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ON INDEPENDENT DOMINATION IN PLANAR CUBIC GRAPHS

Gholamreza Abrishami [a]

Michael A. Henning [b,1]

and

Freydoon Rahbarnia[a,2]

[a] *Department of Applied Mathematics*
Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad 91775, Iran

[b] *Department of Pure and Applied Mathematics*
University of Johannesburg
Auckland Park, 2006 South Africa

e-mail: gh.abrishamimoghadam@mail.um.ac.ir
mahenning@uj.ac.za
rahbarnia@um.ac.ir

Abstract

A set S of vertices in a graph G is an independent dominating set of G if S is an independent set and every vertex not in S is adjacent to a vertex in S. The independent domination number, $i(G)$, of G is the minimum cardinality of an independent dominating set. Goddard and Henning [Discrete Math. 313 (2013) 839–854] posed the conjecture that if $G \notin \{K_{3,3}, C_5 \,\square\, K_2\}$ is a connected, cubic graph on n vertices, then $i(G) \le \frac{3}{8}n$, where $C_5 \,\square\, K_2$ is the 5-prism. As an application of known result, we observe that this conjecture is true when G is 2-connected and planar, and we provide an infinite family of such graphs that achieve the bound. We conjecture that if G is a bipartite, planar, cubic graph of order n, then $i(G) \le \frac{1}{3}n$, and we provide an infinite family of such graphs that achieve this bound.

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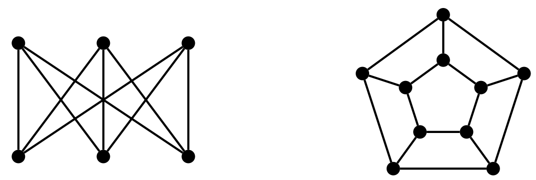
[2]Corresponding author.

1. Introduction

In this note, we continue the study of independent domination in cubic graphs. A set is *independent* in a graph if no two vertices in the set are adjacent. An *independent dominating set*, abbreviated ID-*set*, in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. The *independent domination number* of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set, and an independent dominating set of cardinality $i(G)$ in G is called an $i(G)$-*set*. Independent dominating sets have been studied extensively in the literature (see, for example, [1, 2, 4, 5, 7, 8, 9, 10, 12] and the so-called domination book [6]). A recent survey on independent domination in graphs can be found in [3].

Recall that $K_{3,3}$ denotes the bipartite complete graph with both partite sets on three vertices. The 5-prism, $C_5 \,\square\, K_2$, is the Cartesian product of a 5-cycle with a copy of K_2. The graphs $K_{3,3}$ and $C_5 \,\square\, K_2$ are shown in Figure 1(a) and 1(b), respectively.



(a) $K_{3,3}$ (b) $C_5 \,\square\, K_2$

Figure 1. The graphs $K_{3,3}$ and $C_5 \,\square\, K_2$.

As remarked in [4], the question of best possible bounds on the independent domination number of a connected, cubic graph remains unresolved. Lam, Shiu and Sun [9] established the following upper bound on the independent domination number of a connected, cubic graph. Equality in Theorem 1 holds for the prism $C_5 \,\square\, K_2$ (see Figure 1).

Theorem 1 [9]**.** *For a connected, cubic graph G on n vertices, $i(G) \le \frac{2}{5}n$ except for $K_{3,3}$.*

Goddard and Henning [3] conjectured that the graphs $K_{3,3}$ and $C_5 \,\square\, K_2$ are the only exceptions for an upper bound of $\frac{3}{8}n$. We state their conjecture formally as follows.

Conjecture 2 [3]**.** *If $G \notin \{K_{3,3}, C_5 \,\square\, K_2\}$ is a connected, cubic graph on n vertices, then $i(G) \le \frac{3}{8}n$.*

Dorbec *et al.* [2] proved Conjecture 2 when G does not have a subgraph isomorphic to $K_{2,3}$.

Theorem 3 [2]**.** *If $G \ncong C_5 \,\Box\, K_2$ is a connected, cubic graph on n vertices that does not have a subgraph isomorphic to $K_{2,3}$, then $i(G) \le \frac{3}{8}n$.*

A graph G is k-vertex connected, which we shall simply write as k-connected, if there does not exist a set of $k-1$ vertices whose removal disconnects the graph, i.e., the vertex connectivity of G is at least k. In particular, if a connected graph does not have a cut-vertex, then it is 2-connected. As a simple application of Theorem 3, we observe that Conjecture 2 is true for 2-connected, planar, cubic graphs.

Theorem 4. *If $G \ncong C_5 \,\Box\, K_2$ is a 2-connected, planar, cubic graph on n vertices, then $i(G) \le \frac{3}{8}n$.*

Proof. We show firstly that G has no subgraph isomorphic to $K_{2,3}$. Suppose, to the contrary, that G has a subgraph F, isomorphic to $K_{2,3}$, with partite sets $\{a, f\}$ and $\{b, c, d\}$. Consider an embedding of G in the plane. For every embedding of $K_{2,3}$ in the plane there is a cycle which has a vertex in its interior. Without loss of generality, suppose that c is a vertex in the interior of the cycle C, where $C\colon abfda$. Let x be the neighbor of c different from a and f. Either the vertex x is in the interior of the cycle C or the vertex x belongs to C, in which case $x = b$ or $x = d$. If $x = b$, then the vertex d is a cut-vertex in G, contradicting the 2-connectivity of G. Hence, $x \ne b$. Analogously, $x \ne d$. Therefore, the vertex x is in the interior of C. Renaming vertices, if necessary, we may assume that x is in the interior of cycle $abfca$. Let X be the subgraph of G that lies in the interior of the cycle $abfca$. By assumption, $x \in X$. If the vertex b is adjacent to a vertex of X, then the vertex d is a cut-vertex of G, a contradiction. Therefore, the vertex b is not adjacent to a vertex of X. However, then, the vertex c is a cut-vertex of G, a contradiction. Hence, G has no subgraph isomorphic to $K_{2,3}$. Thus, by Theorem 3, $i(G) \le 3n/8$. ■

We pose the following conjecture.

Conjecture 5. *If $G \ncong C_5 \,\Box\, K_2$ is a connected, planar, cubic graph on n vertices, then $i(G) \le \frac{3}{8}n$.*

The following conjecture was posed by Zhu and Wu [13].

Conjecture 6 [13]**.** *If G is a 2-connected, planar, cubic graph of order n, then $\gamma(G) \le \frac{1}{3}n$.*

We pose the following two conjectures.

Conjecture 7. *If G is a bipartite, planar, cubic graph of order n, then $i(G) \le \frac{1}{3}n$.*

Conjecture 8. *If G is a bipartite, planar, cubic graph of order n, then $\gamma(G) \le \frac{1}{3}n$.*

We remark that every bipartite, cubic graph has no cut-vertex, and therefore each of its components is a 2-connected, cubic (bipartite) graph. Hence, Conjecture 6 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 6. We also remark that Conjecture 7 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 7. A computer search confirms that Conjecture 7 is true when $n \leq 24$.

We have three immediate aims in this paper.

Our first aim is to provide an infinite family, $\mathcal{G}_{\rm cubic}$, of 2-connected, planar, cubic graphs that achieve the upper bound of Theorem 4. The family $\mathcal{G}_{\rm cubic}$ is constructed in Section 2.

Our second aim is to provide an infinite family, $\mathcal{F}_{\rm cubic}$, of connected, planar, cubic graphs that are not 2-connected that achieve the upper bound of Conjecture 5. The family $\mathcal{F}_{\rm cubic}$ is constructed in Section 3.

Our third aim is to provide an infinite family, $\mathcal{H}_{\rm cubic}$, of bipartite, planar, cubic graphs that achieve the upper bound of Conjecture 7 and Conjecture 8. The family $\mathcal{H}_{\rm cubic}$ is constructed in Section 4.

For $k \geq 1$, we use the notation $[k] = \{1, \ldots, k\}$.

2. The Graph Family $\mathcal{G}_{\rm cubic}$

We denote the graph obtained from a 5-prism by deleting an edge that does not belong to a 5-cycle by $(C_5 \,\Box\, K_2)^-$. The graph $(C_5 \,\Box\, K_2)^-$ is illustrated in Figure 2.

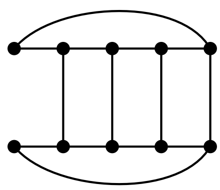


Figure 2. The graph $(C_5 \,\Box\, K_2)^-$.

Let $F \cong (C_5 \,\Box\, K_2)^-$, where $V(F) = \{r_1, r_2, \ldots, r_5, s_1, s_2, \ldots, s_5\}$, where $r_1 r_2 \cdots r_5 r_1$ and $s_1 s_2 \cdots s_5 s_1$ are the two 5-cycles in F and $r_i s_i \in E(F)$ for $i \in \{2, 3, 4, 5\}$. Let $H \cong (C_5 \,\Box\, K_2)^-$, where $V(H) = \{p_1, p_2, \ldots, p_5, q_1, q_2, \ldots, q_5\}$, where $p_1 p_2 \cdots p_5 p_1$ and $q_1 q_2 \cdots q_5 q_1$ are the two 5-cycles in H and $p_i q_i \in E(H)$ for $i \in \{2, 3, 4, 5\}$. An infinite family, $\mathcal{G}_{\rm cubic}$, of 2-connected, planar, cubic graphs can be constructed as follows. For $k \geq 1$, define the graph G_k as described below. Consider two copies of the path P_{4k+2} with respective vertex sequences

$c_0d_0a_1b_1c_1d_1 \cdots a_kb_kc_kd_k$ and $y_0z_0w_1x_1y_1z_1 \cdots w_kx_ky_kz_k$. Join c_0 to z_0, and join d_0 to y_0, and for each $i \in [k]$, join a_i to w_i, b_i to x_i, c_i to z_i, and d_i to y_i. To complete G_k add a disjoint copy of F and H, and join c_0 to r_1, y_0 to s_1, d_k to p_1, and z_k to q_1. We note that the graph G_k has order $8k + 24$. Let $\mathcal{G}_{\text{cubic}} = \{\, G_k : k \geq 1 \,\}$. An embedding of the graph $G_2 \in \mathcal{G}_{\text{cubic}}$ (of order 40) in the plane is illustrated in Figure 3.

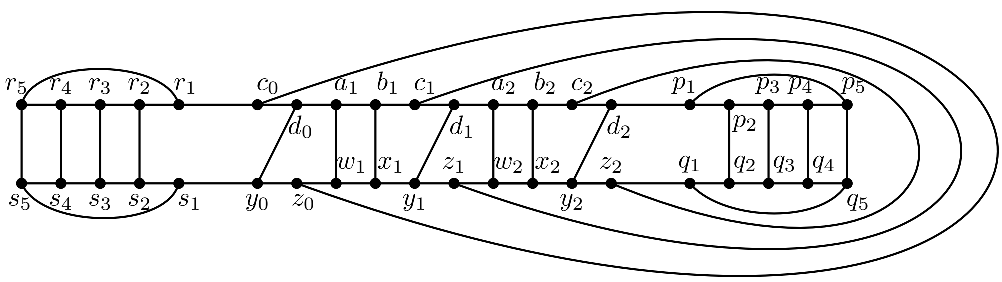


Figure 3. A planar drawing of the graph G_2.

For simplicity, the graph G_2 is redrawn in Figure 4.

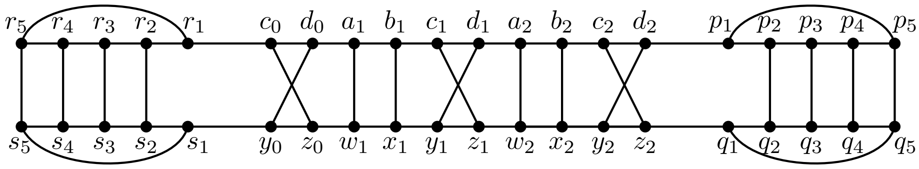


Figure 4. The graph G_2.

We are now in a position to prove the following result.

Proposition 9. *If $G \in \mathcal{G}_{\text{cubic}}$ has order n, then $i(G) = \frac{3}{8}n$.*

Proof. Let $G \in \mathcal{G}_{\text{cubic}}$ have order n. Then, $G = G_k$ for some $k \geq 1$, and so G has order $n = 8k + 24$. We show that $i(G) = 3k + 9$. Let $V_0 = \{c_0, d_0, y_0, z_0\}$, and let $V_i = \{a_i, b_i, c_i, d_i, w_i, x_i, y_i, z_i\}$ for $i \in [k]$. The set

$$\{r_2, r_4, s_1, s_3\} \cup \{p_2, p_4, q_1, q_3\} \cup \{z_0\} \cup \left(\bigcup_{i=1}^{k} \{a_i, c_i, y_i\} \right)$$

is an ID-set of G of cardinality $3k + 9$, implying that $i(G) \leq 3k + 9$. We show next that $i(G) \geq 3k + 9$. We adopt the following notation. If X is a subset of vertices of G, we let $X_F = X \cap V(F)$ and let $X_H = X \cap V(H)$. Further, we let $X_0 = V_0 \cap X$, and for $i \in [k]$, we let $X_i = V_i \cap X$.

Let X be an $i(G)$-set. In order to dominate $\{d_0, z_0\}$, we note that $|X_0| \geq 1$ since at most one of a_1 and w_1 belong to X. In order to dominate $\{b_i, c_i, x_i, y_i\}$, we note that $|X_i| \geq 2$. Let $I_X = \{\, i \in [k] \colon |X_i| = 2 \,\}$. Among all $i(G)$-sets, let X be chosen so that

(1) $|X_F| + |X_H|$ is maximum.

(2) Subject to (1), $|X_0|$ is minimum.

(3) Subject to (2), $|I_X|$ is minimum.

We proceed further with the following series of claims. The statement and proof of our first claim is analogous to the statement and proof of a similar claim in [4]. For completeness, we include the proof of this claim.

Claim A. *If $\{d_i, z_i\} \subseteq X_i$ for some $i \in [k]$, then $|X_i| = 3$ or $|X_i| = 4$. Further, if $|X_i| = 3$, then either a_i or w_i is not dominated by X_i.*

Proof. If $\{a_i, w_i\} \cap X_i \neq \emptyset$, then either $a_i \in X_i$, in which case $x_i \in X_i$ in order to dominate x_i, or $w_i \in X_i$, in which case $b_i \in X_i$ in order to dominate b_i. In both cases, $|X_i| = 4$. On the other hand, if $\{a_i, w_i\} \cap X_i = \emptyset$, then either $b_i \in X_i$, in which case w_i is not dominated by X_i, or $x_i \in X_i$, in which case a_i is not dominated by X_i. □

Claim B. $3 \leq |X_H| \leq 4$. *Further, if $|X_H| = 3$, then neither p_1 nor q_1 belongs to X_H, and exactly one of p_1 and q_1 is not dominated by X_H.*

Proof. Suppose that $\{p_1, q_1\} \subseteq X_H$. In this case, $p_3 \in X_H$ or $q_3 \in X_H$. We may assume, by symmetry, that $p_3 \in X_H$, which forces q_4 to belong to X_H, and so $|X_H| = 4$. Suppose that exactly one of p_1 and q_1 belongs to X_H. We may assume, by symmetry, that $p_1 \in X_H$, and so $q_1 \notin X_H$. In order to dominate q_2, either $q_2 \in X_H$ or $q_3 \in X_H$. If $q_2 \in X_H$, then in order to dominate p_3 and q_5, we note that X_H contains two vertices in addition to p_1 and q_2, and so $|X_H| = 4$. If $q_3 \in X_H$, then $X_H = \{p_1, p_4, q_3, q_5\}$, and once again $|X_H| = 4$. Suppose that neither p_1 nor q_1 belongs to X_H. In this case, either $p_2 \in X_H$ or $q_2 \in X_H$. We may assume, by symmetry, that $p_2 \in X_H$. Now, either $|X_H| = 4$ or $X_H = \{p_2, p_5, q_3\}$ or $X_H = \{p_2, p_5, q_4\}$. In particular, if $|X_H| = 3$, then q_1 is not dominated by X_H. □

By symmetry, the proof of Claim C is analogous to that of Claim B, and is therefore omitted.

Claim C. $3 \leq |X_F| \leq 4$. *Further, if $|X_F| = 3$, then neither r_1 nor s_1 belongs to X_F, and exactly one of r_1 and s_1 is not dominated by X_F.*

Claim D. $|X_F| = |X_H| = 4$.

Proof. Suppose, to the contrary, that $|X_F| \neq 4$ or $|X_H| \neq 4$. By symmetry, we may assume that $|X_H| \neq 4$. Then, by Claim B, $|X_H| = 3$, neither p_1 nor q_1 belongs to X_H, and exactly one of p_1 and q_1 is not dominated by X_H. We may assume, by symmetry, that p_1 is not dominated by X_H. In order to dominate the vertex p_1, we have that $d_k \in X_k$. But then $z_k \in X_k$ in order to dominate z_k, noting that $q_1 \notin X_H$. Thus, $\{d_k, z_k\} \subseteq X_k$. By Claim A, $|X_k| = 3$ or $|X_k| = 4$.

Suppose that $|X_k| = 4$. In this case, either $X_k = \{a_k, d_k, x_k, z_k\}$ or $X_k = \{b_k, d_k, w_k, z_k\}$. We may assume, by symmetry, that $X_k = \{a_k, d_k, x_k, z_k\}$. But then removing the five vertices in $X_H \cup \{d_k, z_k\}$ from X, and replacing them with the five vertices $\{c_k, p_1, p_3, q_1, q_4\}$ produces a new $i(G)$-set X' satisfying $|X'_F| = |X_F|$ and $|X'_H| > |X_H|$, which is contrary to our choice of the set X. Hence, $|X_k| = 3$.

Since $|X_k| = 3$, either $X_k = \{b_k, d_k, z_k\}$ or $X_k = \{x_k, d_k, z_k\}$. We may assume, by symmetry, that $X_k = \{b_k, d_k, z_k\}$. But then removing the five vertices in $X_H \cup \{d_k, z_k\}$ from X, and replacing them with the five vertices $\{y_k, p_1, p_3, q_1, q_4\}$ produces a new $i(G)$-set X' satisfying $|X'_F| = |X_F|$ and $|X'_H| > |X_H|$, which is contrary to our choice of the set X. $\square$

Claim E. $|X_0| = 1$.

Proof. As observed earlier, $|X_0| \geq 1$. Suppose, to the contrary, that $|X_0| \geq 2$. Then, either $X_0 = \{c_0, y_0\}$ or $X_0 = \{d_0, z_0\}$. If $X_0 = \{c_0, y_0\}$, then removing the four vertices in X_F from X, and replacing them with the three vertices $\{r_2, r_5, s_3\}$ produces an ID-set of G of cardinality $|X| - 1$, contradicting the fact that X is an $i(G)$-set. Hence, $X_0 = \{d_0, z_0\}$. This implies that neither a_1 nor w_1 belongs to X, and at most one of b_1 and x_1 belongs to X. By symmetry, we may assume that $b_1 \notin X$. The set $X' = (X \setminus \{d_0\}) \cup \{a_1\}$ produces a new $i(G)$-set satisfying $|X'_F| + |X'_H| = |X_F| + |X_H|$ and $|X'_0| < |X_0|$, which is contrary to our choice of the set X. $\square$

The proof of the following claim uses some of the arguments presented in [4].

Claim F. $I_X = \emptyset$.

Proof. Suppose, to the contrary, that $|I_X| \geq 1$. Let i be the largest integer such that $|X_i| = 2$. In order to dominate $\{b_i, c_i, x_i, y_i\}$, we may assume, by symmetry, that $X_i = \{b_i, y_i\}$ or $X_i = \{b_i, z_i\}$ or $X_i = \{b_i, d_i\}$ or $X_i = \{c_i, y_i\}$. In all four cases, the vertex w_i is not dominated by X_i. If $i = 1$, then this would imply that in order to dominate the vertex w_i, we have that $z_0 \in X_0$. But then $d_0 \in X_0$, and so $X_0 = \{d_0, z_0\}$, contradicting Claim E.

Thus, $i \geq 2$. We now consider the set X_{i-1}. In order to dominate the vertex w_i, we have that $z_{i-1} \in X_{i-1}$. But then $d_{i-1} \in X_{i-1}$ in order to dominate d_{i-1}. Thus, $\{d_{i-1}, z_{i-1}\} \subseteq X_{i-1}$. By Claim A, either $|X_{i-1}| = 3$ or $|X_{i-1}| = 4$.

Suppose that $|X_{i-1}| = 4$. We may assume, by symmetry, that $a_{i-1} \in X_{i-1}$; that is, $X_{i-1} = \{a_{i-1}, d_{i-1}, x_{i-1}, z_{i-1}\}$. But then the set $X' = (X \setminus \{d_{i-1}, x_{i-1}, z_{i-1}\}) \cup \{c_{i-1}, y_{i-1}, w_i\}$ is an $i(G)$-set such that $|X'_F| + |X'_H| = |X_F| + |X_H|$, $|X'_0| = |X_0|$, and $|I_{X'}| < |I_X|$, contradicting our choice of the set X. Hence, $|X_{i-1}| = 3$.

Since $|X_{i-1}| = 3$, either $X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}$ or $X_{i-1} = \{x_{i-1}, d_{i-1}, z_{i-1}\}$. We may assume, by symmetry, that $X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}$. Thus, w_{i-1} is not dominated by X_{i-1}. If $i = 2$, then this would imply that in order to dominate the vertex w_{i-1}, we have that $z_0 \in X_0$. But then $d_0 \in X_0$, and so $X_0 = \{d_0, z_0\}$, contradicting Claim E. Thus, $i \geq 3$. We now consider the set X_{i-2}. In order to dominate the vertex w_{i-1}, we have that $\{d_{i-2}, z_{i-2}\} \subseteq X_{i-2}$.

Continuing this process, there is a smallest positive integer $j < i$ such that $\{d_{i-j}, z_{i-j}\} \subseteq X_{i-j}$ and $|X_{i-j}| = 4$. We may assume, by symmetry, that $a_{i-j} \in X_{i-j}$; that is, $X_{i-j} = \{a_{i-j}, d_{i-j}, x_{i-j}, z_{i-j}\}$. We now define the set X' of vertices of G as follows. For $\ell \in [k]$, let $X'_\ell = V_i \cap X'$ be the set defined as follows. Let $X'_i = X_i \cup \{w_i\}$ and let $X'_{i-j} = \{a_{i-j}, c_{i-j}, y_{i-j}\}$. If $j \geq 2$, then for $i - j + 1 \leq \ell \leq i - 1$, let $X'_\ell = \{a_\ell, c_\ell, y_\ell\}$. If $j \leq i - 1$, then for $0 \leq \ell \leq i - j - 1$, let $X'_\ell = X_\ell$. If $i < k$, then for $i + 1 \leq \ell \leq k$, let $X'_\ell = X_\ell$. Then, $|X'_i| = |X_i| + 1 = 3$, $|X'_{i-j}| = |X_{i-j}| - 1 = 3$, and $|X'_\ell| = |X_\ell|$ for all $\ell \notin \{i, i-j\}$, where $\ell \in [k] \cup \{0\}$. Further, let $X'_F = X_F$ and $X'_H = X_H$. Thus,

$$X' = X'_F \cup X'_H \cup \left(\bigcup_{i=1}^{k} X'_i \right),$$

and $|X'| = |X|$. Since the set X is an ID-set, by construction so too is the set X', implying that the set X' is an $i(G)$-set. However, $|X'_F| = |X_F|$, $|X'_H| = |X_H|$, $|X'_0| = |X_0|$ and $|I_{X'}| < |I_X|$, contradicting our choice of the set X. Consequently, $I_X = \emptyset$. $\square$

By Claim F, $I_X = \emptyset$, implying that $|X_i| \geq 3$ for all $i \in [k]$. Thus, by Claim D and Claim E, we note that $i(G) = |X| \geq 3k + 9$. As observed earlier, $i(G) \leq 3k + 9$. Consequently, $i(G) = 3k + 9 = 3n/8$. This completes the proof of Proposition 9. $\blacksquare$

3. The Graph Family $\mathcal{F}_{\text{cubic}}$

Following the notation introduced in Section 2, we construct an infinite family, $\mathcal{F}_{\text{cubic}}$, of connected, planar, cubic graphs that are not 2-connected as follows. Let G_1^* be the graph obtained from the graph $G_1 \in \mathcal{G}_{\text{cubic}}$ by deleting the vertices in $V(F)$, and adding a new vertex v and adding the edges vc_0 and vy_0. The resulting graph, G_1^*, is illustrated in Figure 5.

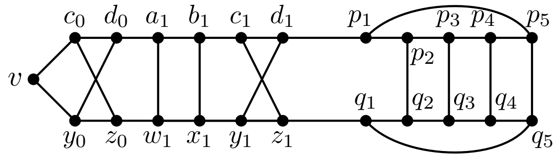


Figure 5. The graph G_1^*.

We note that G_1^* has order 23. An analogous, but simpler, proof than that of Proposition 9 (or simple use a computer) shows that $i(G_1^*) = 9$. The set $\{p_1, p_3, q_1, q_4, c_1, y_1, d_0, z_0, v\}$ is an example of an $i(G_1^*)$-set.

For $k \geq 3$, let $F_1, F_2, \ldots, F_k$ be k vertex-disjoint copies of the graph G_1^*, and let v_i be the vertex of degree 2 in F_i for $i \in [k]$. Let $C\colon u_1u_2 \cdots u_ku_1$ be a k-cycle that has no vertex in common with these k copies of the graph G_1^*. Let F_k^* be the graph obtained from the disjoint union, $F_1 \cup F_2 \cup \cdots \cup F_k \cup C$, of these $k+1$ graphs by adding the k edges u_iv_i for $i \in [k]$. Let $\mathcal{F}_{\text{cubic}} = \{\, F_k^* : k \geq 3 \,\}$. The graph F_4^* (of order 96) in the family $\mathcal{F}_{\text{cubic}}$ is illustrated in Figure 6.

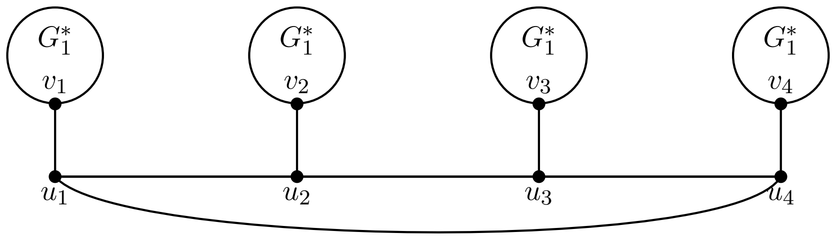


Figure 6. The graph $F_4^* \in \mathcal{F}_{\text{cubic}}$.

For each $k \geq 3$, the graph F_k^* has order $n = 24k$. Further, since $i(G_1^*) = 9$ and there exists an $i(G_1^*)$-set containing the vertex v of degree 2 in G_1^*, we observe that $i(F_k^*) = 9k = 3n/8$. We state this formally as follows.

Proposition 10. *If $G \in \mathcal{F}_{\text{cubic}}$ has order n, then G is a connected, planar, cubic graph satisfying $i(G) = \frac{3}{8}n$.*

4. The Graph Family $\mathcal{H}_{\text{cubic}}$

An infinite family, $\mathcal{H}_{\text{cubic}}$, of bipartite, planar, cubic graphs can be constructed as follows. For $k \geq 2$, define the graph H_k as described below. Consider two copies of the cycle C_{2k} with respective vertex sequences $a_1b_1a_2b_2 \cdots a_kb_ka_1$ and $c_1d_1c_2d_2 \cdots c_kd_kc_1$. To complete H_k, add $2k$ new vertices $e_1, e_2, \ldots, e_k$ and

$f_1, f_2, \ldots, f_k$, and for each $i \in [k]$, join e_i to a_i, c_i and f_i, and join f_i to b_i and d_i. We note that the graph H_k has order $6k$. Let $\mathcal{H}_{\rm cubic} = \{\, H_k : k \geq 2 \,\}$. The graph H_5 (of order 30) in the family $\mathcal{H}_{\rm cubic}$ is illustrated in Figure 7.

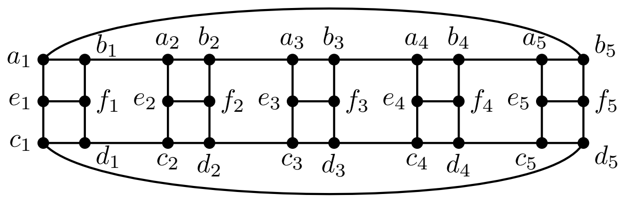


Figure 7. The bipartite, planar, cubic graph H_5.

Let S be a set of vertices in a graph G and let $v \in S$. The *open neighborhood* of v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. The *S-private neighborhood* of v is defined by $\mathrm{pn}[v, S] = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$. A classical result of Ore [11] states that if S is dominating set in a graph G, then S is a minimal dominating set of G if and only if for each $v \in S$, $\mathrm{pn}[v, S] \neq \emptyset$.

We are now in a position to prove the following result.

Proposition 11. *If $G \in \mathcal{H}_{\rm cubic}$ has order n, then $\gamma(G) = i(G) = \frac{1}{3}n$.*

Proof. Let $G \in \mathcal{H}_{\rm cubic}$ have order n. Then, $G = H_k$ for some $k \geq 2$, and so G has order $n = 6k$. We show that $\gamma(G) = i(G) = 2k$. Let $X_i = \{a_i, b_i, c_i, d_i, e_i, f_i\}$ for $i \in [k]$. The set $D_k = \bigcup_{i=1}^{k} \{a_i, d_i\}$ is an ID-set of G of cardinality $2k$, implying that $i(G) \leq |D_k| = 2k$. We show next that $\gamma(G) \geq 2k$. Let S be a $\gamma(G)$-set. By the minimality of S, and by construction of the graph G, we note that $1 \leq |S \cap X_i| \leq 4$ for all $i \in [k]$. For $j \in [4]$, let $S_j = \{i \in [k] : |S \cap X_i| = j\}$. Thus, (S_1, S_2, S_3, S_4) is a (weak) partition of the set $[k]$, where some of the sets may be empty. We note that $|S| = \sum_{i=1}^{4} i|S_i|$ and $k = \sum_{i=1}^{4} |S_i|$.

In what follows, we take addition modulo k. Among all $\gamma(G)$-sets, we choose S so that $|S_4|$ is a minimum. We proceed further with the following two claims.

Claim I. $|S_4| = 0$.

Proof. Suppose, to the contrary, that $|S_4| \geq 1$. Thus, $|S \cap X_i| = 4$ for some $i \in [k]$. By the minimality of the set S, we note that $S \cap X_i = \{a_i, b_i, c_i, d_i\}$. By Ore's Theorem [11] and the structure of the graph G, we note that $\mathrm{pn}[a_i, S] = \{b_{i-1}\}$ and $\mathrm{pn}[c_i, S] = \{d_{i-1}\}$. This implies that $S \cap X_{i-1} = \{e_{i-1}\}$. We now consider the set $S' = (S \setminus \{a_i, c_i\}) \cup \{e_i, f_{i-1}\}$. The resulting set S' is a dominating set of G satisfying $|S'| = |S|$, and is therefore a $\gamma(G)$-set. However, $|S'_4| = |S_4| - 1$, contradicting our choice of the set S. Therefore, $|S_4| = 0$. □

Claim II. $|S_3| \geq |S_1|$.

Proof. Suppose that $i \in S_1$ for some $i \in [k]$, and so $|S \cap X_i| = 1$. In order to dominate e_i and f_i, we note that either $e_i \in S$ or $f_i \in S$. Suppose that $e_i \in S$. In order to dominate b_i, the vertex $a_{i+1} \in S$, while in order to dominate d_i, the vertex $c_{i+1} \in S$. In order to dominate the vertex f_{i+1}, the set S contains a vertex of X_{i+1} different from a_{i+1} and c_{i+1}, implying that $|S \cap X_{i+1}| \geq 3$. By Claim I, $|S \cap X_{i+1}| \leq 3$. Consequently, $|S \cap X_{i+1}| = 3$, and so $i + 1 \in S_3$. Hence, if $e_i \in S$, then $i + 1 \in S_3$, $\{a_{i+1}, c_{i+1}\} \subset S$ and $|S \cap \{b_{i+1}, d_{i+1}\}| \leq 1$. Analogously, if $f_i \in S$, then $i - 1 \in S_3$, $\{b_{i-1}, d_{i-1}\} \subset S$ and $|S \cap \{a_{i-1}, c_{i-1}\}| \leq 1$. This implies that if $i \in S_1$, then either $e_i \in S$, in which case we can uniquely associate $i + 1 \in S_3$ with i, or $f_i \in S$, in which case we can uniquely associate $i - 1 \in S_3$ with i. Therefore, $|S_3| \geq |S_1|$. □

We now return to the proof of Proposition 11. By Claim I, $|S_4| = 0$, and so $|S| = \sum_{i=1}^{3} i|S_i|$ and $k = \sum_{i=1}^{3} |S_i|$.

By Claim II, $|S_3| \geq |S_1|$, and so $k = |S_1| + |S_2| + |S_3| \geq 2|S_1| + |S_2|$, or, equivalently, $k - 2|S_1| - |S_2| \geq 0$. Thus,

$$\begin{aligned} |S| &= |S_1| + 2|S_2| + 3|S_3| = |S_1| + 2|S_2| + 3(k - |S_1| - |S_2|) \\ &= 3k - 2|S_1| - |S_2| = 2k + (k - 2|S_1| - |S_2|) \geq 2k. \end{aligned}$$

Thus, $2k \leq |S| = \gamma(G) \leq i(G) \leq 2k$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(G) = i(G) = 2k = \frac{1}{3}n$. ■

5. Summary of Results

In this paper, we consider five conjectures which we name as Conjectures 2, 5, 6, 7 and 8. We first consider Conjecture 2. We prove in Theorem 4 that Conjecture 2 is true for 2-connected graphs. Our first main result constructs an infinite family, $\mathcal{G}_{\text{cubic}}$, of 2-connected, planar, cubic graphs in Section 2 to show that in this case the bound is tight.

We next consider Conjecture 5. By of our previous result, it suffices to prove Conjecture 5 for connected, planar, cubic graphs that contain cut-vertices. Our second result constructs an infinite family, $\mathcal{F}_{\text{cubic}}$, of connected, planar, cubic graphs that contain cut-vertices in Section 3 to show that if Conjecture 5 is true for graphs with cut-vertices, then the bound is tight.

We finally consider Conjectures 6, 7 and 8. Our third result constructs an infinite family, $\mathcal{H}_{\text{cubic}}$, of bipartite, planar, cubic graphs in Section 4 to show that if Conjectures 7 and 8 are true, then the bounds are tight.

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