p)$, where $0 \le p, q \le b$, $1 \le i \le a$. We reserve the path $s_\ell - (i,0) - t_\ell$ (or $s_\ell - t_\ell$). Let $u \in X \setminus A(i)$ be a terminal closest to $(0,p)$. Now we apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell\}$, $\phi_0(s_\ell) = (i,0)$, and $\phi_0(u) = (0,p)$.

Concerning the choice of a 'closest' terminal u in the blockings above Figure (α) illustrates the case $(X \cap A(j)) \setminus \{t_\ell\} \ne \emptyset$ when u' is a 1-step mate of u; Figure (β) illustrates the case $(X \cap B(p)) \setminus \{t_\ell\} = \emptyset$ when u' is a 3-step mate of u.

Due to (α) and (β), we may assume for every pair of terminals $s_\ell = (i,p), t_\ell = (j,q)$, that $i \ne j$ and $p \ne q$. We also assume $(0,0) \notin X$, since otherwise, Γ can be filled with mates of the remaining $a+b$ terminals using Lemma 3, and we are done.

First we are dealing with the terminals in $X \cap \Gamma$. For each $x = (0,q) \in X$, if $(i,q) \notin X$, for some $1 \le i \le a$, then we replace x with $x^* = (i,q)$. Suppose now that there is a pairing with the modified set X^* of terminals. Let P be the path connecting s_ℓ^* and t_ℓ^*. Then P either contains a subpath from s_ℓ to t_ℓ, or one (or both) of the edges $s_\ell^* - s_\ell$ and $t_\ell^* - t_\ell$ are not used in the pairing, in which case they can be added to P forming a path from s_ℓ to t_ℓ. From a pairing for the pairs in X^* thus we derive a pairing for the original set X of terminals. The same reasoning can be repeated for the terminals lying in $B(0)$. Thus we may assume that

$$(0,q) \in X \text{ implies } (i,q) \in X, \text{ for every } 1 \le i \le a,$$
$$(p,0) \in X \text{ implies } (p,j) \in X, \text{ for every } 1 \le j \le b.$$

Since $2(b+1) > a+b+1$, we have $|X \cap B(0)| \le 1$, and in the case of equality, we also have $m = |X \cap A(0)| \le 1$. Observe that for $X \cap B(0) = \emptyset$, the average number of terminals per column is $(a+b+1)/b \le 2$ with equality for $b = a+1$.

Case I. Every column $B(j)$, $1 \le j \le b$, contains exactly two terminals. Since $a \ge 2$, we have $X \cap \Gamma = \emptyset$ and $a = b-1$. Let H be the $a \times b$ raw-column bipartite graph of $G(a,b)$ where each edge $(i,j) \in H$ corresponds to the terminal $(i,j) \in X$ located at $A(i) \cap B(j)$.

Assume that H is not connected. Then, by Proposition 6, H has a component H_1 which is a cycle or an even path. Let $H_0 = H - H_1$. If there is a pair s_ℓ, t_ℓ separated by H_0 and H_1, say $s_\ell \in H_0$, $t_\ell \in H_1$, then blocking a path between them will be done as follows.

(γ) Let $s_\ell = (i,p), t_\ell = (j,q)$, with $i \ne j, p \ne q$; then $(i,q) \notin X$ (since s_ℓ, t_ℓ are in distinct components). We reserve the path $s_\ell - (i,0) - (i,q) - (0,q) - t_\ell$. Let $u \in (X \cap B(p)) \setminus \{s_\ell\}$ and let $v \in (X \cap B(q)) \setminus \{t_\ell\}$. We apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell, (i,q)\}$, $\phi_0(u) = (j,0)$, $\phi_0(v) = (0,q)$, and $\phi(s_\ell) = (i,0)$.

Notice that (γ) can be used under the weaker condition that $(i,q) \notin X$ and each of $B(p)$ and $B(q)$ contains at least two terminals (if $|X \cap B(p)| > 2$, then u is selected to be the closest terminal to $(j,0)$).

Now we assume that there is no pair of terminals separated by distinct connected components of H. Let s_ℓ, t_ℓ belong to H_1. Assuming that (γ) does not apply, and since H_1 is either an even path or a cycle, we obtain that $H_1 \cong K_{2,2}$ containing another pair, simply denote them u, v. Reserving a path for (s_ℓ, t_ℓ) will be done as follows.

(δ) Let $s_\ell = (i,p), t_\ell = (j,q), u = (i,q), v = (j,p)$ (with $i \ne j, p \ne q$), and let $w = (h,r) \in X$, for $h \notin \{i,j\}$, $r \notin \{p,q\}$. To reserve the path $s_\ell - (i,0) - (i,r) - (0,r) - (j,r) - (j,0) - t_\ell$ we apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell, (i,r), (j,r)\}$, $\phi_0(s_\ell) = (i,0)$, $\phi_0(u) = (0,q)$, $\phi_0(w) = (0,r)$, and $\phi_0(v) = (j,0)$.

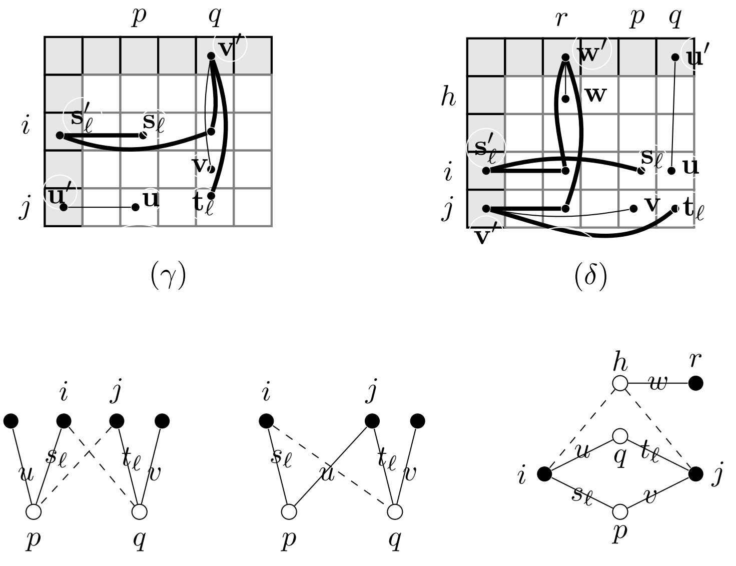


Figure 2. (γ) and (δ) in terms of H.

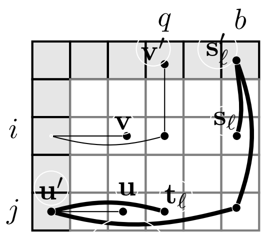
Next let H be connected. By Lemma 5, H is either isomorphic to a base graph $H_0 \in \{\Lambda, \Sigma, \Theta\}$ or obtained from a base graph by even extensions. Observe that $H \ncong \Theta$, since otherwise we have the case $a = 2, b = 3$ excluded in the theorem.

A fairly straightforward case analysis shows that H always contains an appropriate pair s_ℓ, t_ℓ, such that $s_\ell = (i, p)$ is an edge of H belonging to an (eventually subdivided) copy of a $K_{2,2} \subset H_0$, together with $t_\ell = (j, q)$, u, v and w, as shown in Figure 2 (dashed lines indicate non-edges of H). In the first two cases (γ) works, in the third case (δ) applies.

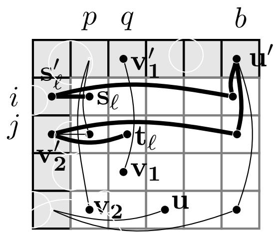
Case II. There is a column not in Γ which contains exactly one terminal. Let $X \cap B(b) = \{s_\ell\}$, $s_\ell = (i, b)$ and $t_\ell = (j, q)$. Observe that $i \neq 0$ and assume $i \neq j$. We reserve the path $s_\ell - (0, b) - (j, b) - (j, 0) - t_\ell$. Let $u \in X \setminus (\{s_\ell, t_\ell\} \cup B(q))$ be a terminal closest to $(j, 0)$. Let $v \in X \setminus (\{s_\ell, t_\ell\} \cup A(j))$ be a terminal different from u and closest to $(0, q)$. (Such u and v exist, because $|X| \geq b + 3$.) Now we apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell, (j, b)\}$, $\phi_0(s_\ell) = (0, b)$, $\phi_0(u) = (j, 0)$, $\phi_0(v) = (0, q)$.

Case III. There is a column containing no terminal. Let $X \cap B(b) = \emptyset$, and assume that Case II does not apply, thus each column has 0 or at least 2 terminals. Recall that $|X \cap B(0)| \leq 1$ and either $|X \cap A(0)| \leq 2$ or $|X \cap \Gamma| = |X \cap A(0)| \geq 3$.

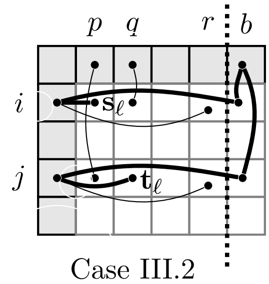
Case III.1. First we assume that $X' = X \setminus \Gamma$ is not contained in the union of two rows. Let $s_\ell = (i, p) \in X'$, $t_\ell = (j, q) \in X'$ (with $i \neq j$, $p \neq q$). We shall reserve the path $s_\ell - (i, 0) - (i, b) - (0, b) - (j, b) - (j, 0) - t_\ell$.



Case II



Case III.1



Case III.2

By assumption, there is a terminal $u \in X' \setminus (A(i) \cup A(j))$; assume without loss of generality that $u \notin B(q)$. Next we select a terminal $v_1 \in (X \cap B(q)) \setminus \{t_\ell\}$ closest to $(0, q)$. There still remain terminals not in $B(q) \cup \{u, s_\ell\}$, let $v_2 \in X \setminus (B(q) \cup \{u, s_\ell\})$ be a terminal closest to $(j, 0)$. Now we apply Lemma 4 with $X \setminus \{t_\ell\}$, $Z = \{t_\ell, (i, b), (j, b)\}$, $\phi_0(s_\ell) = (i, 0)$, $\phi_0(u) = (0, b)$, $\phi_0(v_1) = (0, q)$, and $\phi_0(v_2) = (j, 0)$ (see Figure Case III.1).

Case III.2. Next we assume that $m = |X \cap A(0)| \leq 2$, and $X' = X \setminus \Gamma$ belongs to the union of two rows, $X' \subseteq A(i) \cup A(j)$ $(1 \leq i < j \leq a)$.

For $s_\ell = (i, p)$, $t_\ell = (j, q)$ (with $p \neq q$), we reserve the vertices of the path $s_\ell - (i, 0) - (i, b) - (0, b) - (j, b) - (j, 0) - t_\ell$ not in $B(b)$. Then we remove the pair s_ℓ, t_ℓ from X, and apply Lemma 4 on $G(a, b-1) \cong G(a, b) - B(b)$ with the remaining $(a+b-1)/2$ pairs. For $r \notin \{p, q\}$, let $B(r)$ be a column containing (two) terminals. Let $Z = \{s_\ell, t_\ell\}$, $\phi_0(j, p) = (0, p)$, $\phi_0(i, q) = (0, q)$. Set $\phi_0(i, r) = (i, 0)$ and $\phi_0(j, r) = (j, 0)$, unless one of $(i, 0)$ or $(j, 0)$ is a terminal, in which case we set $\phi_0(i, 0) = (i, 0)$ or $\phi_0(j, 0) = (j, 0)$, respectively (see Figure Case III.2).

Case III.3. Finally we consider the case when $m = |X \cap A(0)| \geq 3$. Since $m(a + 1) \leq a + b + 1$ and $a \geq 2$, we obtain $m < (a + b + 1)/2$. Hence there is a pair of terminals not in $A(0)$, let $s_\ell = (i, p), t_\ell = (j, q)$, where $1 \leq i, j \leq a$ and $1 \leq p, q < b$ (with $i \neq j, p \neq q$). Since $m \geq 3$, there are distinct terminals $u_1, u_2 \in A(i) \setminus \{s_\ell\}$, and there are distinct terminals $v_1, v_2 \in A(j) \setminus \{t_\ell\}$ such that u_2 is closest to $(0, q)$ and v_2 is closest to $(0, p)$.

We reserve the path $s_\ell - (i, 0) - (i, b) - (0, b) - (j, b) - (j, 0) - t_\ell$ as in Case III.2 and remove an appropriate pair of terminals from X. Then we apply Lemma 4 on $G(a, b-1) \cong G(a, b) - B(b)$ with the set of terminals $X \setminus \{s_\ell, t_\ell\}$, and by setting $Z = \{s_\ell, t_\ell\}$, $\phi_0(u_1) = (i, 0)$, $\phi_0(v_1) = (j, 0)$, $\phi_0(u_2) = (0, q)$, $\phi_0(v_2) = (0, p)$. ∎

3. The Product of Two Trees Different From a Star

A k-path-pairable graph G obviously satisfies the condition that for any subset $A \subset V(G)$, $|A| \leq k$, the number of edges from A to $V(G) \setminus A$ is at least k.

Based on this 'cut condition' it follows easily that the Cartesian product of a non-star tree T has $pp(T) = 1$. One would expect that for non-star trees T_1, T_2 $pp(T_1 \Box T_2) \leq c$, with some constant c. Our next result shows that this is not always the case.

Proposition 7. *If $\widehat{K}_{1,m}$ denote a star $K_{1,m}$ with a subdivided edge, then for $a, b \geq 3$, $pp(\widehat{K}_{1,a} \Box \widehat{K}_{1,b}) \geq \lfloor \min\{a/2, b/2\} \rfloor$.*

Proof. Let $x_0 \in \widehat{K}_{1,a}$ and $y_0 \in \widehat{K}_{1,b}$ be the leaf incident to the subdivided edge, and let x_1 and y_1 be their neighbors of degree two, respectively. We use the notations $A(x_0) = \{x_0\} \Box \widehat{K}_{1,b}$, $A(x_1) = \{x_1\} \Box \widehat{K}_{1,b}$, and $B(y_0) = \widehat{K}_{1,a} \Box \{y_0\}$, $B(y_1) = \widehat{K}_{1,a} \Box \{y_1\}$. Let $G = \widehat{K}_{1,a} \Box \widehat{K}_{1,b}$, set $z_0 = \{x_0\} \Box \{y_0\}$, $z_1 = \{x_1\} \Box \{y_1\}$ and let Q be the square induced by z_1, z_0 and their two common neighbors z_2, z_3. For a vertex $v \in (A(x_0) \cup B(y_0)) \setminus \{z_0\}$, let v' be the unique neighbor of v in the subgraph $G' = G - (A(x_0) \cup B(y_0)) \cong K_{1,a} \Box K_{1,b}$ (see Figure 3).

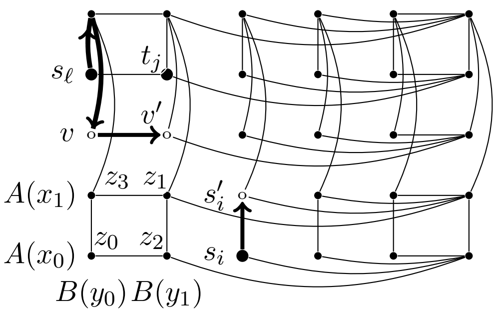


Figure 3. Mating into $K_{1,3} \Box K_{1,4}$.

Given a pairing of $\min\{a, b\}$ (or $\min\{a, b\} - 1$) terminals, our goal is to mate all terminals in $A(x_0) \cup B(y_0)$ into the subgraph $G' = G - (A(x_0) \cup B(y_0)) \cong K_{1,a} \Box K_{1,b}$ using pairwise edge disjoint mating paths and not using edges of G'.

If z_0 is a terminal, then mate it into a free (non-terminal) vertex of Q by using a mating path of length 1 or 2. If every vertex of Q is a terminal, then mate z_0 with z_1 along the mating path (z_0, z_2, z_1).

For $s_i \in (A(x_0) \cup B(y_0)) \setminus \{z_0\}$, if s_i' is free (it is not a terminal or a mate), then let s_i' be the mate of s_i. Let α be the number of pairs w, w' such that $w \in (A(x_0) \cup B(y_0)) \setminus \{z_0\}$ and precisely one of w and w' is a terminal (or mate).

Let $s_\ell \in B(y_0) \setminus \{z_0\}$ and assume that s_ℓ' is a terminal (say $s_\ell' = t_j$, $j \neq \ell$). If there exists an auxiliary vertex $v \in B(y_0) \setminus \{z_0, z_3\}$ such that v and v' are both free, then we use the mating path of length 3 from s_ℓ to v' through v as indicated in Figure 3. One finds a mating path of length 3 in the same way for $s_\ell \in A(x_0) \setminus \{z_0\}$ using a free pair of auxiliary vertices v, v', where $v \in A(x_0) \setminus \{z_0, z_2\}$. Assume that there are β pairs w, w' such that $w \in (A(x_0) \cup B(y_0)) \setminus \{z_0\}$ and both w and w' are terminals (or mates). For $a \le b$, there are at most a terminals in G, hence $\alpha + 2\beta \le a$. This implies that

$$\beta \le a - (\alpha + \beta) \le |B(y_0) \setminus \{z_0, z_3\}| - (\alpha + \beta) = |A(x_0) \setminus \{z_0, z_2\}| - (\alpha + \beta).$$

Therefore, the number of the free pairs of auxiliary vertices in $B(y_0) \setminus \{z_0\, z_3\}$ and also in $A(x_0) \setminus \{z_0, z_2\}$ is not smaller than β, hence the mating procedure succeeds.

Then we are done by induction, unless every vertex of Q is a terminal. If this exceptional case happens, then all the $\min\{a, b\}$ mates and terminals are in $G' \cong K_{1,a} \Box K_{1,b}$ in such a way that the mate of z_0 and terminal z_1 coincide at a vertex of degree two of G'. To handle this case we need a variation of Lemma 3. Recall that Γ is the set of vertices of the $(a + b)$-star subgraph of $K_{1,a} \Box K_{1,b}$.

Lemma 8. *For $n \le \min\{a, b\}$, let X be a set of $n - 1$ distinct terminals in $K_{1,a} \Box K_{1,b}$ plus one additional terminal located at a vertex of degree two occupied by another terminal. Then there is a mapping $\phi : X \longmapsto \Gamma$ such that $|\phi(X)| = n$, $\phi(x) = x$, for $x \in X \cap \Gamma$, $\phi(x)$ is a 1- or 3-step boundary mate of x, for every $x \in X \setminus \Gamma$, and the mating paths determined by ϕ are pairwise edge-disjoint.*

Proof. Let $\Gamma = \Gamma_a \cup \Gamma_b$, where Γ_a induces a $K_{1,a}$ and Γ_b induces a $K_{1,b}$. Let s_i, s_j be the terminals located at the same vertex. We follow the proof of Lemma 3. Observe that during the procedure when $s_i\ s_j$ are to be mated into Γ, there is a free vertex in both Γ_a and Γ_b, since $n - 2 < \min\{a, b\}$. Then we mate s_i into a free vertex in Γ_a, and for s_j, we use a free vertex in Γ_b. □

After applying Lemma 8 the proof of the proposition concludes with the solution of the pairing in the star induced by Γ. ■

Note that if one of the graphs in Proposition 7 is replaced with a star by subdividing one of its edges twice, then the path-pairability of their Cartesian products drops below 6, due to the cut condition.

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A. Appendix

In the tables below the vertices of $G(a,b) = K_{1,a}\Box K_{1,b}$ are arranged in an array such that each row $A(i) = \{(i,j) \mid 0 \le j \le b\}$ induces a copy of $K_{1,b}$ with center $(i,0)$, and each column $B(j) = \{(i,j) \mid 0 \le i \le a\}$ induces a copy of $K_{1,a}$ with center $(0,j)$. Let Γ be the star induced by $A(0) \cup B(0)$.

Proposition 9. For $b \ge 2$, $pp(K_{1,1}\Box K_{1,b}) \le \lfloor (1+b)/2 \rfloor$.

Proof. For b even, we define the terminal pairs s_ℓ, t_ℓ, $1 \le \ell \le b/2+1$, as follows.

	s_1	s_2	$\cdots$	$s_{b/2-1}$	$s_{b/2}$	$s_{b/2+1}$					
					$t_{b/2+1}$	$t_{b/2}$	$t_{b/2-1}$	$\cdots$	t_3	t_2	t_1

There is no solution, since each of the paths joining the terminals $s_{b/2}, t_{b/2}$ and joining the terminals $s_{b/2+1}, t_{b/2+1}$ cannot use a 'vertical' edge different from the edge $(0,0)-(1,0)$. For b odd, take the pairs above for $b-1$, then add a pair $s_{(b+1)/2}, t_{(b+1)/2}$ in the last column $B(b)$. ■

Proposition 10. $pp(K_{1,2}\Box K_{1,3}) \le \lfloor (2+3)/2 \rfloor$.

Proof. The terminals of each pair s_ℓ, t_ℓ, $1 \le \ell \le 3$, below are at distance 4:

	s_1	s_2	s_3
	t_3	t_1	t_2

Any path between the paired terminals must use at least two edges of the boundary star Γ which contains only 5 edges, hence there is no solution. ■

Proposition 11. *For $b \ge 2$, $pp(K_{1,1}\Box K_{1,b}) = \lfloor (b+1)/2 \rfloor$, furthermore, $pp(K_{1,2}\Box K_{1,3}) = 2$.*

Proof. By Lemma 3, any set of at most $b+1$ terminals of $G(1,b)$ has an injection into the path-pairable boundary star Γ. Then the first equality follows by Proposition 9. By Lemma 3, any set of at most 4 terminals of $G(2,3)$ has an injection into the path-pairable star Γ. The second equality follows by Proposition 10. ■