# THE TURÁN NUMBER OF $2 P_{7}$ 

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#### Abstract

The Turán number of a graph $H$, denoted by $e x(n, H)$, is the maximum number of edges in any graph on $n$ vertices which does not contain $H$ as a subgraph. Let $P_{k}$ denote the path on $k$ vertices and let $m P_{k}$ denote $m$ disjoint copies of $P_{k}$. Bushaw and Kettle [Turán numbers of multiple paths and equibipartite forests, Combin. Probab. Comput. 20 (2011) 837-853] determined the exact value of $e x\left(n, k P_{\ell}\right)$ for large values of $n$. Yuan and Zhang [The Turán number of disjoint copies of paths, Discrete Math. 340 (2017) 132-139] completely determined the value of $e x\left(n, k P_{3}\right)$ for all $n$, and also determined $e x\left(n, F_{m}\right)$, where $F_{m}$ is the disjoint union of $m$ paths containing at most one odd path. They also determined the exact value of $e x\left(n, P_{3} \cup P_{2 \ell+1}\right)$ for $n \geq 2 \ell+4$. Recently, Bielak and Kieliszek [The Turán number of the graph $2 P_{5}$, Discuss. Math. Graph Theory 36 (2016) 683-694], Yuan and Zhang [Turán numbers for disjoint paths, arXiv:1611.00981v1] independently determined the exact value of $e x\left(n, 2 P_{5}\right)$. In this paper, we show that $e x\left(n, 2 P_{7}\right)=\max \{[n, 14,7], 5 n-14\}$ for all $n \geq 14$, where $[n, 14,7]=(5 n+91+r(r-6)) / 2, n-13 \equiv r(\bmod 6)$ and $0 \leq r<6$.


Keywords: Turán number, extremal graphs, $2 P_{7}$.
2010 Mathematics Subject Classification: 05C35.

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## 1. Introduction

Throughout this paper, we only consider simple graphs. For a graph $G$ we use $V(G),|G|, E(G), e(G)$ to denote the vertex set, number of vertices, edge set, number of edges, respectively. For $S_{1}, S_{2} \subseteq V(G)$ and $S_{1} \cap S_{2}=\emptyset$, denote by $e\left(S_{1}, S_{2}\right)$ the number of edges between $S_{1}$ and $S_{2}$. Let $G$ and $H$ be two disjoint graphs. By $G \cup H$ denote the disjoint union of graphs $G$ and $H$ and by $k G$ denote the $k$ disjoint copies of $G$. Denote by $G+H$ the graph obtained from $G \cup H$ by joining all vertices of $G$ to all vertices of $H$. Let $\bar{G}$ be the complement of the graph $G$. Denote by $P_{n}, C_{n}$ and $K_{n}$ the path, cycle and complete graph on $n$ vertices, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$ and let $|S|$ denote the cardinality of $S$. For a graph $G$ and its subgraph $H$, by $G-H$ we mean a graph obtained from $G$ by deleting all vertices of $H$ with all incident edges. If $H$ consists of a single vertex $x$, then we simple write $G-x$. For $v \in V(G)$, let $N_{G}(v)$ denote the set of vertices in $G$ which are adjacent to $v$. We define $d_{G}(v)=\left|N_{G}(v)\right|$.

A graph is $H$-free if it does not contain $H$ as a subgraph. The Turán number of a graph $H$, denoted by $e x(n, H)$, is the maximum number of edges in any $H$ free graph on $n$ vertices, i.e.,

$$
e x(n, H)=\max \{e(G): H \nsubseteq G \text { and }|G|=n\} .
$$

Let $E X(n, H)$ denote the family of all $H$-free graphs on $n$ vertices with $e x(n, H)$ edges. A graph in $E X(n, H)$ is called an extremal graph for $H$. Moreover, we denote by $e x_{\text {con }}(n, H)$ the maximum number of edges in any connected $H$ free graph on $n$ vertices. The problem of determining Turán number for assorted graphs traces its history back to 1907, when Mantel (see, e.g., [3]) proved $e x\left(n, C_{3}\right)=\left\lfloor n^{2} / 4\right\rfloor$. In 1941, Turán [13, 14] proved that the extremal graph for $K_{r}$ is the complete ( $r-1$ )-partite graph, which is as balanced as possible (any two part sizes differ at most 1). The balanced complete ( $r-1$ )-partite graph on $n$ vertices is called as the Turán graph denoted by $T_{r-1}(n)$. For sparse graphs, Erdős and Gallai [5] in 1959 proved the following well known result.

Theorem 1.1 [5]. Let $G$ be a $P_{k}$-free graph on $n$ vertices and $n \geq k \geq 2$. Then $e(G) \leq(k-2) n / 2$ with equality if and only if $n=(k-1) t$ and $G=t K_{k-1}$.

For convenience, we first introduce the following symbols.
Definition 1.2. Let $n \geq m \geq \ell \geq 3$ be given three positive integers. If $n$ can be written as $n=(m-1)+t(\ell-1)+r$, where $t \geq 0$ and $0 \leq r<\ell-1$, then we denote

$$
[n, m, \ell]=\binom{m-1}{2}+t\binom{\ell-1}{2}+\binom{r}{2} .
$$

Moreover, if $n \leq m-1$, then we denote

$$
[n, m, \ell]=\binom{n}{2}
$$

Definition 1.3. Let $s=\sum_{i=1}^{m}\left\lfloor k_{i} / 2\right\rfloor$ and $k_{i}$ be positive integers. If $n \geq s$, then we denote

$$
[n, s]=\binom{s-1}{2}+(s-1)(n-s+1)
$$

Later, for all integers $n$ and $k$, Faudree and Schelp [7] characterized all extremal graphs for $P_{k}$.
Theorem 1.4 [7]. Let $G$ be a graph on $n=t(k-1)+r(0 \leq t$ and $0 \leq r<k-1)$ vertices. If $G$ is $P_{k}$-free, then $e(G) \leq[n, k, k]$. Moreover, the equality holds if and only if

- $G=\left(t K_{k-1}\right) \cup K_{r}$ or
- $G=\left((t-s-1) K_{k-1}\right) \cup\left(K_{(k-2) / 2}+\overline{K_{k / 2+s(k-1)+r}}\right)$, where $k$ is even, $t>0$, $r=k / 2$ or $(k-2) / 2$ and $0 \leq s<t$.
Corollary 1.5. For a positive integer $n \equiv r(\bmod k), e x\left(n, P_{k+1}\right)=(n(k-1)+$ $r(r-k)) / 2$.

We see that $e x\left(n, P_{k}\right)$ has been determined for all integers $n \geq k$ and all extremal graphs has also been characterized. For connected graphs, Kopylov [8] and Balister, Györi, Lehel and Schelp [1] determined $e x_{c o n}\left(n, P_{k}\right)$ and characterized all extremal graphs for all integers $n \geq k$. Recently, Lan, Shi and Song [9] studied the Turán number of paths in planar graphs.
Theorem 1.6 $[1,8]$. Let $G$ be a connected $P_{k}$-free graph on $n$ vertices and $n \geq$ $k \geq 4$. Then

$$
e(G) \leq \max \left\{\binom{k-2}{2}+(n-k+2),[n,\lfloor k / 2\rfloor]+c\right\}
$$

$\underline{\text { where } k} \equiv c(\bmod 2)$. Further, the equality holds if and only if $G=\left(K_{k-3} \cup\right.$ $\left.\overline{K_{n-k+2}}\right)+K_{1}$ or $G=\left(K_{1+c} \cup \overline{K_{n-\lfloor(k+1) / 2\rfloor}}\right)+K_{\lfloor k / 2\rfloor-1}$.

In 1962, Erdős [6] first studied the Turán number of $k K_{3}$. And later, Moon [11] and Simonovits [12] studied the case of $k K_{r}$. In 2011, Bushaw and Kettle [4] determined $e x\left(n, k P_{\ell}\right)$ for $n$ sufficiently large.
Theorem 1.7 [4]. For integers $k \geq 2, \ell \geq 4$ and $n \geq 2 \ell+2 k \ell(\lceil\ell / 2\rceil+1)\binom{\ell}{\ell / 2\rfloor}$,

$$
e x\left(n, k P_{\ell}\right)=\left[n, k\left\lfloor\frac{\ell}{2}\right\rfloor\right]+c
$$

where $\ell \equiv c(\bmod 2)$.

Furthermore, their proof shows that their construction is optimal for $n=$ $\Omega\left(k \ell^{3 / 2} 2^{\ell}\right)$. Moreover, Bushaw and Kettle conjectured that their construction is optimal for $n=\Omega(k \ell)$. Recently, Lidický et al. [10] extended Bushaw and Kettle's result and determined $e x\left(n, F_{m}\right)$ for $n$ sufficiently large, where $F_{m}=\bigcup_{i=1}^{m} P_{k_{i}}$ and $k_{1} \geq k_{2} \geq \cdots \geq k_{m}$.

Theorem 1.8 [10]. Let $F_{m}=\bigcup_{i=1}^{m} P_{k_{i}}$ and $k_{1} \geq k_{2} \geq \cdots \geq k_{m}$. If at least one $k_{i}$ is not 3 , then for $n$ sufficiently large,

$$
e x\left(n, F_{m}\right)=\left[n, \sum_{i=1}^{m}\left\lfloor\frac{k_{i}}{2}\right\rfloor\right]+c
$$

where $c=1$ if all $k_{i}$ are odd, and $c=0$ otherwise. Moreover, the extremal graph is unique.

However, they did not consider $e x\left(n, F_{m}\right)$ for smaller $n$. Recently, Yuan and Zhang $[15,16]$ completely determined the value of $e x\left(n, k P_{3}\right)$ and characterized all the extremal graphs for all $n$. Furthermore, they proved the following result in which $F_{m}$ contains at most one odd path and proposed Conjecture 1.10.

Theorem 1.9 [15]. Let $k_{1} \geq k_{2} \geq \cdots \geq k_{m} \geq 3, n \geq \sum_{i=1}^{m} k_{i}$ and $F_{m}=$ $\bigcup_{i=1}^{m} P_{k_{i}}$. If there is at most one odd in $\left\{k_{1}, k_{2}, \ldots, k_{m}\right\}$, then
$e x\left(n, F_{m}\right)=\max \left\{\left[n, k_{1}, k_{1}\right],\left[n, k_{1}+k_{2}, k_{2}\right], \ldots,\left[n, \sum_{i=1}^{m} k_{i}, k_{m}\right],\left[n, \sum_{i=1}^{m}\left\lfloor\frac{k_{i}}{2}\right\rfloor\right]\right\}$.
Moreover, if $k_{1}, k_{2}, \ldots, k_{m}$ are even, then the extremal graphs are characterized.
Conjecture 1.10 [15]. Let $k_{1} \geq k_{2} \geq \cdots \geq k_{m} \geq 3, k_{1}>3$ and $F_{m}=\bigcup_{i=1}^{m} P_{k_{i}}$. Then

$$
\begin{aligned}
& \operatorname{ex}\left(n, F_{m}\right) \\
& =\max \left\{\left[n, k_{1}, k_{1}\right],\left[n, k_{1}+k_{2}, k_{2}\right], \ldots,\left[n, \sum_{i=1}^{m} k_{i}, k_{m}\right],\left[n, \sum_{i=1}^{m}\left\lfloor\frac{k_{i}}{2}\right\rfloor\right]+c\right\}
\end{aligned}
$$

where $c=1$ if all of $k_{1}, k_{2}, \ldots, k_{m}$ are odd, and $c=0$ otherwise. Moreover, the extremal graphs are

$$
\begin{gathered}
E X\left(n, P_{k_{1}}\right), \ldots, K_{\sum_{i=1}^{m} k_{i}-1} \cup H \text { for } H \in E X\left(n-\sum_{i=1}^{m} k_{i}+1, P_{k_{m}}\right), \text { and } \\
K_{\sum_{i=1}^{m}\left\lfloor k_{i} / 2\right\rfloor-1}+\left(K_{1+c} \cup \overline{K_{n-\sum_{i=1}^{m}\left\lfloor k_{i} / 2\right\rfloor-c}}\right)
\end{gathered}
$$

When there are at least two odd integers in $\left\{k_{1}, k_{2}, \ldots, k_{m}\right\}$, Yuan and Zhang also determined $e x\left(n, P_{3} \cup P_{2 \ell+1}\right)$ for $n \geq 2 \ell+4$ and characterized all extremal graphs. Bielak and Kieliszek [2] and Yuan and Zhang [15] independently determined $e x\left(n, 2 P_{5}\right)$ and characterized all extremal graphs. In this paper, we prove the following result, which partially confirms Conjecture 1.10.

Theorem 1.11. For $n \geq 14$,

$$
e x\left(n, 2 P_{7}\right)=\max \{[n, 14,7], 5 n-14\} .
$$

Moreover, the extremal graphs are $K_{13} \cup H$ for $H \in E X\left(n-13, P_{7}\right)$ when $n \leq 21$ and $K_{5}+\left(K_{2} \cup \overline{K_{n-7}}\right)$ when $n \geq 22$.

## 2. Proof of Theorem 1.11

We first present some useful lemmas. In the following, we say that $u$ hits $v$ or $v$ hits $u$ if two vertices $u$ and $v$ are adjacent. Otherwise, we say that $u$ misses $v$ or $v$ misses $u$ if $u$ and $v$ are not adjacent. We say a vertex set $A$ hits (misses) a vertex set $B$, it means that each vertex of $A$ is adjacent (non-adjacent) to each vertex of $B$.

Lemma 2.1 (Observation 2 of [15]). Let $k_{1} \geq k_{2} \geq 3$ be two positive integers. If $n_{1} \geq k_{1}$, then $\left[n_{1}, k_{1}+k_{2}, k_{2}\right]+\left[n_{2}, k_{2}, k_{2}\right] \leq\left[n_{1}+n_{2}, k_{1}+k_{2}, k_{2}\right]$.

Lemma 2.2 (Observation 5 of [15]). Let $k_{1} \geq k_{2} \geq 3$ be two positive integers. If $n_{1} \geq k_{1}+k_{2}$, then $\left[n_{1},\left\lfloor k_{1} / 2\right\rfloor+\left\lfloor k_{2} / 2\right\rfloor\right\rfloor+\left[n_{2}, k_{2}, k_{2}\right]<\left[n_{1}+n_{2},\left\lfloor k_{1} / 2\right\rfloor+\left\lfloor k_{2} / 2\right\rfloor\right]$.
Lemma 2.3. Let $G$ be a connected $2 P_{7}$-free graph on $n \geq 14$ vertices. Then

$$
e(G) \leq \max \{[n, 14,7], 5 n-14\},
$$

with equality only when $n \geq 22$ and $G=K_{5}+\left(\overline{K_{n-7}} \cup K_{2}\right)$.
Proof. Let $G \neq K_{5}+\left(\overline{K_{n-7}} \cup K_{2}\right)$ be any connected $2 P_{7}$-free graph on $n$ vertices with $e(G) \geq \max \{[n, 14,7], 5 n-14\}$ edges. Note that $\max \{[n, 14,7], 5 n-14\}=$ $[n, 14,7]$ when $n \leq 21$ and $\max \{[n, 14,7], 5 n-14\}=5 n-14$ when $n \geq 22$. Since $\max \{[n, 14,7], 5 n-14\} \geq e x_{\text {con }}\left(n, P_{13}\right)$, by Theorem 1.6, $G$ contains $P_{13}$ as a subgraph. Let $P_{13}=x_{1} x_{2} \cdots x_{13}$ be a subgraph of $G$. Then
(*) each vertex of $G-P_{13}$ cannot hit two adjacent vertices in $P_{13}$.
Notice that each vertex in $G-P_{13}$ misses $\left\{x_{1}, x_{6}, x_{8}, x_{13}\right\}$ and cannot hit both $x_{p}$ and $x_{p+8}$ for $p \in\{2,3,4\}$. Moreover, if $y$ is an isolated vertex in $G-P_{13}$, then by $(*),\left|N_{G}(y) \cap V\left(P_{13}\right)\right| \leq 5$; if $y$ is not an isolated vertex in $G-P_{13}$, then $N_{G}(y) \cap V\left(P_{13}\right) \subseteq\left\{x_{3}, x_{4}, x_{7}, x_{10}, x_{11}\right\}$ and so $\left|N_{G}(y) \cap V\left(P_{13}\right)\right| \leq 3$ by (*); if
$P_{k}=y_{1} y_{2} \cdots y_{k} \subseteq G-P_{13}$ and $k \geq 3$ such that $y_{1}$ hits $P_{13}$, then $y_{1}$ can only hit $x_{7}$. Now we will prove the following useful facts.
Fact 1. $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 74$.
Proof. Since $G$ is connected and $n \geq 14$, at least one vertex of $V(G) \backslash V\left(P_{13}\right)$ hits $P_{13}$, say $x_{i}$. Then either $i \geq 6$ or $i \leq 8$. Without loss of generality, we may assume that $i \geq 6$. For $1 \leq j \leq i-2$, if both $x_{13} x_{j} \in E(G)$ and $x_{i+1} x_{j+1} \in E(G)$, then $G$ contains $2 P_{7}$ as a subgraph, a contradiction. Thus $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 74$.

Fact 2. If there exists a $P_{3}=y_{1} y_{2} y_{3} \subseteq G-P_{13}$ such that $y_{1}$ hits $P_{13}$, then we have $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 57$.
Proof. Clearly, $y_{1}$ must hit $x_{7}$ and so $G$ contains a copy of $P_{7}$ with vertices $x_{4}, x_{5}, x_{6}, x_{7}, y_{1}, y_{2}, y_{3}$. Therefore, $\left\{x_{1}, x_{2}, x_{3}, x_{5}, x_{6}\right\}$ misses $\left\{x_{11}, x_{12}, x_{13}\right\}$. Symmetrically, $\left\{x_{8}, x_{9}, x_{11}, x_{12}, x_{13}\right\}$ misses $\left\{x_{1}, x_{2}, x_{3}\right\}$. So $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-(2$. $15-9)=57$.
Fact 3. If there exists a non-isolated vertex in $G-P_{13}$ that hits one vertex of $P_{13}$, then we have e $\left(G\left[V\left(P_{13}\right)\right]\right) \leq 68$.
Proof. Let $y$ be a non-isolated vertex in $G-P_{13}$, that hits one vertex, say $x_{i}$ of $P_{13}$. Recall that $x_{i} \in\left\{x_{3}, x_{4}, x_{7}, x_{10}, x_{11}\right\}$. If $x_{i} \in\left\{x_{3}, x_{4}\right\}$, then $\left\{x_{1}, x_{2}, \ldots, x_{i-1}\right\}$ misses $\left\{x_{i+1}, x_{i+2}, x_{9}, x_{12}, x_{13}\right\}$ and so $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 68$. Symmetrically, if $x_{i} \in$ $\left\{x_{10}, x_{11}\right\}$, then $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 68$. Now assume that $x_{i}=x_{7}$. Then $\left\{x_{1}, x_{2}\right.$, $\left.x_{i-1}, x_{i-2}\right\}$ misses $\left\{x_{12}, x_{13}\right\}$ and symmetrically $\left\{x_{i+1}, x_{i+2}, x_{12}, x_{13}\right\}$ misses $\left\{x_{1}\right.$, $\left.x_{2}\right\}$. So $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-(2 \cdot 8-4)=66$.
Fact 4. If there exists a non-isolated vertex in $G-P_{13}$ that hits two vertices of $P_{13}$, then we have e $\left(G\left[V\left(P_{13}\right)\right]\right) \leq 59$.
Proof. Let $y$ be a non-isolated vertex in $G-P_{13}$, that hits two vertices, say $x_{i}$ and $x_{j}(i<j)$, of $P_{13}$. Recall that $\left\{x_{i}, x_{j}\right\} \subseteq\left\{x_{3}, x_{4}, x_{7}, x_{10}, x_{11}\right\}$ and $\left\{x_{i}, x_{j}\right\} \neq$ $\left\{x_{3}, x_{11}\right\}$. If $x_{i}=x_{3}$, then by $(*), x_{j} \in\left\{x_{7}, x_{10}\right\}$. Thus $\left\{x_{1}, x_{2}\right\}$ misses $\left\{x_{4}, x_{5}, x_{6}\right.$, $\left.x_{8}, x_{9}, x_{11}, x_{12}, x_{13}\right\}$ and $\left\{x_{j-2}, x_{j-1}\right\}$ misses $\left\{x_{12}, x_{13}\right\}$. So $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 58$. Symmetrically, if $x_{j}=x_{11}$, then by $(*), x_{i} \in\left\{x_{4}, x_{7}\right\}$ and so $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 58$. Now we can assume that $x_{i} \neq x_{3}$ and $x_{j} \neq x_{11}$. If $x_{i}=x_{4}$, then $x_{j} \in\left\{x_{7}, x_{10}\right\}$. Thus $\left\{x_{1}, x_{2}, x_{3}\right\}$ misses $\left\{x_{5}, x_{6}, x_{9}, x_{12}, x_{13}\right\}$ and $\left\{x_{j-2}, x_{j-1}\right\}$ misses $\left\{x_{12}, x_{13}\right\}$. So $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 59$. Symmetrically, if $x_{j}=x_{10}$, then $x_{i} \in\left\{x_{4}, x_{7}\right\}$ and so $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 59$.
Fact 5. If there exists an isolated vertex in $G-P_{13}$ that hits five vertices of $P_{13}$, then $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 50$.
Proof. Let $y$ be an isolated vertex in $G-P_{13}$ that hits exactly five vertices, say $x_{i}, x_{j}, x_{k}, x_{\ell}, x_{m}, i<j<k<\ell<m$, of $P_{13}$. Recall that $\left\{x_{i}, x_{j}, x_{k}, x_{\ell}, x_{m}\right\} \subseteq$ $V\left(P_{13}\right) \backslash\left\{x_{1}, x_{6}, x_{8}, x_{13}\right\}$ and $y$ cannot hit both $x_{p}$ and $x_{p+8}$ for $p \in\{2,3,4\}$.

Since $y$ cannot hit two adjacent vertices in $P_{13}$, we have $x_{k}=x_{7},\left\{x_{i}, x_{j}\right\} \subseteq$ $\left\{x_{2}, x_{3}, x_{4}, x_{5}\right\}$ and $\left\{x_{\ell}, x_{m}\right\} \subseteq\left\{x_{9}, x_{10}, x_{11}, x_{12}\right\}$. Let $A=\left\{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}\right.$, $\left.x_{m-1}, x_{13}\right\}$ and $B=\left\{x_{1}, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}, x_{m+1}\right\}$. Then, $A$ and $B$ are independent sets and $|A \cap B|=4$. Since $\left\{x_{3}, x_{11}\right\} \nsubseteq N_{G}(y)$, we have either $i=2$ or $m=12$. If $i=2$ and $m=12$, then $N_{G}(y)=\left\{x_{2}, x_{5}, x_{7}, x_{9}, x_{12}\right\}$, which implies that $x_{5}$ misses $\left\{x_{10}, x_{11}\right\}$. And symmetrically $x_{9}$ misses $\left\{x_{3}, x_{4}\right\}$. If $i=2$ and $m \neq 12$, then $\ell=9$ and $m=11$, which implies that $x_{m}$ misses $\left\{x_{3}, x_{6}\right\}$ and $x_{\ell}$ misses $\left\{x_{q}, x_{q+1}\right\} \subseteq\left\{x_{1}, \ldots, x_{7}\right\} \backslash N_{G}(y)$. If $i \neq 2$ and $m=$ 12, then $i=3$ and $j=5$, which implies that $x_{i}$ misses $\left\{x_{8}, x_{11}\right\}$ and $x_{j}$ misses $\left\{x_{q}, x_{q+1}\right\} \subseteq\left\{x_{7}, \ldots, x_{13}\right\} \backslash N_{G}(y)$. For each of the above cases, we have $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-\left(\binom{|A|}{2}+\binom{|B|}{2}-\binom{|A \cap B|}{2}\right)-4=50$.
Fact 6. If there exists an isolated vertex in $G-P_{13}$ that hits four vertices of $P_{13}$, then $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 59$.
Proof. Let $y$ be an isolated vertex in $G-P_{13}$ that hits exactly four vertices, say $x_{i}, x_{j}, x_{k}, x_{\ell}, i<j<k<\ell$, of $P_{13}$. Recall that $\left\{x_{i}, x_{j}, x_{k}, x_{\ell}\right\} \subseteq V\left(P_{13}\right) \backslash$ $\left\{x_{1}, x_{6}, x_{8}, x_{13}\right\}$ and $y$ cannot hit both $x_{p}$ and $x_{p+8}$ for $p \in\{2,3,4\}$. Let $A=$ $\left\{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}, x_{13}\right\}$ and $B=\left\{x_{1}, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}\right\}$. Then $A$ and $B$ are independent sets and $|A \cap B| \leq 3$. If $|A \cap B| \leq 2$, then $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-$ $\left(\binom{|A|}{2}+\binom{|B|}{2}-1\right)=59$. Now we assume that $|A \cap B|=3$. If $i=2$ and $\ell=12$, then $7 \in\{j, k\}$ which implies that $x_{3}$ misses $x_{11}$ and $x_{p}$ misses $x_{p+9}$ for $p \in\{1,4\}$. If $i=2, \ell \neq 12$ and $7 \in\{j, k\}$, then $x_{11}$ misses $\left\{x_{3}, x_{6}\right\}$. If $i=2, \ell \neq 12$ and $7 \notin\{j, k\}$, then $N_{G}(y)=\left\{x_{2}, x_{4}, x_{9}, x_{11}\right\}$ which implies $x_{11}$ misses $\left\{x_{5}, x_{8}\right\}$. If $\ell=12$ and $i \neq 2$, then it is similar as the case of $i=2$ and $\ell \neq 12$. If $i \neq 2$ and $\ell \neq 12$, then $N_{G}(y)=\left\{x_{3}, x_{5}, x_{7}, x_{9}\right\}$ which implies $x_{11}$ misses $\left\{x_{1}, x_{4}\right\}$. For each of the above cases, $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-\left(\binom{|A|}{2}+\binom{|B|}{2}-\binom{|A \cap B|}{2}\right)-2=59$.

Fact 7. If there exists an isolated vertex in $G-P_{13}$ that hits three vertices of $P_{13}$, then $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 67$.

Proof. Let $y$ be an isolated vertex in $G-P_{13}$ that hits exactly three vertices, say $x_{i}, x_{j}, x_{k}, i<j<k$, of $P_{13}$. Recall that $\left\{x_{i}, x_{j}, x_{k}\right\} \subseteq V\left(P_{13}\right) \backslash$ $\left\{x_{1}, x_{6}, x_{8}, x_{13}\right\}$ and $y$ cannot hit both $x_{p}$ and $x_{p+8}$ for $p \in\{2,3,4\}$. Let $A=$ $\left\{x_{i-1}, x_{j-1}, x_{k-1}, x_{13}\right\}$ and $B=\left\{x_{1}, x_{i+1}, x_{j+1}, x_{k+1}\right\}$. Then both $A$ and $B$ are independent sets and $|A \cap B| \leq 2$. Hence, $e\left(G\left[V\left(P_{13}\right)\right]\right) \leq 78-\left(\binom{|A|}{2}+\binom{|B|}{2}-\right.$ $(\underset{2}{|A \cap B|})) \leq 78-(6+6-1)=67$.

Let $P_{k}=y_{1} y_{2} \cdots y_{k}$, where $k \leq 6$, be the longest path in $G-P_{13}$ such that $y_{1}$ hits $P_{13}$. Let $H_{1}, H_{2}, \ldots, H_{t}$ be connected components of order at least 2 of $G-P_{13}$ and let $H$ be a subgraph of $G$ which consists of all isolated vertices of $G-P_{13}$. Note that $\sum_{i=1}^{t}\left|H_{i}\right|+|H|=n-13$. Let $m\left(H_{i}\right)$ be the number of edges
incident with the vertices of $H_{i}$ and let $H_{1}$ be a component of $G-P_{13}$ which contains $P_{k}$ as a subgraph. We first show the following claim.

Claim. For $1 \leq i \leq t, m\left(H_{i}\right) \leq 4\left|H_{i}\right|$.
Proof. We use induction on $\left|H_{i}\right|$. Recall that each vertex of $H_{i}$ can hit at most three vertices of $P_{13}$. For $\left|H_{i}\right|=2, m\left(H_{i}\right)=e\left(G\left[V\left(H_{i}\right)\right]\right)+e\left(V\left(H_{i}\right), V\left(P_{13}\right)\right) \leq$ $7 \leq 4\left|H_{i}\right|$. If $H_{i}$ has a pendant vertex $x$, then $d_{G}(x) \leq 4$. By induction hypothesis, we have $m\left(H_{i}\right)=m\left(H_{i}-x\right)+d_{G}(x) \leq 4\left(\left|H_{i}\right|-1\right)+4 \leq 4\left|H_{i}\right|$. Next, if $H_{i}$ has no pendant vertex, then each vertex of $H_{i}$ must be an endpoint of a path of length at least two. This implies that each vertex of $H_{i}$ can only hit $x_{7}$ of $P_{13}$. Thus, $m\left(H_{i}\right)=e\left(G\left[V\left(H_{i}\right)\right]\right)+e\left(V\left(H_{i}\right), V\left(P_{13}\right)\right) \leq e x_{c o n}\left(\left|H_{i}\right|, P_{7}\right)+\left|H_{i}\right| \leq \frac{7}{2}\left|H_{i}\right|$ since $H_{i}$ is $P_{7}$-free.

Let $\Delta(H)=\max \left\{d_{G}(v) \mid v \in V(H)\right\}$. Recall that $\Delta(H) \leq 5$. Now we would divide the proof into the following cases (in each case we assume, the previous cases do not hold).

Case 1. $\Delta(H)=5$. Then by Fact 5 and the Claim,

$$
e(G) \leq 50+5(n-13)=5 n-15<\max \{[n, 14,7], 5 n-14\}
$$

a contradiction.
Case 2. $\Delta(H)=4$ or $k \geq 3$ or there exists a non-isolated vertex in $G-P_{13}$ that hits two vertices of $P_{13}(k=2)$. Then by Facts 6,2 and 4 and the Claim,

$$
e(G) \leq 59+4(n-13)=4 n+7<\max \{[n, 14,7], 5 n-14\}
$$

a contradiction.
Case 3. $\Delta(H)=3(k=2)$ or there exists a non-isolated vertex in $G-P_{13}$ that hits one vertex of $P_{13}(k=2)$. For $k=2$, each component of $G-P_{13}$ is a star (with at least three vertices), or an edge, or an isolated vertex. For $1 \leq i \leq t, e\left(G\left[V\left(H_{i}\right)\right]\right) \leq\left|H_{i}\right|-1 . m_{0} \leq \sum_{i=1}^{t}\left(2\left|H_{i}\right|-1\right)+3|H|=3(n-15)+$ $6-\sum_{i=1}^{t}\left|H_{i}\right|-t \leq 3(n-13)$. Then by Facts 7 and 3 , we have

$$
e(G) \leq 68+3(n-13)=3 n+29<\max \{[n, 14,7], 5 n-14\}
$$

a contradiction.
Case 4. $\Delta(H) \leq 2$ and $k=1$. Then by Fact 1,

$$
e(G) \leq 74+2(n-13)=2 n+48<\max \{[n, 14,7], 5 n-14\}
$$

a contradiction.
The proof is thus completed.

Proof of Theorem 1.11. Let $G$ be any $2 P_{7}$-free graph on $n$ vertices with $e(G) \geq \max \{[n, 14,7], 5 n-14\}$. If $G$ is connected, then by Lemma 2.3, $e(G) \leq$ $\max \{[n, 14,7], 5 n-14\}$ when $n \geq 22$ and $e(G)<\max \{[n, 14,7], 5 n-14\}$ when $n \leq 21$. Thus when $G$ is connected, $e(G) \leq \max \{[n, 14,7], 5 n-14\}$ with equality holds if and only if $n \geq 22$ and $G=K_{5}+\left(\overline{K_{n-7}} \cup K_{2}\right)$. Now we may assume that $G$ is disconnected. By Lemma 1.4, $G$ contains $P_{7}$ as a subgraph. Let $C$ be a connected component with $n_{1} \geq 7$ vertices which contains $P_{7}$ as a subgraph. Notice that $C$ is $2 P_{7}$-free and $G-C$ is $P_{7}$-free. If $n_{1} \geq 22$, then by Lemma 2.3, $e(C) \leq 5 n-14$ and by Lemmas 1.4 and 2.2,

$$
e(G)=e(C)+e(G-C) \leq 5 n_{1}-14+\left[n-n_{1}, 7,7\right]<5 n-14,
$$

a contradiction. If $14 \leq n_{1} \leq 21$, then by Lemma $2.3, e(C)<\left[n_{1}, 14,7\right]$ and by Lemmas 1.4 and 2.1,

$$
e(G)=e(C)+e(G-C)<\left[n_{1}, 14,7\right]+\left[n-n_{1}, 7,7\right] \leq[n, 14,7],
$$

a contradiction. If $n_{1} \leq 13$, then $e(G) \leq\binom{ n_{1}}{2}+\left[n-n_{1}, 7,7\right] \leq[n, 14,7]$ with equality holds if and only if $C=K_{13}$ and $G-C \in E X\left(n-13, P_{7}\right)$. But then when $n \geq 22, e(G) \geq \max \{[n, 14,7], 5 n-14\}=5 n-14>[n, 14,7]$, a contradiction. Thus when $G$ is disconnected, $e(G) \leq \max \{[n, 14,7], 5 n-14\}$ with equality holds if and only if $n \leq 21, G=K_{13} \cup H$ for $H \in E X\left(n-13, P_{7}\right)$.

The proof is thus complete.

## Acknowledgment

We wish to thank the two anonymous referees for their valuable suggestions and comments. This work was supported by the National Science Foundation and the Natural Science Foundation of Tianjin (No. 17JCQNJC00300).

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doi:10.1016/j.disc.2016.08.004
Received 4 March 2017
Revised 22 November 2017
Accepted 22 November 2017


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