*Discussiones Mathematicae*
*Graph Theory 39 (2019) 899–908*
*doi:10.7151/dmgt.2109*

# STAR COLORING OUTERPLANAR BIPARTITE GRAPHS

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**Abstract**

A proper coloring of the vertices of a graph is called a *star coloring* if at least three colors are used on every 4-vertex path. We show that all outerplanar bipartite graphs can be star colored using only five colors and construct the smallest known example that requires five colors.

**Keywords:** chromatic number, star coloring, outerplanar bipartite graph.

**2010 Mathematics Subject Classification:** 05C15.

## 1. Introduction

A *proper $r$-coloring* of a graph $G$ is an assignment of labels from $\{1, 2, \ldots, r\}$ to the vertices of $G$ so that adjacent vertices receive distinct colors. The minimum $r$ so that $G$ has a proper $r$-coloring is called the *chromatic number* of $G$, denoted by $\chi(G)$. The chromatic number is one of the most studied parameters in graph theory, and by convention, the term *coloring* of a graph is usually used instead of proper coloring. In 1973, Grünbaum [5] considered proper colorings with the additional constraint that the subgraph induced by every pair of color classes is *acyclic*, i.e., contains no cycles. He called such colorings *acyclic colorings*, and the minimum $r$ such that $G$ has an acyclic $r$-coloring is called the *acyclic chromatic number* of $G$, denoted by $a(G)$. In introducing the notion of an acyclic coloring, Grünbaum noted that the condition that the union of any two color classes induces a forest can be generalized to other bipartite graphs. Among other problems, he suggested requiring that the union of any two color classes induces a star forest, i.e., a proper coloring avoiding 2-colored paths with four vertices. We call such a coloring a *star coloring* and the minimum $r$ such that $G$ has an $r$-star coloring is called the *star chromatic number* of $G$.

Fertin, Raspaud and Reed [4] in 2001 and Nešetřil and Ossona de Mendez [7] in 2003 studied star colorings extensively. In 2004, Albertson, Chappell, Kierstead, Kündgen and Ramamurthi [1] presented further results and questions on the star chromatic number of graphs. In particular, for planar graphs, they proved an upper bound on the star chromatic number of 20 and constructed a planar graph with star chromatic number 10. By restricting the family to bipartite planar graphs, Kierstead, Kündgen and Timmons [6] were able to improve the upper bound to 14, and they also gave an example of a planar bipartite graph requiring 8 colors to star color. In general, determining the exact minimum number of colors required to star color a family of graphs is a challenging problem. Recently Chen, Raspaud and Wang [3] gave an intricate proof that graphs with maximum degree 3 can be star colored with 6 colors. This is best possible since the 3-regular Möbius ladder obtained by joining opposite vertices of an 8-cycle requires 6 colors [4].

This paper investigates star colorings of outerplanar graphs. Both Fertin, Raspaud and Reed [4] and Albertson *et al.* [1] separately established a bound of six on the star chromatic number of an outerplanar graph, and gave constructions showing that the result is best possible. We show that when restricted to the family of outerplanar bipartite graphs, the star chromatic number is at most 5 and that this is best possible (see also [8, 9]) by constructing an example of an outerplanar bipartite graphs that requires 5 colors. Our example has only 30 vertices, making it the smallest known construction that achieves the bound.

In Section 2, we collect some basic definitions and results from previous works that we will use in later sections. In Section 3, we make some observations about the structure of outerplanar bipartite graphs and prove that they are 5-star colorable. In Section 4, we define a family of graphs called the *outerplanar grid*, $OG_n$ and show that for $n \geq 7$, any star coloring of $OG_n$ requires 5 colors.

## 2. Definitions and Preliminaries

All graphs considered in this paper are loopless. The general terminology and definitions used in this paper follow West [10]. For terminology related to star colorings, we follow the paper of Albertson *et al.* [1]. In this section, we collect some definitions and results from that paper as well as from the work of Fertin, Raspaud and Reed [4].

Recall that a *star* is a graph isomorphic to $K_{1,t}$ for some $t \geq 0$ and a graph, all of whose components are stars, is called a *star forest.* In a proper coloring that avoids a 2-colored $P_4$, the union of any two color classes cannot induce a cycle since every even cycle contains $P_4$ as a subgraph. Hence the union induces a star forest (every component must be a star, since otherwise it would contain a 2-colored $P_4$).

**Definition 2.1.** An $r$-coloring of $G$ is called a *star coloring* if there are no 2-colored paths on 4 vertices. The minimum $r$ such that $G$ has a star coloring using $r$ colors is called the *star chromatic number* of $G$ and is denoted by $\chi_s(G)$.

To study star coloring an equivalent digraph coloring notion was introduced in [1].

**Definition 2.2.** A proper coloring of an orientation of a graph $G$ is called an *in-coloring* if within every 2-colored $P_3$ in $G$, the edges are directed towards the middle vertex. We call such a $P_3$ an *in-*$P_3$. A coloring of $G$ is an in-coloring if it is an in-coloring of some orientation of $G$.

The following simple lemma from [1] asserts that to star color $G$, it suffices to in-color a suitable orientation of $G$.

**Lemma 2.3.** *A coloring of a graph $G$ is a star coloring if and only if it is an in-coloring of some orientation of $G$.*

For general graphs, the star chromatic number can be bounded in terms of the acyclic chromatic number. Albertson *et al.* [1] proved that $\chi_s(G) \leq a(2a-1)$, where $a$ is the acyclic chromatic number of $G$. Borodin proved that planar graphs are acyclically 5-colorable [2], which yields a bound of 45. Albertson *et al.* [1] also used Lemma 2.3 to prove more directly that $\chi_S(G) \leq 20$ when $G$ is planar. For special classes of planar graphs, however, it is possible to get the exact answer. For the $m \times n$ grid graph $G_{m,n}$, Fertin, Raspaud and Reed [4] proved the following.

**Theorem 2.4.** $\chi_s(G_{m,n}) = 5$ *when* $m, n \geq 4$.

They state that their proof to show $\chi_s(G_{4,4}) \geq 5$, and hence $\chi_s(G_{m,n}) \geq 5$ for $\min\{m, n\} \geq 4$, was the result of a rather tedious case by case analysis that was confirmed by computer. For the upper bound, the coloring obtained by assigning to a vertex with coordinates $(a, b)$ the color $(a + 2b) \mod 5$ can be easily seen to be a proper coloring with the additional property that any two vertices that are distance at most 2 apart receive different colors. Thus, there are no 2-colored $P_3$'s and hence, there can be no 2-colored $P_4$'s.

In this paper, we restrict our attention to outerplanar graphs. Outerplanar graphs are a special class of planar graphs with many interesting structural properties. An outerplanar graph is characterized by the fact that it has an embedding so that every vertex lies on the outer face. Every simple outerplanar graph has a vertex of degree at most 2, hence outerplanar graphs are 2-degenerate. It follows that they are 3-colorable. In 1973, Grünbaum [5] observed that outerplanar graphs are acyclically 3-colorable. The following theorem was proved by Fertin, Raspaud and Reed [4] and separately by Albertson *et al.* [1].

**Theorem 2.5.** *If $G$ is outerplanar, then $\chi_s(G) \leq 6$ and this is best possible.*

It seems natural to ask if the bound in Theorem 2.5 can be tightened if the family under consideration is restricted to a subset of outerplanar graphs. In the next section, we prove that indeed outerplanar bipartite graphs are 5-star colorable. In Section 4 we give a construction to show that this is best possible.

## 3. 5-Star Coloring Outerplanar Bipartite Graphs

For the purpose of this paper a *weak quadrangulation* is a 2-connected outerplanar bipartite graph with an embedding in the plane such that every face except, possibly, the outer face has length four. Detailed proofs of the following basic facts can be found in [8]. Every bipartite outerplanar graph is the subgraph of a weak quadrangulation. Every weak quadrangulation has two adjacent vertices of degree 2 whose removal yields a weak quadrangulation (consider a face that is a leaf in the tree obtained from the dual of the weak quadrangulation by ignoring the outer face). Thus it follows by induction that every weak quadrangulation has an even number of vertices.

To prove that the vertices of any outerplanar bipartite graph can be 5-star colored it thus suffices to prove the following theorem.

**Theorem 3.1.** *Every weak quadrangulation $G$ has an orientation $D$ of maximum out-degree at most $2$ such that $D$ can be in-colored with $5$ colors.*

***Proof.*** We will prove this result by induction on the number of vertices of $G$, $n$. For the base case $n = 4$, $G$ is a 4-cycle which can be arbitrarily oriented to obtain $D$ since we can chose a different color for each vertex.

Since $n$ must be even, we may now assume that $n \geq 6$, and let $u, v$ be adjacent vertices of degree 2 in $G$. Since $u, v$ must be on the same faces, it follows that there is a facial 4-cycle $u, v, x, y, u$. Since $G' = G - \{u, v\}$ is a weak quadrangulation on $n - 2$ vertices, it has an orientation $D'$ of maximum out-degree 2, and we can assume without loss of generality that the edge $xy$ is oriented so that its tail is at $x$. $D'$ can be extended to the desired orientation $D$ of $G$ by orienting $vx$ and $uv$ away from $v$ and $uy$ away from $u$. Observe that this does not change the outdegree of any vertex in $D'$ and that $u, v$ have outdegrees 1 and 2 respectively.

It remains to see that $D$ can be in-colored with 5 colors. By the induction hypothesis we may assume that $D'$ is in-colored with 5 colors. Next we color $u$, avoiding the colors of $x, y$ and the (at most two) outneighbors of $y$. Finally color $v$, avoiding the colors of $u, x, y$ and the at most 1 outneighbor of $x$ that is different from $y$. It can be easily verified that this yields an in-coloring of $D$. ■

**Corollary 3.2.** *If $G$ is a bipartite outerplanar graph, then $G$ is $5$ star-colorable.*

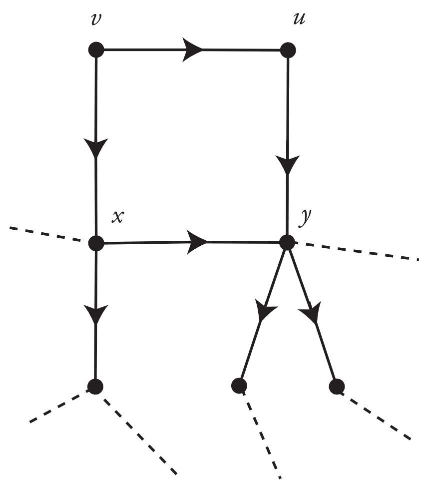


Figure 1. Extending the in-coloring to $u$ and $v$.

***Proof.*** By the remarks from the first paragraph of this section we know that $G$ is the subgraph of a weak quadrangulation $G'$. By Theorem 3.1 $G'$ can be in-colored using only the colors $1, 2, 3, 4, 5$. By Lemma 2.3 this is a star-coloring with 5 colors for $G'$, and thus for $G$. ∎

**Remark 3.3.** The proofs above do not change if we color each vertex from a list of size 5, rather than using some 5 fixed colors. Thus without additional effort we obtain the more general result that the vertices of every outerplanar bipartite graph can be star-colored from lists of size 5, that is these graphs are 5-star-choosable.

## 4. A Lower Bound Construction

In this section, we give the construction of a bipartite outerplanar graph, similar to the $4 \times n$ grid, that requires 5 colors to star color.

**Definition 4.1.** The *n-rung outerplanar grid* $OG_n$ is the outerplanar bipartite graph shown in Figure 2. It can be constructed in the following manner.

We begin with a *ladder* on $n$ rungs, that is a $2 \times n$ grid, consisting of two $n$ vertex paths $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ called the *rails* and $n$ edges of the form $x_i y_i$, called the *rungs* of the ladder. Extend each edge $x_i x_{i+1}$ to a 4-cycle by adding new vertices $u'_i$ and $u_{i+1}$ and edges $x_i u'_i, u'_i u_{i+1}, x_{i+1} u_{i+1}$. Similarly extend each edge $y_i y_{i+1}$ to a 4-cycle $y_i, y_{i+1}, v_{i+1}, v'_i$ by adding 2 new vertices.

The resulting graph $OG_n$ is bipartite and outerplanar, and we call the $n-1$ faces $x_i x_{i+1} y_{i+1} y_i$ the *ladder faces.* The four faces of length 4 containing $u'_1, u_n, v'_1$,

$v_n$ respectively, are the *corner faces*, and every face of length 4 that is not a corner or ladder face is a *side face*. The *reduced grid* $OG_n^-$ is $OG_n$ with the corner faces removed, i.e. $OG_n - \{u'_1, v'_1, u_2, v_2, u'_{n-1}, v'_{n-1}, u_n, v_n\}$.

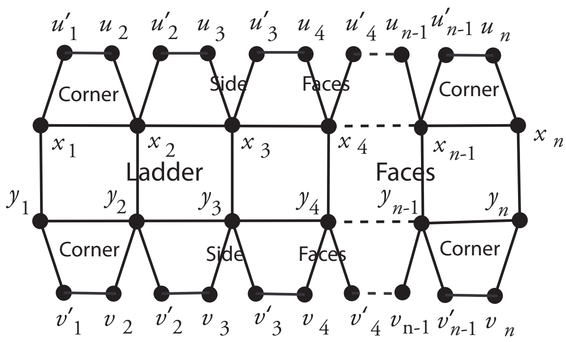


Figure 2. $OG_n$.

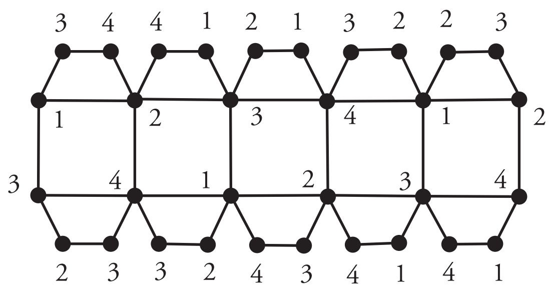


Figure 3. A star coloring of $OG_6$ with 4 colors.

The main aim of this section is to prove that Theorem 3.1 is best possible by establishing the lower bound in the following theorem.

**Theorem 4.2.** $\chi_s(OG_7^-) = 5$.

Figure 3 shows that $OG_n$ for $n \leq 6$ requires at most 4 colors. We will prove Theorem 4.2 by contradiction, assuming that there is a star coloring $c : V(OG_7^-) \to \{1, 2, 3, 4\}$, via a series of lemmas.

**Lemma 4.3.** *A 4-star coloring of $OG_7^-$ has no ladder face in which a color is repeated.*

***Proof.*** Aiming for a contradiction, consider a possible 4-star coloring of $OG_7^-$ in which some ladder face has a repeated color. Such an $OG_7^-$ would contain a 4-colored subgraph isomorphic to $OG_5^-$ in which there is a repeated color on the first ladder face. Without loss of generality we may assume that here $c(x_1) = c(y_2) = 1$, $c(x_2) = 2$ and $c(y_1) = 3$. In order to avoid a 2-colored $P_4$ involving these vertices it follows that $y_3$ does not have color 2 or 3, and in order to avoid adjacent vertices of color 1 it follows that $y_3$ does not have color 1, and thus $c(y_3) = 4$. Similarly $c(v_2') = 4$. Since $x_3$ has neighbors of color 2 and 4, and to avoid a 2-colored $P_4$ $v_2'y_2y_3x_3$ we conclude $c(x_3) = 3$. Similar reasoning now implies that $c(v_3) = 2$ and thus $c(v_3') = 3 = c(y_4)$. Next it follows that $c(x_4) = 1$ and then $c(v_4) = 2$. The coloring up to this point is shown in Figure 4. Now none of the 4 available colors can be used on $y_5$ without creating a 2-colored $P_4$ or adjacent vertices of the same color. ∎

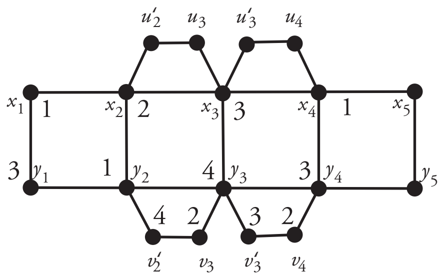


Figure 4. A partial coloring of $OG_5^-$.

**Lemma 4.4.** *A* 4*-star coloring of* $OG_7^-$ *has no side face in which a color is repeated.*

***Proof.*** If a side face has two vertices of the same color we can assume without loss of generality that $OG_7^-$ contains a subgraph isomorphic to $OG_4^-$ with $c(x_3) = c(u_2') = 1$, $c(x_2) = 2$ and $c(u_3) = 3$. To avoid a 2-colored $P_4$ we now need $c(y_3) = 4 = c(x_4)$, contradicting the fact that no ladder face has a repeated color. ∎

**Lemma 4.5.** *There is no* 4*-star coloring of* $OG_7^-$ *in which a rail contains a* 2*-colored* $P_3$*.*

***Proof.*** Aiming for a contradiction, we may suppose that $OG_7^-$ contains a 4-star colored $OG_5^-$ such that $x_1$ and $x_3$ receive the same color. Without loss of generality, we may assume that $c(x_1) = c(x_3) = 1$ and $c(x_2) = 2$. By Lemma 4.3 we conclude that $y_1$ and $y_2$ must use colors 3 and 4, say $y_1$ has color 3 and $y_2$ color 4. Lemma 4.3 now also implies $c(y_3) = 3$, and that $x_4, y_4$ must use colors 2 and 4. To avoid a $P_4$ in colors 1 and 2 it follows that $c(x_4) = 4$ and $c(y_4) = 2$. Figure 5 shows the coloring of this $OG_5$ we have argued to this point.

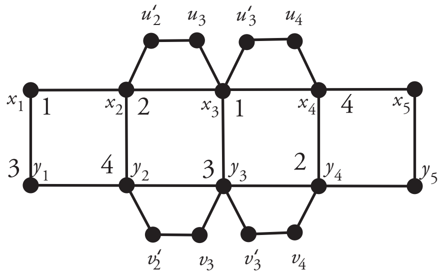


Figure 5. A partial coloring of $OG_5^-$.

As every side face must be 4-colored, $u'_3$ must have color 2 or 3. If $c(u'_3) = 2$, then $x_1, x_2, x_3, u'_3$ would be a 2-colored $P_4$. Thus $c(u'_3) = 3$. Similarly we can argue that $c(v'_3) = 1$, so that $u'_3, x_3, y_3, v'_3$ is a 2-colored $P_4$. ■

The previous three lemmas imply that we can assume without loss of generality that the rails are colored as in Figure 6, and in the next lemma we show that the remaining vertices also have the indicated colors.

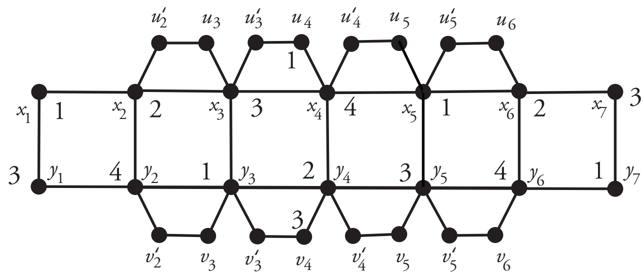


Figure 6. A partial coloring of $OG_7^-$.

**Lemma 4.6.** *In a 4-star coloring of* $OG_7^-$, $c(u_4) \neq c(y_4)$ *and* $c(v_4) \neq c(x_4)$.

***Proof.*** We will use the coloring of the rails shown in Figure 6 as the starting point of our proof by contradiction. By symmetry, it suffices to consider the case $c(u_4) = c(y_4) = 2$. By Lemma 4.4, $u'_3$ now has color 1, and the color of $v'_4$ must be 1 or 4. Since $c(v'_4) = 4$ would imply that $v'_4, y_4, x_4, u_4$ is 2-colored, it follows that $c(v'_4) = 1$. By Lemma 4.4, $v_3$ must be assigned color 2 or color 3. However,

$v_3$ is adjacent to the ends of two 2-colored $P_3$'s using the colors 1,2 and color 1,3. Hence, there is no color available for $v_3$. ∎

Since Lemma 4.4 implies that $u_4$ has color 1 or 2, and that $v_4$ has color 3 or 4, it now follows from Lemma 4.6 that the coloring is as shown in Figure 6. To complete the proof of Theorem 4.2 we observe that $u'_5$ must have color 3 or 4 by Lemma 4.4, but color 4 would cause a 2-colored $P_4$ ending in $u_4$. Thus $c(u'_5) = 3$, and a similar argument shows that $c(v'_5) = 1$, so that $u'_5, x_5, y_5, v'_5$ is 2-colored, the final contradiction.

## 5. Concluding Remarks

Several questions remain regarding star-colorings of outerplanar bipartite graphs.

**Problem 5.1.** *Find a characterization for outerplanar bipartite graphs that are* 4*-star-colorable.*

Apart from potentially providing smaller graphs than those in the previous section, a characterization would shed light on the question if determining if an outerplanar bipartite graph is 4-star-colorable is NP-complete. Since only star-forests are 2-star-colorable, the corresponding 2-colorability dedision problem can be solved in linear time. The following problem seems more tractable.

**Problem 5.2.** *Is there a polynomial time algorithm for deciding if an outerplanar bipartite graph is* 3*-star-colorable?*

### Acknowledgements

The authors thank the referees for helpful suggestions, and André Kündgen for helping to write this paper.

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Received 7 August 2014
Revised 19 December 2017
Accepted 20 December 2017