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TOTAL ROMAN REINFORCEMENT IN GRAPHS

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Abstract

A total Roman dominating function on a graph G is a labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2 and the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex. The minimum weight of a total Roman dominating function on a graph G is called the total Roman domination number of G. The total Roman reinforcement number $r_{tR}(G)$ of a graph G is the minimum number of edges that must be added to G in order to decrease the total Roman domination number. In this paper, we investigate the properties of total Roman reinforcement number in graphs, and we present some sharp bounds for $r_{tR}(G)$. Moreover, we show that the decision problem for total Roman reinforcement is NP-hard for bipartite graphs.

Keywords: total Roman domination number, total Roman reinforcement number.

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1. Introduction

Throughout this paper, G denotes a simple graph without isolated vertex, with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. An S-*external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the S-*external private neighborhood* of v and is denoted by $\operatorname{epn}(v, S)$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. A *leaf* is a vertex of degree 1, and a *support vertex* is a vertex adjacent to a leaf. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

We write K_n for the *complete graph* of order n, P_n for the *path* of order n, C_n for the *cycle* of length n, and $\overline{G}$ for the *complement graph* of G. A tree obtained from a star on at least three vertices by subdividing every edge exactly once is called a *subdivided star*. A tree containing exactly two vertices that are not leaves (which are necessarily adjacent) is called a *double star*. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$. The *corona* of a graph H, denoted $\operatorname{cor}(H)$ or $H \circ K_1$ in the literature, is the graph obtained from H by adding a pendant edge to each vertex of H. The *complete bipartite graph* with partite sets A, B such that $|A| = p$ and $|B| = q$ is denoted by $K_{p,q}$.

A *total dominating set*, abbreviated TD-set, of a graph G without isolated vertex is a set S of vertices such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [12]. A previous survey on total domination in graphs can also be found in [10]. The *total reinforcement number* $r_t(G)$ of a graph G with no isolated vertex is the minimum cardinality of all sets $E' \subseteq E(\overline{G})$ for which $\gamma_t(G + E') < \gamma_t(G)$. In the case that there is no subset of edges E' such that $\gamma_t(G + E') < \gamma_t(G)$, we define $r_t(G) = 0$. The concept of total reinforcement in graphs was introduced by Sridharan *et al.* [19] and has been studied by several authors [11].

A *Roman dominating function* on a graph G, abbreviated RD-function, is a function $f\colon V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G)) = \sum_{v \in V(G)} f(v)$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight among all RD-functions in G.

An RD-function with minimum weight $\gamma_R(G)$ in G is called a $\gamma_R(G)$-*function.* For an RD-function f, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$. Note that $\omega(f) = |V_1| + 2|V_2|$. The concept of Roman dominating function was first defined by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [7] and was motivated by Ian Stewart [20]. Roman domination in graphs is now well studied [8, 14, 15, 17, 18, 21].

A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRD-function, is a Roman dominating function f on G with the additional property that the subgraph of G induced by the set of all vertices of positive weight under f has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRD-function on G. A TRD-function with minimum weight $\gamma_{tR}(G)$ in G is called a $\gamma_{tR}(G)$-function. The concept of total Roman domination in graphs was introduced by Liu and Chang [16] and has been studied in [1, 2, 3, 4, 5].

The *total Roman reinforcement number* $r_{tR}(G)$ of a graph G with no isolated vertex is the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_{tR}(G + E') < \gamma_{tR}(G)$. In the case that there is no such a subset of edges, we define $r_{tR}(G) = 0$. A subset $E' \subseteq E\left(\overline{G}\right)$ is called an $r_{tR}(G)$-set if $|E'| = r_{tR}(G)$ and $\gamma_{tR}(G + E') < \gamma_{tR}(G)$. The following observation is therefore clear and immediate.

Observation 1. *Let G be a graph of order n. If $\Delta(G) = n-1$, then $r_{tR}(G) = 0$.*

Our purpose in this paper is to initiate a study of total Roman reinforcement number in graphs. We first investigate basic properties and bounds for the total Roman reinforcement number of a graph. In the last section, we will show that the decision problem associated to the total Roman reinforcement problem is NP-hard even when restricted to bipartite graphs.

We make use of the following results.

Proposition 2 [2]**.** *If G is a graph with no isolated vertex, then*

$$\gamma_t(G) \leq \gamma_{tR}(G) \leq 2\gamma_t(G).$$

Let $\mathcal{G}$ be the family of graphs that can be obtained from a 4-cycle $(v_1v_2v_3v_4)$ by adding $k_1 + k_2 \geq 1$ vertex-disjoint paths P_2 and joining v_1 to the end of k_1 such paths and joining v_2 to the end of k_2 such paths (possibly, $k_1 = 0$ or $k_2 = 0$). Let $\mathcal{H}$ be the family of graphs that can be obtained from a double star by subdividing each pendant edge once and subdividing the non-pendant edge $r \geq 0$ times.

Proposition 3 [2]**.** *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{tR}(G) = n$ if and only if one of the following holds.*

(1) *G is a path or a cycle.*

(2) *G is a corona,* $\operatorname{cor}(F)$*, of some connected graph F.*

(3) *G is a subdivided star.*

(4) *$G \in \mathcal{G} \cup \mathcal{H}$.*

Proposition 4 [2]**.** *If G is a graph with no isolated vertex, then there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that either V_2 is a dominating set in G, or the set S of vertices not dominated by V_2 satisfies $G[S] = kK_2$ for some $k \geq 1$, where $S \subseteq V_1$ and $N_G(S) \setminus S \subseteq V_0$.*

Proposition 5 [2]**.** *Let G be a connected graph of order at least 3 and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. If x is a leaf and y a support vertex in G, then $x \notin V_2$ and $y \notin V_0$.*

Proposition 6. *For a graph G of order $n \geq 4$, $\gamma_{tR}(G) = 4$ if and only if $G = 2K_2$ or $\Delta(G) = n-2$ or there are two adjacent vertices $u, v \in V(G)$ such that $N[u] \cup N[v] = V(G)$.*

Proof. $\Rightarrow$ Let $f = (V_0, V_1, V_2)$ be a γ_{tR}-function on G with $\gamma_{tR}(G) = 4$. Clearly $|V_2| \leq 2$. If $\Delta(G) = n-1$, then $\gamma_{tR}(G) = 3$ which is a contradiction. Hence $\Delta(G) \leq n-2$. Assume first that $V_2 = \emptyset$. Then $|V_1| = 4$ and thus $n = 4$. Since G is nonempty and $1 \leq \Delta(G) \leq n-2 = 2$, we have $\Delta(G) = n-2 = 2$ or $G = 2K_2$. Now assume that $|V_2| = 1$, say $V_2 = \{v\}$. Then $|V_1| = 2$. Since v is adjacent to all V_0 and at least one vertex of V_1, we deduce that $n-2 \leq d(v) \leq \Delta(G) \leq n-2$ and so we have $\Delta(G) = n-2$. Finally, assume that $|V_2| = 2$, say $V_2 = \{u, v\}$. Clearly $V_1 = \emptyset$, $uv \in E(G)$ and $N[u] \cup N[v] = V(G)$.

$\Leftarrow$ Obviously, if $G = 2K_2$, then $\gamma_{tR}(G) = 4$. Also, it is easy to see that if $\Delta(G) \leq n-2$, then $\gamma_{tR}(G) \geq 4$. If $\Delta(G) = n-2$, then let x be a vertex of maximum degree, y be the non-neighbor of x and z be a common neighbor of x and y. Define $f : V(G) \to \{0, 1, 2\}$ by $f(x) = 2$, $f(y) = f(z) = 1$ and $f(s) = 0$ otherwise. Clearly, f is a TRD-function of G with weight 4 and so $\gamma_{tR}(G) = 4$. Likewise, if there are two adjacent vertices $u, v \in V(G)$ such that $N[u] \cup N[v] = V(G)$, then define a TRD-function on G as follows: $f = (V(G) \setminus \{u, v\}, \emptyset, \{u, v\})$. Since $\omega(f) = 4$, we deduce that $\gamma_{tR}(G) = 4$. ■

2. Graphs G with Small $r_{tR}(G)$

In this section, we study graphs with total Roman reinforcement number at most two.

Lemma 7. *If G is a connected graph of order $n \geq 4$ with $\gamma_{tR}(G) = n$, then $r_{tR}(G) = 1$.*

Proof. Since $\gamma_{tR}(G) = n$, G satisfies one of the four conditions (1)–(4) in the statement of Proposition 3. If G is a cycle, then let $G = (v_1, v_2, \ldots, v_n)$; if G is a path, then let $G = v_1, v_2, \ldots, v_n$; if $G \in \mathcal{H}$ or G is a subdivided star, then let $v_1v_2v_3 \cdots v_k$ be a diametrical path in G, and if $G = \text{cor}(F)$, then let $v_2 \in V(F)$ whose removal from F leaves F connected, $v_3 \in N_F(v_2)$ and v_1 the leaf adjacent to v_2. Then the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(v_3) = 2, f(v_1) = f(v_2) = 0$ and $f(x) = 1$ otherwise, is a TRD-function of $G + \{v_1v_3\}$ of weight $n - 1$ and this implies that $r_{tR}(G) = 1$. ■

Theorem 8. *Let G be a connected graph of order $n \geq 4$. Then $r_{tR}(G) = 1$ if and only if $\gamma_{tR}(G) = n$ or there exists a function $f : V(G) \to \{0, 1, 2\}$ with partition (V_0, V_1, V_2) of weight less than $\gamma_{tR}(G)$ such that one of the following conditions holds.*

(i) *V_2 dominates V_0 and $G[V_1 \cup V_2]$ has at most two isolated vertices.*

(ii) *$G[V_1 \cup V_2]$ has no isolated vertices and exactly one vertex of V_0 is not dominated by V_2.*

Proof. If $\gamma_{tR}(G) = n$, then $r_{tR}(G) = 1$ by Lemma 7. Assume that $\gamma_{tR}(G) \leq n - 1$, and let $f = (V_0, V_1, V_2)$ be a function on G with weight less than $\gamma_{tR}(G)$ satisfying (i) or (ii). Since $\omega(f) \leq n - 2$, we have $V_2 \neq \emptyset$. If f satisfies (ii) and $u \in V_0$ is the vertex not dominated by V_2, then f is a TRD-function of $G + uv$, where $v \in V_2$. Next, assume that f satisfies (i). If $G[V_1 \cup V_2]$ has two isolated vertices u, v, then f is a TRD-function of $G + \{uv\}$ and if $G[V_1 \cup V_2]$ has exactly one isolated vertex, say u, then f is a TRD-function of $G + \{uv\}$, where $v \in V_1 \cup V_2$. This implies that $r_{tR}(G) = 1$.

Conversely, assume that $r_{tR}(G) = 1$, and let $F = \{e = xy\}$ be an $r_{tR}(G)$-set. If $\gamma_{tR}(G) = n$, then we are done. Hence assume that $\gamma_{tR}(G) < n$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G + e)$-function. If $x, y \in V_1 \cup V_2$, then f satisfies item (i) and if $x \in V_0$ or $y \in V_0$, then f satisfies item (ii). This complete the proof. ■

Proposition 9. *Let G be a graph. If there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that $\left(\bigcup_{u \in V_2} epn(u, V_2)\right) \cap V_1 \neq \emptyset$, then $r_{tR}(G) \leq 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function such that $\left(\bigcup_{u \in V_2} epn(u, V_2)\right) \cap V_1 \neq \emptyset$. Let v be a vertex of $V_1 \cap epn(u, V_2)$ for some $u \in V_2$. Thus u is the unique neighbor of v in V_2. Let $I = N(v) \cap V_1$. Suppose first that $I = \emptyset$. If $V_2 = \{u\}$, then either $\Delta(G) = n - 1$ and so $r_{tR}(G) = 0$ (by Observation 1), or $\Delta(G) < n - 1$ and thus the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 0$ and $g(y) = f(y)$ otherwise, is a TRD-function of $G + uz$ of weight $\gamma_{tR}(G) - 1$, where $z \in V_1 \setminus \{v\}$. Therefore $r_{tR}(G) \leq 1$. Hence we can assume that $|V_2| \geq 2$. Then V_2 contains a vertex, say s, such that $us \notin E$, for otherwise reassigning to v a 0 instead of 1 provides a TRD-function with weight less than $\gamma_{tR}(G)$. Hence the function $g : V(G) \to \{0,1,2\}$ defined by $g(v) = 0$ and $g(y) = f(y)$ otherwise, is a TRD-function of $G + us$ of weight $\gamma_{tR}(G) - 1$, and thus $r_{tR}(G) = 1$.

Secondly, assume that $I \neq \emptyset$. Clearly, $G[I \cup \{u\}]$ contains an isolated vertex for otherwise reassigning a 0 to v provides a TRD-function of G with weight less than $\gamma_{tR}(G)$. Let x be an isolated vertex in $G[I \cup \{u\}]$. If $x \neq u$, then the function $g : V(G) \to \{0,1,2\}$ defined by $g(x) = 0$ and $g(y) = f(y)$ otherwise, is a TRD-function of $G + ux$ of weight $\gamma_{tR}(G) - 1$. If $x = u$, then we may assume that u is the unique isolated vertex in $G[I \cup \{u\}]$. Let s be any vertex of I. Again the function $g : V(G) \to \{0,1,2\}$ defined by $g(v) = 0$ and $g(y) = f(y)$ otherwise, is a TRD-function of $G + us$ of weight $\gamma_{tR}(G) - 1$. In any case, we have $r_{tR}(G) = 1$. ∎

Proposition 10. *Let G be a graph. Then $r_{tR}(G) \leq 2$ or there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is a dominating set in G.*

Proof. Suppose there is no $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is dominating set of G. Then by Proposition 4, there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that the set S of vertices not dominated by V_2 satisfies $G[S] = kK_2$ for some $k \geq 1$, where $S \subseteq V_1$ and $N_G(S) \setminus S \subseteq V_0$. If $|V_2| = 0$, then $\gamma_{tR}(G) = n$ and by Lemma 7 we have $r_{tR}(G) = 1$. Hence we assume that $|V_2| \geq 1$, and let $w \in V_2$. If u, v are the vertices of a component in $G[S]$, then the function $g : V(G) \to \{0,1,2\}$ defined by $g(u) = g(v) = 0$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{u, v\}$ is a TRD-function of $G + \{uw, vw\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 2$. ∎

Proposition 11. *If G is a graph having a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| \leq 1$, then $r_{tR}(G) \leq 1$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function with $|V_2| \leq 1$. If $|V_2| = 0$, then $\gamma_{tR}(G) = n$ and thus $r_{tR}(G) = 1$ (by Lemma 7). Hence assume that $|V_2| = 1$, say $V_2 = \{u\}$. If $d(u) = n - 1$, then Observation 1 implies that $r_{tR}(G) = 0$. Thus, suppose that $d(u) < n - 1$. Since there exists a vertex $w \in V_1 \setminus N(u)$, let G_1 be a component of $G[V_1]$ containing a neighbor of u. If $|V(G_1)| = 1$ and $V(G_1) = \{x\}$, then the function $g : V(G) \to \{0,1,2\}$ defined by $g(x) = 0$ and $g(y) = f(y)$ for $y \in V(G) \setminus \{x\}$ is a TRD-function of $G + \{uw\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 1$. Let $|V(G_1)| \geq 2$. As f is a $\gamma_{tR}(G)$-function, u is not adjacent to all vertices in $V(G_1)$. Let $z \in V(G_1)$ be a vertex with maximum distance from u in the induced subgraph $G[V(G_1) \cup \{u\}]$ and define $g : V(G) \to \{0,1,2\}$ by $g(z) = 0$ and $g(y) = f(y)$ for $y \in V(G) \setminus \{z\}$. Clearly, g is a TRD-function of $G + \{uz\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 1$. This completes the proof. ∎

Proposition 12. *If G has a support vertex of degree two, then $r_{tR}(G) \leq 2$.*

Proof. Let v be a support vertex of degree two and u the leaf adjacent to v. By Proposition 10, $r_{tR}(G) \leq 2$ or there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is a dominating set of G. If $r_{tR}(G) \leq 2$, then we are done. Hence we can assume that $r_{tR}(G) \geq 3$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function such that V_2 is a dominating set of G. Clearly, $V_2 \neq \emptyset$. If $|V_2| = 1$, then by Proposition 11, $r_{tR}(G) \leq 1$, a contradiction. Henceforth we can assume that $|V_2| \geq 2$. Since V_2 is a dominating set of G, we conclude from Proposition 5 that $v \in V_2$. Since f is a total Roman dominating function, we may assume that $u \in epn(v, V_2)$. Let $w \in V_2 \setminus \{v\}$ and define $g : V(G) \to \{0, 1, 2\}$ by $g(v) = 1, g(u) = \min\{0, f(u)\}$ and $g(y) = f(y)$ for $y \in V(G) \setminus \{v, u\}$. Clearly, g is a TRD-function of either $G + \{uw\}$ or $G + \{vw\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 1$, a contradiction. ■

Proposition 13. *If G is a connected graph containing a path $v_1v_2v_3v_4v_5$ in which $d(v_i) = 2$ for $i \in \{2, 3, 4\}$, then $r_{tR}(G) \leq 2$.*

Proof. If v_1 or v_5 is a leaf, then the desired result follows by Proposition 12. So we assume that G does not have a support vertex of degree 2. As discussed in Proposition 11, we assume that $r_{tR}(G) \geq 3$ and there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is a dominating set of G. If $|V_2| = 1$, then by Proposition 11, $r_{tR}(G) \leq 1$, a contradiction. Henceforth we can assume $|V_2| \geq 2$. Since V_2 is a dominating set of G, we have $V_2 \cap \{v_2, v_3, v_4\} \neq \emptyset$. If $f(v_2) + f(v_3) + f(v_4) = 2$, then we may assume, without loss of generality, that $f(v_2) = 2$, $f(v_3) = f(v_4) = 0$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v_2) = 1$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{v_2\}$ is a function of G of weight less than $\gamma_{tR}(G)$ satisfying Condition (ii) of Theorem 8, and thus $r_{tR}(G) = 1$, a contradiction. Now let $f(v_2) + f(v_3) + f(v_4) \geq 3$. If $f(v_3) = 2$, then we may assume that $f(v_2) \geq 1$ and $f(v_4) = 0$. Clearly, $v_4 \in epn(v_3, V_2)$ for otherwise we can reduce the weight of f by reassigning a 1 to v_3 instead of 2. The function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v_3) = 1$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{v_3\}$ is function of G with weight less than $\gamma_{tR}(G)$ satisfying Condition (ii) of Theorem 8 and thus $r_{tR}(G) = 1$, a contradiction. If $f(v_3) \leq 1$, then we suppose $f(v_2) = 2$ because V_2 must dominate v_3. In this case, using the fact that $epn(v_2, V_2) \neq \emptyset$, one can see that the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v_2) = 1$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{v_2\}$ is a function of G of weight less than $\gamma_{tR}(G)$ satisfying Condition (ii) of Theorem 8 and so $r_{tR}(G) = 1$, a contradiction. This completes the proof. ■

3. PROPERTIES AND BOUNDS

In this section we investigate basic properties of $r_{tR}(G)$ and establish sharp bounds on the total Roman reinforcement number of a graph.

Theorem 14. *Let G be a connected graph of order $n \geq 4$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. Then*

$$r_{tR}(G) \leq \max\{2, \min\{|epn(u, V_2)| : u \in V_2\}\}.$$

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. If $|V_2| \leq 1$, then $r_{tR}(G) \leq 1$ (by Proposition 11). Henceforth, we assume that $|V_2| \geq 2$. If $epn(u, V_2) \cap V_1 \neq \emptyset$ for some $u \in V_2$, then by Proposition 9 we have $r_{tR}(G) \leq 1$ as desired. Thus assume that $epn(u, V_2) \subseteq V_0$ for each $u \in V_2$. Now let $u \in V_2$ with $epn(u, V_2) = \{x_1, \ldots, x_t\}$ and let $v \in V_2 \setminus \{u\}$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(u) = 1$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{u\}$ is a TRD-function of $G + \{vx_1, \ldots, vx_t\}$ of weight $\gamma_{tR}(G) - 1$. Therefore $r_{tR}(G) \leq |epn(u, V_2)|$. The results follows from the fact that u is an arbitrary vertex of $|V_2|$. ■

Since every vertex of V_2 has at most $d(u) - 1$ neighbors in V_0, the following result is immediate from Theorem 14.

Corollary 15. *Let G be a connected graph of order $n \geq 4$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. Then*

$$r_{tR}(G) \leq \max\{2, \min\{d(u) - 1 : u \in V_2\}\}.$$

The next result is an immediate consequence from Corollary 15 and Lemma 7.

Corollary 16. *For any connected graph G, $r_{tR}(G) \leq \Delta(G) - 1$.*

The bounds of Theorem 14 and Corollary 16 are sharp for the double star $DS_{p,p}$ with $p \geq 2$.

Corollary 17. *For any graph G of order $n \geq 3$, we have $r_{tR}(G) \leq n - 3$.*

Proof. If $\Delta(G) = n - 1$, then by Observation 1 we have $r_{tR}(G) = 0$. Hence, let $\Delta(G) \leq n - 2$. By Corollary 16, $r_{tR}(G) \leq \Delta(G) - 1 \leq n - 3$. ■

Proposition 18. *Let G be a graph with $\Delta(G) \geq 3$. If there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_1| \neq 0$, then $r_{tR}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.*

Proof. If $\gamma_{tR}(G) = 3$, then $r_{tR}(G) = 0 < \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Hence assume that $\gamma_{tR}(G) \geq 4$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function such that $|V_1| \neq 0$. If $|V_2| = 0$, then by Lemma 7 we have $r_{tR}(G) = 1 < \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Thus assume that $|V_2| \geq 1$. For each $x \in V_1$, we define

$$B_x = \{u \in V_1 \cup V_2 : u \text{ is an isolated vertex in } G[V_1 \cup V_2 \setminus \{x\}]\}.$$

Let $v \in V_1$ and $B_v = \{x_1, \ldots, x_t\}$. Clearly, $|B_v| \leq d(v) \leq \Delta(G)$.

Suppose $t = 0$, and let $u \in V_2$. Then $uv \notin E(G)$ for otherwise $(V_0 \cup \{v\}, V_1 \setminus \{v\}, V_2)$ is a TRD-function of G, a contradiction. Thus the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 0$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{v\}$ is a TRD-function of $G + uv$ of weight less than $\omega(f)$, and thus $r_{tR}(G) = 1$.

Now, suppose that $t = 1$. If $x_1 \in V_1$, then for a vertex $u \in V_2$ the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = g(x_1) = 0$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{v, x_1\}$ is a TRD-function of $G + \{uv, ux_1\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 2 \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Let $x_1 \in V_2$. Since $\gamma_{tR}(G) \geq 4$, let $w \in V_1 \cup V_2 \setminus \{v, x_1\}$ and define $g : V(G) \to \{0, 1, 2\}$ by $g(v) = 0$ and $g(x) = f(x)$ for $x \in V(G) \setminus \{x_1\}$. Clearly, g is a TRD-function of $G + \{wx_1\}$ of weight less than $\omega(f)$ yielding $r_{tR}(G) \leq 1 < \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Suppose that $t = 2$. If one of x_1 and x_2 belongs to V_2, say x_1, then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{v\}$ is a TRD-function of $G + x_1x_2$ with weight less than $\omega(f)$ yielding $r_{tR}(G) = 1$. Hence assume that $\{x_1, x_2\} \subset V_1$. For a vertex $u \in V_2$, the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x_1) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{x_1\}$ is a TRD-function of $G + x_1u$ with weight less than $\omega(f)$ yielding $r_{tR}(G) = 1$.

Finally, assume that $t \geq 3$. We claim that v has a neighbor in V_2. Suppose, to contrary, that $N(v) \subseteq V_0 \cup V_1$. In particular, we have $B_v \subseteq V_1$. Since $t \geq 3$, the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 2, g(x) = 0$ for $x \in B_v \setminus \{x_1\}$ and $g(x) = f(x)$ for each $x \in V(G) \setminus (B_v \setminus \{x_1\})$ is a TRD-function of G with weight at most $\gamma_{tR}(G) - 1$, which is a contradiction. Hence $N(v) \cap V_2 \neq \emptyset$. Let $E' = \{x_1x_2, x_3x_4, \ldots, x_{t-1}x_t\}$ if t is even, and $E' = \{x_1x_2, x_1x_3, x_4x_5, \ldots, x_{t-1}x_t\}$ if t is odd. Observe that $|E'| = \lceil \frac{t}{2} \rceil \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Now, the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(v) = 0$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{v\}$ is a TRD-function of $G + E'$ with weight less than $\gamma_{tR}(G)$, and therefore $r_{tR}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. ■

Our next result gives a characterization of connected graphs G with $\Delta(G) \geq 4$ such that $r_{tR}(G) = \Delta(G) - 1$.

Theorem 19. *Let G be a connected graph with $\Delta(G) \geq 4$. Then, $r_{tR}(G) = \Delta(G) - 1$ if and only if for each $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ we have $|V_1| = 0$ and $|epn(u, V_2)| = \Delta(G) - 1$ for every vertex $u \in V_2$.*

Proof. Let $r_{tR}(G) = \Delta(G) - 1$. Since $\Delta(G) \geq 4$, we deduce from Proposition 18 that for each $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ we have $|V_1| = 0$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. Then $|V_2| \geq 2$. By Theorem 14 we have

$$\Delta(G) - 1 = r_{tR}(G) \leq \min\{|epn(u, V_2)| \ : \ u \in V_2\} \leq \Delta(G) - 1$$

and so $|epn(u, V_2)| = \Delta(G) - 1$ for every vertex $u \in V_2$.

Conversely, Let $f = \left(V_0^f, V_1^f, V_2^f\right)$ be a $\gamma_{tR}(G)$-function. Then $|V_1^f| = 0$ and $\left|epn\left(u, V_2^f\right)\right| = \Delta(G) - 1$ for every vertex $u \in V_2^f$. It follows that $V_0^f = \bigcup_{v \in V_2^f} epn\left(v, V_2^f\right)$ and so

$$n - \left|V_2^f\right| = \left|\bigcup_{v \in V_2^f} epn\left(v, V_2^f\right)\right| = (\Delta(G) - 1)\left|V_2^f\right|. \tag{1}$$

Let S be an $r_{tR}(G)$-set and assume that $g = (V_0^g, V_1^g, V_2^g)$ is a $\gamma_{tR}(G+S)$-function. Then

$$\Delta\left|V_2^f\right| = \left|V_0^f\right| + \left|V_2^f\right| = n = |V_0^g| + |V_1^g| + |V_2^g|. \tag{2}$$

Let A be the set of vertices of G that belong to $V_1^g \cup V_2^g$ or having a neighbor in V_2^g, and let $|A| = \ell$. Then $\ell \leq |V_1^g| + \Delta(G)|V_2^g|$. Moreover, $\gamma_{tR}(G+S) \leq \gamma_{tR}(G) - 1$ implies that $|V_1^g| + 2|V_2^g| \leq 2|V_2^f| - 1$, and thus $|V_2^g| \leq |V_2^f| - \frac{1+|V_1^g|}{2}$ if $|V_1^g|$ is odd, and $|V_2^g| \leq |V_2^f| - \frac{2+|V_1^g|}{2}$ if $|V_1^g|$ is even. Therefore,

$$\begin{aligned}\ell &\leq |V_1^g| + \left(\left|V_2^f\right| - \tfrac{2+|V_1^g|}{2}\right)\Delta(G) = n - \Delta(G) - |V_1^g|\tfrac{\Delta(G)-2}{2} \\ &\leq n - \Delta(G),\end{aligned}$$

if $|V_1^g|$ is even, and

$$\begin{aligned}\ell &\leq |V_1^g| + \left(|V_2^f| - \tfrac{(1+|V_1^g|)}{2}\right)\Delta(G) = n - (\Delta(G) - 1) - (|V_1^g| - 1)\tfrac{\Delta(G)-2}{2} \\ &\leq n - (\Delta(G) - 1),\end{aligned}$$

if $|V_1^g|$ is odd. Hence, there are at least $\Delta(G) - 1$ vertices in V_0^g which are not Roman dominated by g in G, that is, those vertices of G that do not belong to A. Since these vertices are Roman dominated by g in $G + S$, we conclude that $r_{tR}(G) = |S| \geq \Delta(G) - 1$ and this completes the proof. ■

In the aim to characterize all trees T with $r_{tR}(T) = \Delta - 1$, we introduce for $\Delta \geq 4$ the family $\mathcal{T}_\Delta$ of trees that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 = DS_{\Delta-1,\Delta-1}$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by adding a double star $DS_{\Delta-1,\Delta-1}$ and joining one of its leaves to a vertex of degree less than Δ of T_i.

Theorem 20. *A tree T of order $n \geq 6$ and maximum degree $\Delta \geq 4$ satisfies $r_{tR}(T) = \Delta - 1$ if and only if $T \in \mathcal{T}_\Delta$.*

Proof. Let $T \in T_\Delta$. Then T can be obtained by a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where $T_1 = DS_{\Delta-1,\Delta-1}$, $T = T_k$, and, if $k \geq 2$, T_{i+1} is obtained recursively from T_i by adding a double star $DS_{\Delta-1,\Delta-1}$ and joining one of its leaves to a vertex of degree less than Δ of T_i. We proceed by induction on the number of operations performed to construct T. If $k = 1$, then it is not difficult to observe that $r_{tR}(T_1) = \Delta - 1$. This establishes the base case. Assume now that $k \geq 2$ and that the result is true for every tree $T' = T_{k-1}$ of the family T_Δ constructed by $k-1$ operations. Let $T = T_k$ be a tree of T_Δ constructed by k operations. Clearly, any total Roman domination function of T_{k-1}, can be extended to a TRD-function of T_k by assigning the weight 2 to a and b, where a and b are the support vertices of the added double star $DS_{\Delta-1,\Delta-1}$. Hence $\gamma_{tR}(T_k) \leq \gamma_{tR}(T_{k-1}) + 4$.

Now let $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(T_k)$-function. We may assume that $f(a) = f(b) = 2$ and every neighbor x of a or b has $f(x) = 0$. Clearly the function f, restricted to T_{k-1} is a total Roman domination function of T_{k-1} and so

$$\gamma_{tR}(T_k) = \omega(f) \geq 4 + \omega\left(f_{T_{k-1}}\right) \geq 4 + \gamma_{tR}(T_{k-1}).$$

Therefore $\gamma_{tR}(T_k) = \gamma_{tR}(T_{k-1}) + 4$. Since $r_{tR}(T_{k-1}) = \Delta - 1$, by Theorem 19 and $\gamma_{tR}(T_k) = \gamma_{tR}(T_{k-1}) + 4$, for each $\gamma_{tR}(T_k)$-function $f = (V_0, V_1, V_2)$ we have $|V_1| = 0$ and $|epn(u, V_2)| = \Delta - 1$ for every vertex $u \in V_2$. Thus $r_{tR}(T_k) = \Delta - 1$.

Now assume that T is a tree with $r_{tR}(T) = \Delta - 1$. We proceed by induction on n. Note that $\operatorname{diam}(T) \geq 2$, since $n \geq 6$. If $\operatorname{diam}(T) = 2$, then T is a star and so $r_{tR}(T) = 0$ which is a contradiction. If $\operatorname{diam}(T) = 3$, then T is a double star $DS_{p,q} : (p \geq q \geq 3)$. Observe that if $p \neq q$, then $r_{tR}(T) < \Delta - 1$. Hence $p = q = \Delta - 1$ and so $T \in \mathcal{T}_\Delta$. Assume that $\operatorname{diam}(T) = 4$ and let $u_1u_2u_3u_4u_5$ be a diametrical path. Note that every support vertex of T has at least two leaves, otherwise there exists a $\gamma_{tR}(T)$-function $f = (V_0, V_1, V_2)$ with $|V_1| \geq 1$ which contradicts Theorem 19. Let f be a $\gamma_{tR}(T)$-function. By Theorem 19, $f(u_2) = f(u_3) = f(u_4) = 2$. If u_3 is not a support vertex, then every neighbor of u_3 is a support vertex and assigned a 2 under f. But in that case $f(u_3) = 1$, a contradiction. Thus u_3 is a support vertex. But then $|epn(u_3, V_2)| < \Delta - 1$, contradicting Theorem 19. Hence $\operatorname{diam}(T) \geq 5$. Let $P : u_1u_2u_3 \cdots u_d$ be a diametrical path. If u_3 is not a support vertex, then u_3 may be assigned a 1, a contradiction. Thus u_3 is a support vertex. We claim that $d(u_4) = 2$ and no support vertex besides u_2 is adjacent to u_3. Indeed, if $d(u_4) \geq 3$ or u_3 is adjacent to a support vertex, then clearly $|epn(u_3, V_2)| < \Delta - 1$, which contradicts Theorem 19. Moreover, since $f(u_2) = f(u_3) = 2$, we must have $d(u_2) = d(u_3) = \Delta$. Also, since $|epn(u_3, V_2)| = \Delta - 1$ we have $u_5 \notin V_2$. Now, let T' be the tree containing u_5 obtained by removing the edge u_4u_5. Clearly the other component containing u_4 is a double star $DS_{\Delta-1,\Delta-1}$. Now if $r_{tR}(T') < \Delta - 1$, then we can easily obtain $r_{tR}(T) < \Delta - 1$, a contradiction. Thus $r_{tR}(T') = \Delta - 1$, and by induction on T', we have $T' \in \mathcal{T}_{\Delta}$. Note that u_5 has degree less than Δ in T'. Since T is obtained from T' by adding a $DS_{\Delta-1,\Delta-1}$ attached from its leaf to a vertex of T' of degree less than Δ, we deduce that $T \in T_{\Delta}$. ■

Theorem 21. *For any graph G with order n, $r_{tR}(G) \leq n - \Delta(G) - 1$.*

Proof. If $\Delta(G) = n - 1$, then by Observation 1, we have $r_{tR}(G) = 0$. Thus assume that $\Delta(G) < n - 1$. Clearly $\gamma_{tR}(G) \geq 4$. Let v be the vertex of degree $\Delta(G)$ and let $X = V(G) \setminus N[v] = \{y_1, \ldots, y_k\}$. Note that $|X| = n - \Delta(G) - 1$. Since $\Delta(G + \{vy_1, \ldots, vy_k\}) = n - 1$, we have $\gamma_{tR}(G + \{vy_1, \ldots, vy_k\}) = 3$ yielding $r_{tR}(G) \leq n - \Delta(G) - 1$. ■

Theorem 22. *Let G be a graph of order $n \geq 5$ such that $r_{tR}(G) > 1$. Then $r_{tR}(G) = n - \Delta(G) - 1$ if and only if $\gamma_{tR}(G) = 4$.*

Proof. Let $\gamma_{tR}(G) = 4$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G)$-function. Then the only way to reduce $\gamma_{tR}(G)$, is to reduce it to 3. This can be accomplished in a minimum way by making the max degree vertex to be adjacent to all the vertices, hence $r_{tR}(G) = n - \Delta(G) - 1$

Conversely, let G be a graph such that $r_{tR}(G) = n - \Delta(G) - 1 > 1$. Suppose to the contrary that, $\gamma_{tR}(G) \neq 4$. If $\gamma_{tR}(G) = 3$, then there exists a vertex of degree $\Delta(G) = n - 1$ and so $r_{tR}(G) = 0$ which is a contradiction. Hence $\gamma_{tR}(G) \geq 5$. By Proposition 10, $r_{tR}(G) \leq 2$ or there exists a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is a dominating set of G. If $r_{tR}(G) \leq 2$, then since $r_{tR}(G) > 1$, we have $r_{tR}(G) = 2$, and thus $\Delta(G) = n - 3$. Let v be a vertex of degree $n - 3$ in G. Then by adding an edge $uv \in \overline{G}$, we obtain $\Delta(G + uv) = n - 2$, and Proposition 6 implies that $\gamma_{tR}(G + uv) = 4$. Hence $r_{tR}(G) = 1$ which is a contradiction. Therefore we conclude that there is a $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$ such that V_2 is a dominating set of G. It follows from Proposition 11 that $|V_2| \geq 2$. Let x be a vertex of degree $\Delta(G)$ and let $X = V(G) \setminus N[x]$. If $f(x) = 2$, then for $y \in V_2 \setminus \{x\}$, add the set F_1 of edges from x to vertices in $epn(y, V_2)$. The function $g : V(G) \to \{0, 1, 2\}$ defined by $g(y) = 1$ and $g(x) = f(x)$ for each $x \in V(G) \setminus \{y\}$, is a TRD-function of $G + F_1$ of weight less than $\omega(f)$. Since $|epn(y, V_2)| < n - \Delta - 1$, we have $r_{tR}(G) < n - \Delta(G) - 1$, a contradiction. Suppose now that $f(x) = 1$. If there exists a vertex $z \in V_2$ such that $epn(z, V_2) \cap V_0 \subseteq N(x)$, then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = 2, g(z) = 1$ and $g(u) = f(u)$ for $u \in V(G) \setminus \{x, z\}$ is a $\gamma_{tR}(G)$-function and so we are in a previous considered case and this leads to the desired result. Assume that for any $z \in V_2$, $epn(z, V_2) \cap V_0 \subsetneq N(x)$. Let $y \in V_2 \cap N(x)$. Note that such a vertex y exists since V_2 dominates G. Let G' be the graph obtained from G by adding edges from x to $epn(y, V_2) \cap V_0 \cap X$ and to vertices in $(V_1 \cup V_2) \cap X$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = 2, g(y) = 0$ and $g(w) = f(w)$ for each $w \in V(G) \setminus \{x, y\}$, is a TRD-function of G' of weight less than $\omega(f)$, implying that $r_{tR}(G) < n - \Delta(G) - 1$, a contradiction. Finally, let $f(x) = 0$ and consider the following two cases.

Case 1. $|V_2| = 2$. Let $V_2 = \{u_1, u_2\}$. If $(N(u_i) \cap V_0) \subseteq N(x)$ for $i = 1, 2$, then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = 2, g(u_1) = g(u_2) = 1$ and $g(w) = f(w)$ for each $w \in V(G) \setminus \{x, u_1, u_2\}$, is a γ_{tR}-function of G and we are in a previous considered case that leads to a contradiction. Suppose, without loss of generality, that $(N(u_2) \cap V_0) \not\subseteq N(x)$. Since $\gamma_{tR}(G) \geq 5$, we have $V_1 \neq \emptyset$ and so we assume $c \in V_1$. Let G' be the graph obtained from G by joining x to all vertices in $epn(u_1, V_2) \cap V_0 \cap X$ and all vertices in $X \cap (V_1 \cup V_2)$ and define $g : V(G) \to \{0, 1, 2\}$ by $g(x) = 2, g(u_1) = g(c) = 0$ and $g(w) = f(w)$ for each $w \in V(G) \setminus \{x, u_1, c\}$. Clearly, g is a TRD-function of G' of weight less than $\omega(f)$ and this implies that $r_{tR}(G) < n - \Delta(G) - 1$, a contradiction.

Case 2. $|V_2| \geq 3$. Let $V_2 = \{u_1, \ldots, u_k\}$. If $(N(u_i) \cap V_0) \subseteq N(x)$ for $i = 1, \ldots, k$, then we get a contradiction as above. Assume, without loss of generality, that $(N(u_k) \cap V_0) \not\subseteq N(x)$. Let G' be the graph obtained from G by joining x to all vertices in $epn(u_i, V_2) \cap V_0 \cap X$ for $i = 1, \ldots, k-1$, and all vertices in $X \cap (V_1 \cup V_2)$ and define $g : V(G) \to \{0, 1, 2\}$ by $g(x) = 2, g(u_1) = 0, g(u_2) = 1$ and $g(w) = f(w)$ for each $w \in V(G) \setminus \{x, u_1, u_2\}$. Clearly, g is a TRD-function of G' of weight less than $\omega(f)$ and this implies that $r_{tR}(G) < n - \Delta(G) - 1$, a contradiction. ∎

The following result due to Cockayne *et al.* [6] will be useful for the next.

Theorem 23 [6]**.** *If G is a connected graph of order $n \geq 3$ and $\Delta(G) \leq n - 2$, then $\gamma_t(G) \leq n - \Delta(G)$.*

Theorem 24. *Let G be a graph of order n. If $r_{tR}(G) \neq 0$, then $r_{tR}(G) \leq n - \Delta(G) - \left\lfloor \frac{\gamma_{tR}(G)}{2} \right\rfloor + 1$.*

Proof. We shall show that $\gamma_{tR}(G) \leq 2n - 2\Delta(G) - 2r_{tR}(G) + 2$. Since $\Delta(G) \leq n - 1 - r_{tR}(G)$, by Theorem 21, we add $r_{tR}(G) - 1$ edges incident with a vertex of maximum degree and call such a set of edges S. Clearly, $\gamma_{tR}(G) = \gamma_{tR}(G + S)$. Since $r_{tR}(G) \neq 0$, we have $\Delta(G) < n - 1$ and so by Theorem 23, $\gamma_t(G) \leq n - \Delta(G)$. Therefore, by Proposition 2 we obtain

$$\begin{aligned}\gamma_{tR}(G) &= \gamma_{tR}(G + S) \leq 2\gamma_t(G + S) \\ &\leq 2n - 2\Delta(G + S) \leq 2n - 2\Delta(G) - 2r_{tR}(G) + 2.\end{aligned}$$

This proves the result. ∎

4. NP-Hardness of Total Roman Reinforcement

Our aim in this section is to show that the decision problem associated with the total Roman reinforcement is NP-hard even when restricted to bipartite graphs.

Total Roman Reinforcement problem (TR-reinforcement)

Instance: A nonempty graph G and a positive integer k.
Question: Is $r_{tR}(G) \leq k$?

We show the NP-hardness of TR-reinforcement problem by transforming the 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [9].

3-SAT problem

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

Theorem 25. *The TR-reinforcement problem is NP-hard for bipartite graphs.*

Proof. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We will construct a graph G and a positive integer k such that $\mathcal{C}$ is satisfiable if and only if $r_{tR}(G) \leq k$.

For each $i \in \{1, 2, \ldots, n\}$, let H_i be the connected graph obtained from a $K_{2,4}$ with partite sets $\{u_i, \overline{u_i}\}$ and $\{a_i, b_i, r_i, s_i\}$ by adding two new vertices attached to vertex s_i. Also let F be the graph obtained from a cycle $C_4 : (v_1v_2v_3v_4)$ by adding three new vertices v', v'' and v''' that we join to v_1. Now corresponding to the variable $u_i \in U$, associate the graph H_i. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Next add the graph F and join v_2 and v_4 to each c_j, and let G be the resulting graph. Clearly, G is a bipartite graph of order $8n + m + 7$ and size $10n + 5m + 7$. Set $k = 1$. Also, for every $\gamma_{tR}(G)$-function $f = (V_0, V_1, V_2)$, we have $f(V(H_i)) \geq 4$ for each $i \in \{1, 2, \ldots, n\}$. In particular, $f(s_i) = 2$ and $f(x) = 2$ for $x \in \{u_i, \overline{u_i}\}$. Moreover, to total Roman dominate vertices of F, we need, without loss of generality, that $f(V(F)) \geq 4$. Therefore $\gamma_{tR}(G) \geq 4n + 4$. The equality is obtained since one can easily construct a TRD-function of G with weight $4n + 4$.

We shall show that $\mathcal{C}$ is satisfiable if and only if $r_{tR}(G) = 1$. Assume that $\mathcal{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then put the vertices u_i and s_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and s_i in D. Hence $|D| = 2n$. Define the function $g : V(G) \to \{0,1,2\}$ by $g(x) = 2$ for every $x \in D, g(v_1) = 2, g(v_2) = 1$ and $g(y) = 0$ for the remaining vertices $y \in V(G)$. It is easy to check that g is a TRD-function of $G + v_3s_1$ of weight $4n + 3 < \gamma_{tR}(G) = 4n + 4$. Therefore $r_{tR}(G) = 1$.

Conversely, assume that $r_{tR}(G) = 1$. Then there is an edge $e \in E\left(\overline{G}\right)$ such that $\gamma_{tR}(G+e) < 4n+4$. Let $h = (V_0, V_1, V_2)$ be a $\gamma_{tR}(G+e)$-function such that every leaf is assigned 0. Clearly such a $\gamma_{tR}(G+e)$-function exists. Then $h(V(H_i)) \geq 4$ for each i. In particular $f(s_i) = 2$ and $f(x) = 2$ for $x \in \{u_i, \overline{u_i}\}$. Also since v_1 is a support vertex with at least two leaves (in case e is incident with the third leaf neighbor of v_1) we have $h(v_1) = 2$. Now since $\gamma_{tR}(G+e) < 4n+4$, we deduce that $h(v_2) + h(v_3) + h(v_4) \leq 1$. Moreover, using the fact that every leaf is assigned a 0 under h, then whatever the added edge e, we must have $h(v_2) + h(v_3) + h(v_4) \geq 1$, and the equality is obtained. Note that if $h(v_3) = 1$, then $e = v_1v_3$, and if $h(v_3) = 0$, then $e = v_3z$, where $z \in \{s_i, u_i, \overline{u_i}\}$ for some i. Therefore $\gamma_{tR}(G+e) = 4n+3$, where $|\{u_i, \overline{u_i}\} \cap V_2| = 1$ for every i. Since every vertex of $\{c_1, c_2, \ldots, c_m\}$ must have a neighbor in V_2, and so it must be dominated by a vertex of $\{u_i, \overline{u_i}\}$ for some $i \in \{1, 2, \ldots, n\}$.

Let $t' : U \to \{T, F\}$ be a mapping defined by $t'(u_i) = T$ if $h(u_i) = 2$ and $t'(u_i) = F$ if $h(\overline{u_i}) = 2$. Assume that $h(u_i) = 2$ and c_j is dominated by u_i. By the construction of G, the literal u_i is in the clause C_j . Then $t'(u_i) = T$, implying that the clause C_j is satisfied by t'. Next assume that $h(\overline{u_i}) = 2$ and c_j is dominated by $\overline{u_i}$. By the construction of G, the literal $\overline{u_i}$ is in the clause C_j. Then $t'(u_i) = F$, and thus t' assigns $\overline{u_i}$ the true value T. Hence t' satisfies the clause C_j. Therefore $\mathcal{C}$ is satisfiable. Since the construction of the total Roman reinforcement instance is straightforward from a 3-SAT instance, the size of the total Roman reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. Consequently, we obtain a polynomial transformation. ■

We conclude this paper with two open problems.

Problem 1. Characterize all graphs G of maximum degree 3 with $r_{tR}(G) = 2$.

Let F_1 be the tree obtained from three copies of $K_{1,2}$ by adding a new vertex and joining it to the centers of $K_{1,2}$. Assume $F_2 = DS_{2,2}$ and $F_3 = K_2$. Let $\mathcal{T}_3$ be the family of trees that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees such that $T_1 = F_1$ or F_2 and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by adding one of the trees F_1, F_2 or F_3 and joining one of its leaves to a vertex of degree less than 3 of T_i.

Problem 2. Prove or disprove: A tree T of order $n \geq 6$ and maximum degree 3 satisfies $r_{tR}(T) = 2$ if and only if $T \in \mathcal{T}_3$.

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