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THE TURÁN NUMBER OF $2P_7$

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Abstract

The Turán number of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in any graph on n vertices which does not contain H as a subgraph. Let P_k denote the path on k vertices and let mP_k denote m disjoint copies of P_k. Bushaw and Kettle [*Turán numbers of multiple paths and equibipartite forests*, Combin. Probab. Comput. 20 (2011) 837–853] determined the exact value of $ex(n, kP_\ell)$ for large values of n. Yuan and Zhang [*The Turán number of disjoint copies of paths*, Discrete Math. 340 (2017) 132–139] completely determined the value of $ex(n, kP_3)$ for all n, and also determined $ex(n, F_m)$, where F_m is the disjoint union of m paths containing at most one odd path. They also determined the exact value of $ex(n, P_3 \cup P_{2\ell+1})$ for $n \geq 2\ell + 4$. Recently, Bielak and Kieliszek [*The Turán number of the graph* $2P_5$, Discuss. Math. Graph Theory 36 (2016) 683–694], Yuan and Zhang [*Turán numbers for disjoint paths*, arXiv:1611.00981v1] independently determined the exact value of $ex(n, 2P_5)$. In this paper, we show that $ex(n, 2P_7) = \max\{[n, 14, 7], 5n - 14\}$ for all $n \geq 14$, where $[n, 14, 7] = (5n + 91 + r(r - 6))/2$, $n - 13 \equiv r \pmod 6$ and $0 \leq r < 6$.

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1. Introduction

Throughout this paper, we only consider simple graphs. For a graph G we use $V(G)$, $|G|$, $E(G)$, $e(G)$ to denote the vertex set, number of vertices, edge set, number of edges, respectively. For $S_1, S_2 \subseteq V(G)$ and $S_1 \cap S_2 = \emptyset$, denote by $e(S_1, S_2)$ the number of edges between S_1 and S_2. Let G and H be two disjoint graphs. By $G \cup H$ denote the disjoint union of graphs G and H and by kG denote the k disjoint copies of G. Denote by $G + H$ the graph obtained from $G \cup H$ by joining all vertices of G to all vertices of H. Let $\overline{G}$ be the complement of the graph G. Denote by P_n, C_n and K_n the path, cycle and complete graph on n vertices, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and let $|S|$ denote the cardinality of S. For a graph G and its subgraph H, by $G - H$ we mean a graph obtained from G by deleting all vertices of H with all incident edges. If H consists of a single vertex x, then we simple write $G - x$. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices in G which are adjacent to v. We define $d_G(v) = |N_G(v)|$.

A graph is *H-free* if it does not contain H as a subgraph. The *Turán number* of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in any H-free graph on n vertices, i.e.,

$$ex(n, H) = \max\left\{e(G) : H \nsubseteq G \text{ and } |G| = n\right\}.$$

Let $EX(n, H)$ denote the family of all H-free graphs on n vertices with $ex(n, H)$ edges. A graph in $EX(n, H)$ is called an *extremal graph* for H. Moreover, we denote by $ex_{con}(n, H)$ the maximum number of edges in any connected H-free graph on n vertices. The problem of determining Turán number for assorted graphs traces its history back to 1907, when Mantel (see, e.g., [3]) proved $ex(n, C_3) = \lfloor n^2/4 \rfloor$. In 1941, Turán [13, 14] proved that the extremal graph for K_r is the complete $(r-1)$-partite graph, which is as balanced as possible (any two part sizes differ at most 1). The balanced complete $(r-1)$-partite graph on n vertices is called as the Turán graph denoted by $T_{r-1}(n)$. For sparse graphs, Erdős and Gallai [5] in 1959 proved the following well known result.

Theorem 1.1 [5]**.** *Let G be a P_k-free graph on n vertices and $n \geq k \geq 2$. Then $e(G) \leq (k-2)n/2$ with equality if and only if $n = (k-1)t$ and $G = tK_{k-1}$.*

For convenience, we first introduce the following symbols.

Definition 1.2. Let $n \geq m \geq \ell \geq 3$ be given three positive integers. If n can be written as $n = (m-1) + t(\ell-1) + r$, where $t \geq 0$ and $0 \leq r < \ell - 1$, then we denote

$$[n, m, \ell] = \binom{m-1}{2} + t\binom{\ell-1}{2} + \binom{r}{2}.$$

Moreover, if $n \le m-1$, then we denote

$$[n, m, \ell] = \binom{n}{2}.$$

Definition 1.3. Let $s = \sum_{i=1}^{m} \lfloor k_i/2 \rfloor$ and k_i be positive integers. If $n \ge s$, then we denote

$$[n, s] = \binom{s-1}{2} + (s-1)(n-s+1).$$

Later, for all integers n and k, Faudree and Schelp [7] characterized all extremal graphs for P_k.

Theorem 1.4 [7]. *Let G be a graph on $n = t(k-1)+r$ $(0 \le t$ and $0 \le r < k-1)$ vertices. If G is P_k-free, then $e(G) \le [n, k, k]$. Moreover, the equality holds if and only if*

- $G = (tK_{k-1}) \cup K_r$ *or*
- $G = ((t-s-1)K_{k-1}) \cup (K_{(k-2)/2} + \overline{K_{k/2+s(k-1)+r}})$, *where k is even, $t > 0$, $r = k/2$ or $(k-2)/2$ and $0 \le s < t$.*

Corollary 1.5. *For a positive integer $n \equiv r \pmod{k}$, $ex(n, P_{k+1}) = (n(k-1) + r(r-k))/2$.*

We see that $ex(n, P_k)$ has been determined for all integers $n \ge k$ and all extremal graphs has also been characterized. For connected graphs, Kopylov [8] and Balister, Győri, Lehel and Schelp [1] determined $ex_{con}(n, P_k)$ and characterized all extremal graphs for all integers $n \ge k$. Recently, Lan, Shi and Song [9] studied the Turán number of paths in planar graphs.

Theorem 1.6 [1, 8]. *Let G be a connected P_k-free graph on n vertices and $n \ge k \ge 4$. Then*

$$e(G) \le \max\left\{\binom{k-2}{2} + (n-k+2), [n, \lfloor k/2 \rfloor] + c\right\},$$

where $k \equiv c \pmod 2$. Further, the equality holds if and only if $G = (K_{k-3} \cup \overline{K_{n-k+2}}) + K_1$ or $G = (K_{1+c} \cup \overline{K_{n-\lfloor (k+1)/2 \rfloor}}) + K_{\lfloor k/2 \rfloor - 1}$.

In 1962, Erdős [6] first studied the Turán number of kK_3. And later, Moon [11] and Simonovits [12] studied the case of kK_r. In 2011, Bushaw and Kettle [4] determined $ex(n, kP_\ell)$ for n sufficiently large.

Theorem 1.7 [4]. *For integers $k \ge 2$, $\ell \ge 4$ and $n \ge 2\ell + 2k\ell(\lceil \ell/2 \rceil + 1)\binom{\ell}{\lfloor \ell/2 \rfloor}$,*

$$ex(n, kP_\ell) = \left[n, k\left\lfloor \frac{\ell}{2} \right\rfloor\right] + c,$$

where $\ell \equiv c \pmod 2$.

Furthermore, their proof shows that their construction is optimal for $n = \Omega\big(k\ell^{3/2}2^{\ell}\big)$. Moreover, Bushaw and Kettle conjectured that their construction is optimal for $n = \Omega(k\ell)$. Recently, Lidický *et al.* [10] extended Bushaw and Kettle's result and determined $ex(n, F_m)$ for n sufficiently large, where $F_m = \bigcup_{i=1}^m P_{k_i}$ and $k_1 \geq k_2 \geq \cdots \geq k_m$.

Theorem 1.8 [10]. *Let $F_m = \bigcup_{i=1}^m P_{k_i}$ and $k_1 \geq k_2 \geq \cdots \geq k_m$. If at least one k_i is not 3, then for n sufficiently large,*

$$ex(n, F_m) = \left[n, \sum_{i=1}^m \left\lfloor \frac{k_i}{2} \right\rfloor\right] + c,$$

where $c = 1$ if all k_i are odd, and $c = 0$ otherwise. Moreover, the extremal graph is unique.

However, they did not consider $ex(n, F_m)$ for smaller n. Recently, Yuan and Zhang [15, 16] completely determined the value of $ex(n, kP_3)$ and characterized all the extremal graphs for all n. Furthermore, they proved the following result in which F_m contains at most one odd path and proposed Conjecture 1.10.

Theorem 1.9 [15]. *Let $k_1 \geq k_2 \geq \cdots \geq k_m \geq 3$, $n \geq \sum_{i=1}^m k_i$ and $F_m = \bigcup_{i=1}^m P_{k_i}$. If there is at most one odd in $\{k_1, k_2, \ldots, k_m\}$, then*

$$ex(n, F_m) = \max\left\{[n, k_1, k_1], [n, k_1 + k_2, k_2], \ldots, \left[n, \sum_{i=1}^m k_i, k_m\right], \left[n, \sum_{i=1}^m \left\lfloor \frac{k_i}{2} \right\rfloor\right]\right\}.$$

Moreover, if $k_1, k_2, \ldots, k_m$ are even, then the extremal graphs are characterized.

Conjecture 1.10 [15]. *Let $k_1 \geq k_2 \geq \cdots \geq k_m \geq 3$, $k_1 > 3$ and $F_m = \bigcup_{i=1}^m P_{k_i}$. Then*

$$\begin{aligned} &ex(n, F_m) \\ &= \max\left\{[n, k_1, k_1], [n, k_1 + k_2, k_2], \ldots, \left[n, \sum_{i=1}^m k_i, k_m\right], \left[n, \sum_{i=1}^m \left\lfloor \frac{k_i}{2} \right\rfloor\right] + c\right\}, \end{aligned}$$

where $c = 1$ if all of $k_1, k_2, \ldots, k_m$ are odd, and $c = 0$ otherwise. Moreover, the extremal graphs are

$$EX\big(n, P_{k_1}\big), \ldots, K_{\sum_{i=1}^m k_i - 1} \cup H \text{ for } H \in EX\left(n - \textstyle\sum_{i=1}^m k_i + 1, P_{k_m}\right), \text{ and}$$

$$K_{\sum_{i=1}^m \lfloor k_i/2 \rfloor - 1} + \left(K_{1+c} \cup \overline{K_{n - \sum_{i=1}^m \lfloor k_i/2 \rfloor - c}}\right).$$

When there are at least two odd integers in $\{k_1, k_2, \ldots, k_m\}$, Yuan and Zhang also determined $ex(n, P_3 \cup P_{2\ell+1})$ for $n \geq 2\ell + 4$ and characterized all extremal graphs. Bielak and Kieliszek [2] and Yuan and Zhang [15] independently determined $ex(n, 2P_5)$ and characterized all extremal graphs. In this paper, we prove the following result, which partially confirms Conjecture 1.10.

Theorem 1.11. *For $n \geq 14$,*

$$ex(n, 2P_7) = \max\{[n, 14, 7], 5n - 14\}.$$

Moreover, the extremal graphs are $K_{13} \cup H$ for $H \in EX(n-13, P_7)$ when $n \leq 21$ and $K_5 + \left(K_2 \cup \overline{K_{n-7}}\right)$ when $n \geq 22$.

2. PROOF OF THEOREM 1.11

We first present some useful lemmas. In the following, we say that u *hits* v or v *hits* u if two vertices u and v are adjacent. Otherwise, we say that u *misses* v or v *misses* u if u and v are not adjacent. We say a vertex set A *hits* (*misses*) a vertex set B, it means that each vertex of A is adjacent (non-adjacent) to each vertex of B.

Lemma 2.1 (Observation 2 of [15])**.** *Let $k_1 \geq k_2 \geq 3$ be two positive integers. If $n_1 \geq k_1$, then $[n_1, k_1 + k_2, k_2] + [n_2, k_2, k_2] \leq [n_1 + n_2, k_1 + k_2, k_2]$.*

Lemma 2.2 (Observation 5 of [15])**.** *Let $k_1 \geq k_2 \geq 3$ be two positive integers. If $n_1 \geq k_1 + k_2$, then $[n_1, \lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor] + [n_2, k_2, k_2] < [n_1 + n_2, \lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor]$.*

Lemma 2.3. *Let G be a connected $2P_7$-free graph on $n \geq 14$ vertices. Then*

$$e(G) \leq \max\{[n, 14, 7], 5n - 14\},$$

with equality only when $n \geq 22$ and $G = K_5 + \left(\overline{K_{n-7}} \cup K_2\right)$.

Proof. Let $G \neq K_5 + \left(\overline{K_{n-7}} \cup K_2\right)$ be any connected $2P_7$-free graph on n vertices with $e(G) \geq \max\{[n, 14, 7], 5n - 14\}$ edges. Note that $\max\{[n, 14, 7], 5n - 14\} = [n, 14, 7]$ when $n \leq 21$ and $\max\{[n, 14, 7], 5n - 14\} = 5n - 14$ when $n \geq 22$. Since $\max\{[n, 14, 7], 5n - 14\} \geq ex_{con}(n, P_{13})$, by Theorem 1.6, G contains P_{13} as a subgraph. Let $P_{13} = x_1 x_2 \cdots x_{13}$ be a subgraph of G. Then

$(*)$ each vertex of $G - P_{13}$ cannot hit two adjacent vertices in P_{13}.

Notice that each vertex in $G - P_{13}$ misses $\{x_1, x_6, x_8, x_{13}\}$ and cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Moreover, if y is an isolated vertex in $G - P_{13}$, then by $(*)$, $|N_G(y) \cap V(P_{13})| \leq 5$; if y is not an isolated vertex in $G - P_{13}$, then $N_G(y) \cap V(P_{13}) \subseteq \{x_3, x_4, x_7, x_{10}, x_{11}\}$ and so $|N_G(y) \cap V(P_{13})| \leq 3$ by $(*)$; if

$P_k = y_1y_2\cdots y_k \subseteq G - P_{13}$ and $k \geq 3$ such that y_1 hits P_{13}, then y_1 can only hit x_7. Now we will prove the following useful facts.

Fact 1. $e(G[V(P_{13})]) \leq 74$.

Proof. Since G is connected and $n \geq 14$, at least one vertex of $V(G)\backslash V(P_{13})$ hits P_{13}, say x_i. Then either $i \geq 6$ or $i \leq 8$. Without loss of generality, we may assume that $i \geq 6$. For $1 \leq j \leq i-2$, if both $x_{13}x_j \in E(G)$ and $x_{i+1}x_{j+1} \in E(G)$, then G contains $2P_7$ as a subgraph, a contradiction. Thus $e(G[V(P_{13})]) \leq 74$. □

Fact 2. *If there exists a $P_3 = y_1y_2y_3 \subseteq G - P_{13}$ such that y_1 hits P_{13}, then we have $e(G[V(P_{13})]) \leq 57$.*

Proof. Clearly, y_1 must hit x_7 and so G contains a copy of P_7 with vertices $x_4, x_5, x_6, x_7, y_1, y_2, y_3$. Therefore, $\{x_1, x_2, x_3, x_5, x_6\}$ misses $\{x_{11}, x_{12}, x_{13}\}$. Symmetrically, $\{x_8, x_9, x_{11}, x_{12}, x_{13}\}$ misses $\{x_1, x_2, x_3\}$. So $e(G[V(P_{13})]) \leq 78 - (2 \cdot 15 - 9) = 57$. □

Fact 3. *If there exists a non-isolated vertex in $G - P_{13}$ that hits one vertex of P_{13}, then we have $e(G[V(P_{13})]) \leq 68$.*

Proof. Let y be a non-isolated vertex in $G - P_{13}$, that hits one vertex, say x_i of P_{13}. Recall that $x_i \in \{x_3, x_4, x_7, x_{10}, x_{11}\}$. If $x_i \in \{x_3, x_4\}$, then $\{x_1, x_2, \ldots, x_{i-1}\}$ misses $\{x_{i+1}, x_{i+2}, x_9, x_{12}, x_{13}\}$ and so $e(G[V(P_{13})]) \leq 68$. Symmetrically, if $x_i \in \{x_{10}, x_{11}\}$, then $e(G[V(P_{13})]) \leq 68$. Now assume that $x_i = x_7$. Then $\{x_1, x_2, x_{i-1}, x_{i-2}\}$ misses $\{x_{12}, x_{13}\}$ and symmetrically $\{x_{i+1}, x_{i+2}, x_{12}, x_{13}\}$ misses $\{x_1, x_2\}$. So $e(G[V(P_{13})]) \leq 78 - (2 \cdot 8 - 4) = 66$. □

Fact 4. *If there exists a non-isolated vertex in $G - P_{13}$ that hits two vertices of P_{13}, then we have $e(G[V(P_{13})]) \leq 59$.*

Proof. Let y be a non-isolated vertex in $G - P_{13}$, that hits two vertices, say x_i and x_j $(i < j)$, of P_{13}. Recall that $\{x_i, x_j\} \subseteq \{x_3, x_4, x_7, x_{10}, x_{11}\}$ and $\{x_i, x_j\} \neq \{x_3, x_{11}\}$. If $x_i = x_3$, then by $(*)$, $x_j \in \{x_7, x_{10}\}$. Thus $\{x_1, x_2\}$ misses $\{x_4, x_5, x_6, x_8, x_9, x_{11}, x_{12}, x_{13}\}$ and $\{x_{j-2}, x_{j-1}\}$ misses $\{x_{12}, x_{13}\}$. So $e(G[V(P_{13})]) \leq 58$. Symmetrically, if $x_j = x_{11}$, then by $(*)$, $x_i \in \{x_4, x_7\}$ and so $e(G[V(P_{13})]) \leq 58$. Now we can assume that $x_i \neq x_3$ and $x_j \neq x_{11}$. If $x_i = x_4$, then $x_j \in \{x_7, x_{10}\}$. Thus $\{x_1, x_2, x_3\}$ misses $\{x_5, x_6, x_9, x_{12}, x_{13}\}$ and $\{x_{j-2}, x_{j-1}\}$ misses $\{x_{12}, x_{13}\}$. So $e(G[V(P_{13})]) \leq 59$. Symmetrically, if $x_j = x_{10}$, then $x_i \in \{x_4, x_7\}$ and so $e(G[V(P_{13})]) \leq 59$. □

Fact 5. *If there exists an isolated vertex in $G - P_{13}$ that hits five vertices of P_{13}, then $e(G[V(P_{13})]) \leq 50$.*

Proof. Let y be an isolated vertex in $G - P_{13}$ that hits exactly five vertices, say $x_i, x_j, x_k, x_\ell, x_m$, $i < j < k < \ell < m$, of P_{13}. Recall that $\{x_i, x_j, x_k, x_\ell, x_m\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$.

Since y cannot hit two adjacent vertices in P_{13}, we have $x_k = x_7$, $\{x_i, x_j\} \subseteq \{x_2, x_3, x_4, x_5\}$ and $\{x_\ell, x_m\} \subseteq \{x_9, x_{10}, x_{11}, x_{12}\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}, x_{m-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}, x_{m+1}\}$. Then, A and B are independent sets and $|A \cap B| = 4$. Since $\{x_3, x_{11}\} \subsetneq N_G(y)$, we have either $i = 2$ or $m = 12$. If $i = 2$ and $m = 12$, then $N_G(y) = \{x_2, x_5, x_7, x_9, x_{12}\}$, which implies that x_5 misses $\{x_{10}, x_{11}\}$. And symmetrically x_9 misses $\{x_3, x_4\}$. If $i = 2$ and $m \neq 12$, then $\ell = 9$ and $m = 11$, which implies that x_m misses $\{x_3, x_6\}$ and x_ℓ misses $\{x_q, x_{q+1}\} \subseteq \{x_1, \ldots, x_7\} \setminus N_G(y)$. If $i \neq 2$ and $m = 12$, then $i = 3$ and $j = 5$, which implies that x_i misses $\{x_8, x_{11}\}$ and x_j misses $\{x_q, x_{q+1}\} \subseteq \{x_7, \ldots, x_{13}\} \setminus N_G(y)$. For each of the above cases, we have $e(G[V(P_{13})]) \leq 78 - \left(\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A\cap B|}{2}\right) - 4 = 50$. $\square$

Fact 6. *If there exists an isolated vertex in $G - P_{13}$ that hits four vertices of P_{13}, then $e(G[V(P_{13})]) \leq 59$.*

Proof. Let y be an isolated vertex in $G - P_{13}$ that hits exactly four vertices, say x_i, x_j, x_k, x_ℓ, $i < j < k < \ell$, of P_{13}. Recall that $\{x_i, x_j, x_k, x_\ell\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}\}$. Then A and B are independent sets and $|A \cap B| \leq 3$. If $|A \cap B| \leq 2$, then $e(G[V(P_{13})]) \leq 78 - \left(\binom{|A|}{2} + \binom{|B|}{2} - 1\right) = 59$. Now we assume that $|A \cap B| = 3$. If $i = 2$ and $\ell = 12$, then $7 \in \{j, k\}$ which implies that x_3 misses x_{11} and x_p misses x_{p+9} for $p \in \{1, 4\}$. If $i = 2$, $\ell \neq 12$ and $7 \in \{j, k\}$, then x_{11} misses $\{x_3, x_6\}$. If $i = 2$, $\ell \neq 12$ and $7 \notin \{j, k\}$, then $N_G(y) = \{x_2, x_4, x_9, x_{11}\}$ which implies x_{11} misses $\{x_5, x_8\}$. If $\ell = 12$ and $i \neq 2$, then it is similar as the case of $i = 2$ and $\ell \neq 12$. If $i \neq 2$ and $\ell \neq 12$, then $N_G(y) = \{x_3, x_5, x_7, x_9\}$ which implies x_{11} misses $\{x_1, x_4\}$. For each of the above cases, $e(G[V(P_{13})]) \leq 78 - \left(\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A\cap B|}{2}\right) - 2 = 59$. $\square$

Fact 7. *If there exists an isolated vertex in $G - P_{13}$ that hits three vertices of P_{13}, then $e(G[V(P_{13})]) \leq 67$.*

Proof. Let y be an isolated vertex in $G - P_{13}$ that hits exactly three vertices, say x_i, x_j, x_k, $i < j < k$, of P_{13}. Recall that $\{x_i, x_j, x_k\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}\}$. Then both A and B are independent sets and $|A \cap B| \leq 2$. Hence, $e(G[V(P_{13})]) \leq 78 - \left(\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A\cap B|}{2}\right) \leq 78 - (6 + 6 - 1) = 67$. $\square$

Let $P_k = y_1 y_2 \cdots y_k$, where $k \leq 6$, be the longest path in $G - P_{13}$ such that y_1 hits P_{13}. Let $H_1, H_2, \ldots, H_t$ be connected components of order at least 2 of $G - P_{13}$ and let H be a subgraph of G which consists of all isolated vertices of $G - P_{13}$. Note that $\sum_{i=1}^{t} |H_i| + |H| = n - 13$. Let $m(H_i)$ be the number of edges incident with the vertices of H_i and let H_1 be a component of $G - P_{13}$ which contains P_k as a subgraph. We first show the following claim.

Claim. *For* $1 \le i \le t$, $m(H_i) \le 4|H_i|$.

Proof. We use induction on $|H_i|$. Recall that each vertex of H_i can hit at most three vertices of P_{13}. For $|H_i| = 2$, $m(H_i) = e(G[V(H_i)]) + e(V(H_i), V(P_{13})) \le 7 \le 4|H_i|$. If H_i has a pendant vertex x, then $d_G(x) \le 4$. By induction hypothesis, we have $m(H_i) = m(H_i - x) + d_G(x) \le 4(|H_i| - 1) + 4 \le 4|H_i|$. Next, if H_i has no pendant vertex, then each vertex of H_i must be an endpoint of a path of length at least two. This implies that each vertex of H_i can only hit x_7 of P_{13}. Thus, $m(H_i) = e(G[V(H_i)]) + e(V(H_i), V(P_{13})) \le ex_{con}(|H_i|, P_7) + |H_i| \le \frac{7}{2}|H_i|$ since H_i is P_7-free. □

Let $\Delta(H) = \max\{d_G(v)|v \in V(H)\}$. Recall that $\Delta(H) \le 5$. Now we would divide the proof into the following cases (in each case we assume, the previous cases do not hold).

Case 1. $\Delta(H) = 5$. Then by Fact 5 and the Claim,

$$e(G) \le 50 + 5(n - 13) = 5n - 15 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 2. $\Delta(H) = 4$ or $k \ge 3$ or there exists a non-isolated vertex in $G - P_{13}$ that hits two vertices of P_{13} ($k = 2$). Then by Facts 6, 2 and 4 and the Claim,

$$e(G) \le 59 + 4(n - 13) = 4n + 7 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 3. $\Delta(H) = 3$ ($k = 2$) or there exists a non-isolated vertex in $G - P_{13}$ that hits one vertex of P_{13} ($k = 2$). For $k = 2$, each component of $G - P_{13}$ is a star (with at least three vertices), or an edge, or an isolated vertex. For $1 \le i \le t$, $e(G[V(H_i)]) \le |H_i| - 1$. $m_0 \le \sum_{i=1}^{t}(2|H_i| - 1) + 3|H| = 3(n - 15) + 6 - \sum_{i=1}^{t} |H_i| - t \le 3(n - 13)$. Then by Facts 7 and 3, we have

$$e(G) \le 68 + 3(n - 13) = 3n + 29 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 4. $\Delta(H) \le 2$ and $k = 1$. Then by Fact 1,

$$e(G) \le 74 + 2(n - 13) = 2n + 48 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

The proof is thus completed. ■

***Proof of Theorem* 1.11.** Let G be any $2P_7$-free graph on n vertices with $e(G) \geq \max\{[n, 14, 7], 5n-14\}$. If G is connected, then by Lemma 2.3, $e(G) \leq \max\{[n, 14, 7], 5n-14\}$ when $n \geq 22$ and $e(G) < \max\{[n, 14, 7], 5n-14\}$ when $n \leq 21$. Thus when G is connected, $e(G) \leq \max\{[n, 14, 7], 5n-14\}$ with equality holds if and only if $n \geq 22$ and $G = K_5 + (\overline{K_{n-7}} \cup K_2)$. Now we may assume that G is disconnected. By Lemma 1.4, G contains P_7 as a subgraph. Let C be a connected component with $n_1 \geq 7$ vertices which contains P_7 as a subgraph. Notice that C is $2P_7$-free and $G - C$ is P_7-free. If $n_1 \geq 22$, then by Lemma 2.3, $e(C) \leq 5n - 14$ and by Lemmas 1.4 and 2.2,

$$e(G) = e(C) + e(G - C) \leq 5n_1 - 14 + [n - n_1, 7, 7] < 5n - 14,$$

a contradiction. If $14 \leq n_1 \leq 21$, then by Lemma 2.3, $e(C) < [n_1, 14, 7]$ and by Lemmas 1.4 and 2.1,

$$e(G) = e(C) + e(G - C) < [n_1, 14, 7] + [n - n_1, 7, 7] \leq [n, 14, 7],$$

a contradiction. If $n_1 \leq 13$, then $e(G) \leq \binom{n_1}{2} + [n - n_1, 7, 7] \leq [n, 14, 7]$ with equality holds if and only if $C = K_{13}$ and $G - C \in EX(n-13, P_7)$. But then when $n \geq 22$, $e(G) \geq \max\{[n, 14, 7], 5n-14\} = 5n - 14 > [n, 14, 7]$, a contradiction. Thus when G is disconnected, $e(G) \leq \max\{[n, 14, 7], 5n-14\}$ with equality holds if and only if $n \leq 21$, $G = K_{13} \cup H$ for $H \in EX(n-13, P_7)$.

The proof is thus complete. ■

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