# ON THE METRIC DIMENSION OF DIRECTED AND UNDIRECTED CIRCULANT GRAPHS 

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#### Abstract

The undirected circulant graph $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ consists of vertices $v_{0}, v_{1}, \ldots, v_{n-1}$ and undirected edges $v_{i} v_{i+j}$, where $0 \leq i \leq n-1,1 \leq j \leq t$ ( $2 \leq t \leq \frac{n}{2}$ ), and the directed circulant graph $C_{n}(1, t)$ consists of vertices $v_{0}, v_{1}, \ldots, v_{n-1}$ and directed edges $v_{i} v_{i+1}, v_{i} v_{i+t}$, where $0 \leq i \leq n-1(2 \leq$ $t \leq n-1$ ), the indices are taken modulo $n$. Results on the metric dimension of undirected circulant graphs $C_{n}( \pm 1, \pm t)$ are available only for special values of $t$. We give a complete solution of this problem for directed graphs $C_{n}(1, t)$ for every $t \geq 2$ if $n \geq 2 t^{2}$. Grigorious et al. [On the metric dimension of circulant and Harary graphs, Appl. Math. Comput. 248 (2014) 47-54] presented a conjecture saying that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right)=t+p-1$ for $n=2 t k+t+p$, where $3 \leq p \leq t+1$. We disprove it by showing that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\frac{p+1}{2}$ for $n=2 t k+t+p$, where $t \geq 4$ is even, $p$ is odd, $1 \leq p \leq t+1$ and $k \geq 1$.


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## 1. Introduction

Let $V(G)$ be vertex set of a connected (undirected or directed) graph $G$. The distance $d(u, v)$ between two vertices $u, v$ in an undirected graph is the number of edges in a shortest path between $u$ and $v$. In a directed graph $G$ the distance $d(u, v)$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest directed path from $u$ to $v$.

A vertex $w$ resolves two vertices $u$ and $v$ if $d(u, w) \neq d(v, w)$. For an ordered set of vertices $W=\left\{w_{1}, w_{2}, \ldots, w_{z}\right\}$, the representation of distances of $v$ with respect to $W$ is the ordered $z$-tuple

$$
r(v \mid W)=\left(d\left(v, w_{1}\right), d\left(v, w_{2}\right), \ldots, d\left(v, w_{z}\right)\right)
$$

A set $W \subset V(G)$ is a resolving set of $G$ if every two distinct vertices of $G$ have different representations of distances with respect to $W$ (if every two vertices of $G$ are resolved by some vertex in $W$ ). The metric dimension of $G$ is the number of vertices in a smallest resolving set and it is denoted by $\operatorname{dim}(G)$. The $i$-th coordinate in $r(v \mid W)$ is 0 if and only if $v=w_{i}$. Thus in order to prove that $W$ is a resolving set of $G$, it suffices to show that $r(u \mid W) \neq r(v \mid W)$ for every two different vertices $u, v \in V(G) \backslash W$.

The metric dimension is an invariant, which has applications in robot navigation [9], pharmaceutical chemistry [2], pattern recognition and image processing [10]. It has been extensively studied. For example, Imran [5] studied barycentric subdivisions of Cayley graphs and Saputro et al. [12] gave bounds on the metric dimension of the lexicographic product of graphs

Let $n, m$ and $a_{1}, a_{2}, \ldots, a_{m}$ be positive integers such that $1 \leq a_{1}<a_{2}<$ $\cdots<a_{m} \leq \frac{n}{2}$. The undirected circulant graph $C_{n}\left( \pm a_{1}, \pm a_{2}, \ldots, \pm a_{m}\right)$ consists of the vertices $v_{0}, v_{1}, \ldots, v_{n-1}$ and undirected edges $v_{i} v_{i+a_{j}}$, where $0 \leq i \leq n-1$, $1 \leq j \leq m$; the indices are taken modulo $n$.

For generators $a_{1}, a_{2}, \ldots, a_{m}$ such that $1 \leq a_{1}<a_{2}<\cdots<a_{m} \leq n-1$, the directed circulant graph $C_{n}\left(a_{1}, a_{2}, \ldots, a_{m}\right)$ consists of the vertices $v_{0}, v_{1}, \ldots, v_{n-1}$ and directed edges $v_{i} v_{i+a_{j}}$, where $0 \leq i \leq n-1,1 \leq j \leq m$; the indices are taken modulo $n$. The directed circulant graph $C_{n}\left(-a_{1},-a_{2}, \ldots,-a_{m}\right)$ contains the directed edges $v_{i} v_{i-a_{j}}$.

Circulant graphs form an important family of Cayley graphs. The metric dimension of undirected circulant graphs $C_{n}( \pm 1, \pm t)$ was studied for special values of $t$. Javaid, Rahim and Ali [8] proved that if $n \equiv 0,2,3(\bmod 4)$, then $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2)\right)=3$. Borchert and Gosselin [1] showed that if $n \equiv 1(\bmod 4)$, then $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2)\right)=4$. The undirected circulant graphs $C_{n}( \pm 1, \pm 3)$ were considered in [7] and the graphs $C_{n}\left( \pm 1, \pm \frac{n}{2}\right)$ for even $n$ were investigated in [11]. We study the metric dimension for directed circulant graphs with 2 generators. We give a complete solution of this problem for directed graphs $C_{n}(1, t)$ for every $t \geq 2$ if $n \geq 2 t^{2}$.

Exact values of the metric dimension of undirected graphs $C_{n}( \pm 1, \pm 2, \pm 3)$ were given in [1] and [6]. Grigorious et al. [4] showed that $t+1$ vertices $v_{0}, v_{1}, \ldots, v_{t}$ resolve the graph $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ if $n \equiv r(\bmod 2 t)$, where $2 \leq r \leq t+2$ and they gave the bound $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq r-1$ if $n \equiv r(\bmod 2 t)$, where $t+3 \leq r \leq 2 t+1$. They presented a conjecture saying that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right)=t+p-1$ for $n=2 t k+t+p$, where
$3 \leq p \leq t+1$. We disprove it for even $t \geq 4$ and odd $p \geq 5$ by showing that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\frac{p+1}{2}$ for $n=2 t k+t+p$ where $t \geq 4$ is even, $p$ is odd, $1 \leq p \leq t+1$ and $k \geq 1$. Note that Chau and Gosselin [3] recently proved that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right)=t+1$ if $n \equiv 2(\bmod 2 t)$ and $n \equiv t+1(\bmod 2 t)$. They also showed that $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right)=\operatorname{dim}\left(C_{n+2 t}( \pm 1, \pm 2, \ldots, \pm t)\right)$ for large $n$, which implies that the metric dimension of the graphs $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ is completely determined by the congruence class of $n$ modulo $2 t$.

## 2. Directed Circulant Graphs

We study the metric dimension of directed circulant graphs $C_{n}(1, t)$. It is easy to see that the graph $C_{n}(1, t)$ is isomorphic to the graph $C_{n}(-1,-t)$ for $2 \leq t \leq n-1$. We present Theorems 1 and 2 for the graph $C_{n}(-1,-t)$, because it is easier to express distances from vertices in a graph to vertices in chosen resolving sets if we consider $C_{n}(-1,-t)$ (especially in the proof of Theorem 2 ).

The distance from the vertex $v_{j}$ to the vertex $v_{i}$ in $C_{n}(-1,-t)$, where $i, j \in$ $\{0,1, \ldots, n-1\}$, is

$$
d\left(v_{j}, v_{i}\right)=\left\{\begin{array}{lll}
\left\lfloor\frac{j-i}{t}\right\rfloor+p, & p \equiv(j-i)(\bmod t), & \text { if } j \geq i  \tag{1}\\
\left\lfloor\frac{n+j-i}{t}\right\rfloor+p, & p \equiv(n+j-i)(\bmod t), & \text { if } j<i
\end{array}\right.
$$

where $0 \leq p \leq t-1$.
Theorem 1. Let $t \geq 2$ and $n \geq 2 t^{2}$. Then $\operatorname{dim}\left(C_{n}(-1,-t)\right) \geq t$.
Proof. We prove the result by contradiction. Assume that $\operatorname{dim}\left(C_{n}(-1,-t)\right) \leq$ $t-1$. Let $W=\left\{v_{i_{1}}, v_{i_{2}}, \ldots, v_{i_{t-1}}\right\}$ be a resolving set of $C_{n}(-1,-t)$, where $0 \leq i_{1} \leq i_{2} \leq \cdots \leq i_{t-1}$. Since we have at most $t-1$ different vertices in $W$ and the graph $C_{n}(-1,-t)$ has at least $2 t^{2}$ vertices, $C_{n}(-1,-t)$ contains a set of $2 t$ consecutive vertices $V^{\prime}=\left\{v_{j}, v_{j+1}, \ldots, v_{j+2 t-1}\right\}$, where $0 \leq j \leq n-1$, such that no vertex of $W$ is in $V^{\prime}$. Without loss of generality we can assume that $j=n-2 t$, which means that $V^{\prime}=\left\{v_{n-2 t}, v_{n-2 t+1}, \ldots, v_{n-1}\right\}$ and $i_{t-1}<n-2 t$.

Since $|W| \leq t-1$, there is a $k \in\{0,1, \ldots, t-1\}$, such that no vertex $v_{i} \in W$ satisfies $i \equiv k(\bmod t)$. So we can write any vertex of $W$ in the form $v_{t r+s}$, where $0 \leq s \leq t-1, s \neq k$ and $r \geq 0$.

Let $v_{l}$ be any vertex in the set of $t$ vertices $\left\{v_{n-2 t}, v_{n-2 t+1}, \ldots, v_{n-t-1}\right\}$, such that $l \equiv k(\bmod t)$. Then we can write $l=t x+k$, where $0 \leq k \leq t-1$. We show that the vertices $v_{t x+k}, v_{t x+k+t-1} \in V^{\prime}$ are not resolved by $W$. Note that
$t x+k>t r+s$. By (1) we have

$$
\begin{gathered}
d\left(v_{t x+k}, v_{t r+s}\right)= \begin{cases}\left\lfloor\frac{t x+k-(t r+s)}{t}\right\rfloor+k-s=x-r+\left\lfloor\frac{k-s}{t}\right\rfloor+k-s \\
=x-r+k-s & \text { if } k>s \\
x-r+\left\lfloor\frac{k-s}{t}\right\rfloor+k-s+t \\
=x-r+k-s+t-1\end{cases} \\
d\left(v_{t x+k+t-1}, v_{t r+s}\right)= \begin{cases}\left\lfloor\frac{t x+k+t-1-(t r+s)}{t}\right\rfloor+k-1-s \\
=x-r+k-s \\
x+1-r+\left\lfloor\frac{k-1-s}{t}\right\rfloor+k-1-s+t \\
=x-r+k-s+t-1\end{cases} \\
\text { if } k>s,
\end{gathered}
$$

Since $d\left(v_{t x+k}, v_{t r+s}\right)=d\left(v_{t x+k+t-1}, v_{t r+s}\right)$ for any vertex $v_{t r+s} \in W$, the graph $C_{n}(-1,-t)$ is not resolved by $W$. A contradiction.

Let us present an upper bound on the metric dimension of directed circulant graphs with 2 generators.

Theorem 2. Let $2 \leq t<n$. Then $\operatorname{dim}\left(C_{n}(-1,-t)\right) \leq t$.
Proof. We prove that $W=\left\{v_{0}, v_{1}, \ldots, v_{t-1}\right\}$ is a resolving set of $C_{n}(-1,-t)$. First we find all vertices $v_{j}(1 \leq j \leq n-1)$ of $C_{n}(-1,-t)$ such that $d\left(v_{j}, v_{0}\right)=x$ for any $x \geq 1$. We can write $j=t r+p$ where $r \geq 0$ and $0 \leq p \leq t-1$. Since by $(1), d\left(v_{t r+p}, v_{0}\right)=r+p$, we have $r+p=x$. Thus $r=x-p(\geq 0)$ and then $v_{t(x-p)+p}$ for $0 \leq p \leq t-1$ and $1 \leq t(x-p)+p \leq n-1$ are the vertices of $C_{n}(1, t)$ such that $d\left(v_{t(x-p)+p}, v_{0}\right)=x$.

It remains to show that these vertices are resolved by $v_{i}, i=1,2, \ldots, t-1$. It suffices to consider only those vertices $v_{t(x-p)+p}$ which are not in $W$, so we can assume that $t(x-p)+p>i$. For $i=1,2, \ldots, t-1$, by (1),

$$
d\left(v_{t(x-p)+p}, v_{i}\right)= \begin{cases}x-p+\left\lfloor\frac{p-i}{t}\right\rfloor+p-i=x-i & \text { if } p \geq i  \tag{3}\\ x-p+\left\lfloor\frac{p-i}{t}\right\rfloor+p-i+t=x+t-1-i & \text { if } p<i\end{cases}
$$

We know that the first entry of $r\left(v_{t(x-p)+p} \mid W\right)$ is $x$. From (3) it follows that the next $p$ entries (where $0 \leq p \leq t-1$ ) are $x-i$ and the last $t-1-p$ entries of $r\left(v_{t(x-p)+p} \mid W\right)$ are $x+t-1-i$.

So if $p=0$ (and if $v_{t x}$ exists), the first entry of $r\left(v_{t x} \mid W\right)$ is $x$ and the other entries are $x+t-1-i$ which means that $r\left(v_{t x} \mid W\right)=(x, x+t-2, x+$ $t-3, \ldots, x+t-1-(t-1))$. If $p=1$, the first entry of $r\left(v_{t(x-1)+1} \mid W\right)$ is $x$, the second entry is $x-1$ and the other entries are $x+t-1-i$, so $r\left(v_{t(x-1)+1} \mid W\right)=$
$(x, x-1, x+t-3, x+t-4, \ldots, x+t-1-(t-1))$. Similarly $r\left(v_{t(x-2)+2} \mid W\right)=$ $(x, x-1, x-2, x+t-4, \ldots, x+t-1-(t-1)), \ldots, r\left(v_{t(x-(t-1))+(t-1)} \mid W\right)=$ $(x, x-1, x-2, \ldots, x-(t-1))$.

Since all vertices $v_{j}, 1 \leq j \leq n-1$, such that $d\left(v_{j}, v_{0}\right)=x$ are resolved by $W$, we have $\operatorname{dim}\left(C_{n}(-1,-t)\right) \leq|W|=t$.

From Theorems 1 and 2 we obtain Corollary 3.
Corollary 3. Let $t \geq 2$ and $n \geq 2 t^{2}$. Then $\operatorname{dim}\left(C_{n}(-1,-t)\right)=t$.
Since the graphs $C_{n}(-1,-t)$ and $C_{n}(1, t)$ are isomorphic, we get the following corollary.

Corollary 4. Let $t \geq 2$ and $n \geq 2 t^{2}$. Then $\operatorname{dim}\left(C_{n}(1, t)\right)=t$.

## 3. Undirected Circulant Graphs

We give an upper bound on the metric dimension of undirected circulant graphs $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ for $n \equiv r(\bmod 2 t)$, where $r=1$ and $r=t+1, t+3, \ldots, 2 t-1$.

The distance between two vertices $v_{i}$ and $v_{j}$ in $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$, where $0 \leq i<j<n$, is

$$
\begin{equation*}
d\left(v_{i}, v_{j}\right)=\min \left\{\left\lceil\frac{j-i}{t}\right\rceil,\left\lceil\frac{n-(j-i)}{t}\right\rceil\right\} . \tag{4}
\end{equation*}
$$

This equation can be simplified as

$$
d\left(v_{i}, v_{j}\right)= \begin{cases}\left\lceil\frac{j-i}{t}\right\rceil & \text { if } 0 \leq j-i \leq \frac{n}{2}  \tag{5}\\ \left\lceil\frac{n-(j-i)}{t}\right\rceil & \text { if } \frac{n}{2}<j-i<n\end{cases}
$$

Theorem 5. Let $n=2 t k+t+p$ where $t \geq 4$ is even, $p$ is odd, $1 \leq p \leq t+1$ and $k \geq 1$. Then

$$
\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\frac{p+1}{2}
$$

Proof. Let $n=2 t k+t+p$ where $k \geq 1, t \geq 4$ is even and $p=1,3, \ldots, t+1$. Let

$$
\begin{aligned}
& W_{1}=\left\{v_{0}, v_{2}, \ldots, v_{t-2}\right\}, \quad W_{2}=\left\{v_{t-1}, v_{t+1}, \ldots, v_{2 t-3}\right\}, \\
& W_{3}=\left\{v_{t k+t-1}, v_{t k+t+1}, \ldots, v_{t k+t+p-2}\right\} .
\end{aligned}
$$

We have $\left|W_{1}\right|=\left|W_{2}\right|=\frac{t}{2}$ and $\left|W_{3}\right|=\frac{p+1}{2}$. Let us prove that $W=W_{1} \cup W_{2} \cup W_{3}$ is a resolving set of the graph $C_{n}(1,2, \ldots, t)$.

We divide the vertex set of $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ into four disjoint sets:

$$
\begin{array}{ll}
V_{1}=\left\{v_{0}, v_{1}, \ldots, v_{t}\right\}, & V_{2}=\left\{v_{t+1}, v_{t+2}, \ldots, v_{t k+t}\right\}, \\
V_{3}=\left\{v_{t k+t+1}, v_{t k+t+2}, \ldots, v_{t k+t+p-1}\right\}, & V_{4}=\left\{v_{t k+t+p}, v_{t k+t+p+1}, \ldots, v_{n-1}\right\} .
\end{array}
$$

First we prove that any two vertices of $V_{2}$ have different representations of distances with respect to $W$. For $x=1,2, \ldots, k-1 ; j=1,2, \ldots, t ; i=$ $0,2, \ldots, t-2$, we have $v_{i} \in W_{1}$ and by (5),

$$
d\left(v_{t x+j}, v_{i}\right)=x+\left\lceil\frac{j-i}{t}\right\rceil= \begin{cases}x+1 & \text { if } i<j \\ x & \text { if } i \geq j\end{cases}
$$

and if $x=k ; j=1,2, \ldots, t$, by (4), we get

$$
\begin{aligned}
d\left(v_{t k+j}, v_{i}\right) & =\min \left\{\left\lceil\frac{(t k+j)-i}{t}\right\rceil,\left\lceil\frac{n-[(t k+j)-i]}{t}\right\rceil\right\}, \\
& =\min \left\{k+\left\lceil\frac{j-i}{t}\right\rceil, k+1+\left\lceil\frac{p+i-j}{t}\right\rceil\right\}= \begin{cases}k+1 & \text { if } i<j, \\
k & \text { if } i \geq j\end{cases}
\end{aligned}
$$

Since $j$ (where $1 \leq j \leq t$ ) is greater than $\left\lceil\frac{j}{2}\right\rceil$ elements from the set $\{0,2, \ldots, t-2\}$, the first $\left\lceil\frac{j}{2}\right\rceil$ entries of $r\left(v_{t x+j} \mid W_{1}\right)$ for $x=1,2, \ldots, k$ are equal to $x+1$ and the other $\frac{t}{2}-\left\lceil\frac{j}{2}\right\rceil$ entries are $x ; r\left(v_{t x+j} \mid W_{1}\right)=(x+1, \ldots, x+1, x, \ldots, x)$. Therefore the only vertices in $V_{2}$ with the same representations of distances with respect to $W_{1}$ are the pairs $\left(v_{t+1}, v_{t+2}\right),\left(v_{t+3}, v_{t+4}\right), \ldots,\left(v_{t k+t-1}, v_{t k+t}\right)$. But since for $x=1,2, \ldots, k$ and $j=1,3, \ldots, t-3$, we obtain $v_{t+j} \in W_{2}$ and by (5),

$$
d\left(v_{t x+j}, v_{t+j}\right)=x-1, \quad d\left(v_{t x+j+1}, v_{t+j}\right)=x-1+\left\lceil\frac{1}{t}\right\rceil=x
$$

and for $v_{t-1} \in W_{2}$, we have

$$
d\left(v_{t x+t-1}, v_{t-1}\right)=x, \quad d\left(v_{t x+t}, v_{t-1}\right)=x+\left\lceil\frac{1}{t}\right\rceil=x+1
$$

vertices in $W_{2}$ resolve the pairs $\left(v_{t+1}, v_{t+2}\right),\left(v_{t+3}, v_{t+4}\right), \ldots,\left(v_{t k+t-1}, v_{t k+t}\right)$. Thus no two vertices in $V_{2}$ have the same representations of distances with respect to $W$.

Let us study representations of distances of the vertices in $V_{4}$. For $x=$ $1,2, \ldots, k-1 ; j=0,1, \ldots, t-1 ; i=0,2, \ldots, t-2 ;$ we have $v_{i} \in W_{1}$ and by (6),

$$
d\left(v_{n-t x+j}, v_{i}\right)=\left\lceil\frac{n-[(n-t x+j)-i]}{t}\right\rceil=x+\left\lceil\frac{i-j}{t}\right\rceil= \begin{cases}x & \text { if } i \leq j, \\ x+1 & \text { if } i>j\end{cases}
$$

and if $x=k$, we get

$$
\begin{aligned}
d\left(v_{n-t k+j}, v_{i}\right) & =\min \left\{\left\lceil\frac{(n-t k+j)-i}{t}\right\rceil,\left\lceil\frac{n-[(n-t k+j)-i]}{t}\right\rceil\right\} \\
& =\min \left\{k+1+\left\lceil\frac{p+j-i}{t}\right\rceil, k+\left\lceil\frac{i-j}{t}\right\rceil\right\}= \begin{cases}k & \text { if } i \leq j, \\
k+1 & \text { if } i>j\end{cases}
\end{aligned}
$$

Since $j$ (where $0 \leq j \leq t-1$ ) is greater than or equal to $\left\lfloor\frac{j}{2}\right\rfloor+1$ elements from the set $\{0,2, \ldots, t-2\}$, the first $\left\lfloor\frac{j}{2}\right\rfloor+1$ entries of $r\left(v_{n-t x+j} \mid W_{1}\right)$ (for $x=$ $1,2, \ldots, k)$ are equal to $x$ and the other entries are $x+1$. The only vertices in $V_{4}$ with the same representations of distances with respect to $W_{1}$ are the pairs $\left(v_{n-t k}, v_{n-t k+1}\right),\left(v_{n-t k+2}, v_{n-t k+3}\right), \ldots,\left(v_{n-2}, v_{n-1}\right)$. We show that most of these pairs are resolved by vertices in $W_{2}$. For $x=1,2, \ldots, k-1$ and $j=1,3, \ldots, t-3$, we have $v_{t+j} \in W_{2}$ and by (6),

$$
d\left(v_{n-t x+j}, v_{t+j}\right)=x+1, d\left(v_{n-t x+j-1}, v_{t+j}\right)=x+1+\left\lceil\frac{1}{t}\right\rceil=x+2,
$$

and for $v_{t-1} \in W_{2}, x=1,2, \ldots, k$, by (6),

$$
d\left(v_{n-t x+t-1}, v_{t-1}\right)=x, d\left(v_{n-t x+t-2}, v_{t-1}\right)=x+\left\lceil\frac{1}{t}\right\rceil=x+1,
$$

so vertices of $W_{2}$ resolve all pairs of vertices $\left(v_{n-t k+t-2}, v_{n-t k+t-1}\right),\left(v_{n-t k+t}\right.$, $\left.v_{n-t k+t+1}\right), \ldots,\left(v_{n-2}, v_{n-1}\right)$, which are the pairs $\left(v_{t k+2 t+p-2}, v_{t k+2 t+p-1}\right)$, $\left(v_{t k+2 t+p}, v_{t k+2 t+p+1}\right), \ldots,\left(v_{n-2}, v_{n-1}\right)$. It remains to resolve the pairs $\left(v_{t k+t+p}\right.$, $\left.v_{t k+t+p+1}\right),\left(v_{t k+t+p+2}, v_{t k+t+p+3}\right), \ldots,\left(v_{t k+2 t+p-4}, v_{t k+2 t+p-3}\right)$.

For $j=0,2, \ldots, t-p-3$, we have $v_{t+p+j} \in W_{2}$ and by (5),

$$
d\left(v_{t k+t+p+j}, v_{t+p+j}\right)=k, \quad d\left(v_{t k+t+p+j+1}, v_{t+p+j}\right)=k+\left\lceil\frac{1}{t}\right\rceil=k+1,
$$

so the pairs $\left(v_{t k+t+p}, v_{t k+t+p+1}\right), \ldots,\left(v_{t k+2 t-3}, v_{t k+2 t-2}\right)$ are resolved by $W_{2}$.
For $j=t-p-1, t-p+1, \ldots, t-4$, we have $v_{t k+p+j} \in W_{3}$ and by (5),

$$
d\left(v_{t k+t+p+j}, v_{t k+p+j}\right)=1, d\left(v_{t k+t+p+j+1}, v_{t k+p+j}\right)=1+\left\lceil\frac{1}{t}\right\rceil=2,
$$

so the pairs $\left(v_{t k+2 t-1}, v_{t k+2 t}\right), \ldots,\left(v_{t k+2 t+p-4}, v_{t k+2 t+p-3}\right)$ are resolved by $W_{3}$. Thus all pairs of vertices in $V_{4}$ are resolved by $W$.

A vertex $v \in V_{2}$ and a vertex in $V_{4}$ can have the same representations of distances with respect to $W_{1}$ only if all entries of $r\left(v \mid W_{1}\right)$ are the same numbers. For $x=1,2, \ldots, k$, we have $v_{t x+t-1}, v_{t x+t} \in V_{2}$ and $r\left(v_{t x+t-1} \mid W_{1}\right)=r\left(v_{t x+t} \mid W_{1}\right)=$ $(x+1, \ldots, x+1)$. For $v_{n-t x+t-2}, v_{n-t x+t-1} \in V_{4}$ we have $r\left(v_{n-t x+t-2} \mid W_{1}\right)=$ $r\left(v_{n-t x+t-1} \mid W_{1}\right)=(x, \ldots, x)$, which implies that for $x=1,2, \ldots, k-1$, we obtain $r\left(v_{t x+t-1} \mid W_{1}\right)=r\left(v_{t x+t} \mid W_{1}\right)=r\left(v_{n-t x-2} \mid W_{1}\right)=r\left(v_{n-t x-1} \mid W_{1}\right)$. Since for $v_{2 t-3} \in W_{2}$, by (5),

$$
d\left(v_{t x+t-1}, v_{2 t-3}\right)=x-1+\left\lceil\frac{2}{t}\right\rceil=x, \quad d\left(v_{t x+t}, v_{2 t-3}\right)=x-1+\left\lceil\frac{3}{t}\right\rceil=x,
$$

and by (6),
$d\left(v_{n-t x-2}, v_{2 t-3}\right)=x+2+\left\lceil\frac{-1}{t}\right\rceil=x+2, d\left(v_{n-t x-1}, v_{2 t-3}\right)=x+2+\left\lceil\frac{-2}{t}\right\rceil=x+2$,
any vertex in $V_{2}$ and any vertex in $V_{4}$ have different representations of distances with respect to $W$.

We study representations of the vertices in $V_{3}$. For $j=1,2, \ldots, p-1$ and $i=0,2, \ldots, t-2$, we have $v_{i} \in W_{1}$ and by (4),

$$
d\left(v_{t k+t+j}, v_{i}\right)=\min \left\{k+1+\left\lceil\frac{j-i}{t}\right\rceil, k+\left\lceil\frac{p+i-j}{t}\right\rceil\right\}=k+1
$$

thus $r\left(v_{t k+t+j} \mid W_{1}\right)=(k+1, \ldots, k+1)$. The only vertices in $V_{2} \cup V_{4}$ with the same representations of distances with respect to $W_{1}$ are $v_{t k+t-1}$ and $v_{t k+t}$.

Let us prove that any two vertices in $V_{3} \cup\left\{v_{t k+t-1}, v_{t k+t}\right\}$ have different representations with respect to $W$. It suffices to consider the vertices in $V^{\prime}=$ $\left(V_{3} \cup\left\{v_{t k+t-1}, v_{t k+t}\right\}\right) \backslash W_{3}=\left\{v_{t k+t}, v_{t k+t+2}, \ldots, v_{t k+t+p-1}\right\}$. For $j=0,2, \ldots, p-$ 1 and $i=1,3, \ldots, t-3$, we have $v_{t+i} \in W_{2}$ and by (5)

$$
d\left(v_{t k+t+j}, v_{t+i}\right)=k+\left\lceil\frac{j-i}{t}\right\rceil= \begin{cases}k & \text { if } i \geq j \\ k+1 & \text { if } i<j\end{cases}
$$

Since $j$ (for $j \leq t-2$ ) is greater than $\frac{j}{2}$ elements from the set $\{1,3, \ldots, t-3\}$, the first $\frac{j}{2}$ entries of $r\left(v_{t k+t+j} \mid W_{2}^{\prime}\right)$ where $W_{2}^{\prime}=W_{2} \backslash\left\{v_{t-1}\right\}$ are equal to $k+1$ and the other $\frac{t}{2}-\frac{j}{2}-1$ entries are $k$. If $p=t+1$ and $j=t$, we obtain $r\left(v_{t k+t+j} \mid W_{2}^{\prime}\right)=r\left(v_{t k+2 t} \mid W_{2}^{\prime}\right)=(k+1, \ldots, k+1)$. It follows that the only vertices of $V^{\prime}$ having the same representations of distances with respect to $W_{2}^{\prime}$ are $v_{t k+2 t}$ and $v_{t k+2 t-2}$ if $p=t+1$. These vertices are resolved by $v_{t k+t-1} \in W_{3}$, since by (5), $d\left(v_{t k+2 t}, v_{t k+t-1}\right)=1+\left\lceil\frac{1}{t}\right\rceil=2$ and $d\left(v_{t k+2 t-2}, v_{t k+t-1}\right)=1+\left\lceil\frac{-1}{t}\right\rceil=1$. Thus all vertices of $V_{3}$ are resolved by $W$.

We consider the vertices in $V_{1}$. For $j=1,3, \ldots, t-1$ and $t ; i=0,2, \ldots, t-2$, we have $v_{i} \in W_{1}$ and $d\left(v_{j}, v_{i}\right)=\left\lceil\frac{|j-i|}{t}\right\rceil=1$, thus $r\left(v_{j} \mid W_{1}\right)=(1, \ldots, 1)$ for $v_{j} \in V_{1} \backslash W_{1}$. From the previous part of this proof we know that the only vertices in $V_{2} \cup V_{3} \cup V_{4}$ having the representation with respect to $W_{1}$ equal to $(1, \ldots, 1)$ are $v_{n-2}$ and $v_{n-1}$. Since $v_{t-1} \in W_{2}$, it remains to resolve all pairs of vertices in the set $V^{\prime \prime}=\left\{v_{1}, v_{3}, \ldots, v_{t-3} ; v_{t}, v_{n-2}, v_{n-1}\right\}$.

We study their representations with respect to $W_{2}$. For $j=1,3, \ldots, t-3$ and $i=-1,1, \ldots, t-3$, we have $v_{t+i} \in W_{2}$ and by (5),

$$
d\left(v_{j}, v_{t+i}\right)=1+\left\lceil\frac{i-j}{t}\right\rceil= \begin{cases}1 & \text { if } i \leq j, \\ 2 & \text { if } i>j\end{cases}
$$

Since $j$ is greater than or equal to $\frac{j+3}{2}$ elements from the set $\{-1,1, \ldots, t-3\}$, the first $\frac{j+3}{2}$ entries of $r\left(v_{j} \mid W_{2}\right)$ are equal to 1 and the other $\frac{t}{2}-\frac{j+3}{2}$ entries
are 2. The first two entries of $r\left(v_{j} \mid W_{3}\right)$ are always 1 . For $v_{t}$ and any $v_{t+i} \in W_{2}$, $d\left(v_{t}, v_{t+i}\right)=\left\lceil\frac{|i|}{t}\right\rceil=1$, therefore $r\left(v_{t} \mid W_{2}\right)=(1, \ldots, 1)$.

For $i=-1,1, \ldots, t-3$, by ( 6 ),
$d\left(v_{n-1}, v_{t+i}\right)=1+\left\lceil\frac{i+1}{t}\right\rceil= \begin{cases}1 & \text { if } i=-1, \\ 2 & \text { if } i \geq 1,\end{cases}$
so $r\left(v_{n-1} \mid W_{2}\right)=(1,2, \ldots, 2)$. We have $d\left(v_{n-2}, v_{t+i}\right)=1+\left\lceil\frac{i+2}{t}\right\rceil=2$, thus $r\left(v_{n-2} \mid W_{2}\right)=(2, \ldots, 2)$.

The only pair of vertices in $V^{\prime \prime}$ having the same representations with respect to $W_{2}$ is $\left(v_{t-3}, v_{t}\right)$, which is resolved by $v_{t k+t-1} \in W_{3}$, since by (5) we have $d\left(v_{t-3}, v_{t k+t-1}\right)=k+\left\lceil\frac{2}{t}\right\rceil=k+1$ and $d\left(v_{t}, v_{t k+t-1}\right)=k+\left\lceil\frac{-1}{t}\right\rceil=k$.

Every two distinct vertices of the graph $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$ have different representations of distances with respect to $W$, thus $W$ is a resolving set of $C_{n}( \pm 1, \pm 2, \ldots, \pm t)$. Hence $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq|W|=t+\frac{p+1}{2}$.

## 4. Conclusion

We studied the metric dimension of undirected and directed circulant graphs. Results on the metric dimension of undirected circulant graphs $C_{n}( \pm 1, \pm t)$ are available only for special values of $t$. In Section 2 we found exact values of the metric dimension for directed circulant graphs $C_{n}(1, t)$ by showing that if $t \geq 2$ and $n \geq 2 t^{2}$, then $\operatorname{dim}\left(C_{n}(1, t)\right)=t$.

In Section 3 we presented a bound on the metric dimension of undirected circulant graphs. We proved that for $n=2 t k+t+p$, where $t \geq 4$ is even, $p$ is odd, $1 \leq p \leq t+1$ and $k \geq 1, \operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\frac{p+1}{2}$. Note that by [13], $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\frac{p}{2}$ if $t$ and $p$ are even, $2 \leq p \leq t$, thus we have $\operatorname{dim}\left(C_{n}( \pm 1, \pm 2, \ldots, \pm t)\right) \leq t+\left\lceil\frac{p}{2}\right\rceil$ for $n=2 t k+t+p$, where $t \geq 4$ is even, $1 \leq p \leq t+1$ and $k \geq 1$,

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