

## ON INDEPENDENT DOMINATION IN PLANAR CUBIC GRAPHS

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### Abstract

A set  $S$  of vertices in a graph  $G$  is an independent dominating set of  $G$  if  $S$  is an independent set and every vertex not in  $S$  is adjacent to a vertex in  $S$ . The independent domination number,  $i(G)$ , of  $G$  is the minimum cardinality of an independent dominating set. Goddard and Henning [Discrete Math. 313 (2013) 839–854] posed the conjecture that if  $G \notin \{K_{3,3}, C_5 \square K_2\}$  is a connected, cubic graph on  $n$  vertices, then  $i(G) \leq \frac{3}{8}n$ , where  $C_5 \square K_2$  is the 5-prism. As an application of known result, we observe that this conjecture is true when  $G$  is 2-connected and planar, and we provide an infinite family of such graphs that achieve the bound. We conjecture that if  $G$  is a bipartite, planar, cubic graph of order  $n$ , then  $i(G) \leq \frac{1}{3}n$ , and we provide an infinite family of such graphs that achieve this bound.

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## 1. INTRODUCTION

In this note, we continue the study of independent domination in cubic graphs. A set is *independent* in a graph if no two vertices in the set are adjacent. An *independent dominating set*, abbreviated *ID-set*, in a graph is a set that is both dominating and independent. Equivalently, an independent dominating set is a maximal independent set. The *independent domination number* of a graph  $G$ , denoted by  $i(G)$ , is the minimum cardinality of an independent dominating set, and an independent dominating set of cardinality  $i(G)$  in  $G$  is called an  $i(G)$ -set. Independent dominating sets have been studied extensively in the literature (see, for example, [1, 2, 4, 5, 7, 8, 9, 10, 12] and the so-called domination book [6]). A recent survey on independent domination in graphs can be found in [3].

Recall that  $K_{3,3}$  denotes the bipartite complete graph with both partite sets on three vertices. The 5-prism,  $C_5 \square K_2$ , is the Cartesian product of a 5-cycle with a copy of  $K_2$ . The graphs  $K_{3,3}$  and  $C_5 \square K_2$  are shown in Figure 1(a) and 1(b), respectively.

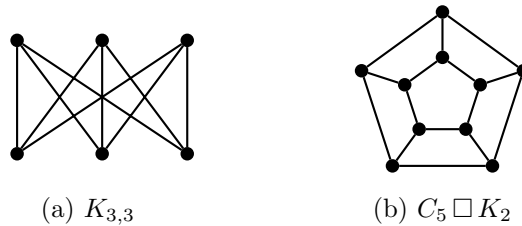


Figure 1. The graphs  $K_{3,3}$  and  $C_5 \square K_2$ .

As remarked in [4], the question of best possible bounds on the independent domination number of a connected, cubic graph remains unresolved. Lam, Shiu and Sun [9] established the following upper bound on the independent domination number of a connected, cubic graph. Equality in Theorem 1 holds for the prism  $C_5 \square K_2$  (see Figure 1).

**Theorem 1** [9]. *For a connected, cubic graph  $G$  on  $n$  vertices,  $i(G) \leq \frac{2}{5}n$  except for  $K_{3,3}$ .*

Goddard and Henning [3] conjectured that the graphs  $K_{3,3}$  and  $C_5 \square K_2$  are the only exceptions for an upper bound of  $\frac{3}{8}n$ . We state their conjecture formally as follows.

**Conjecture 2** [3]. *If  $G \notin \{K_{3,3}, C_5 \square K_2\}$  is a connected, cubic graph on  $n$  vertices, then  $i(G) \leq \frac{3}{8}n$ .*

Dorbec *et al.* [2] proved Conjecture 2 when  $G$  does not have a subgraph isomorphic to  $K_{2,3}$ .

**Theorem 3** [2]. *If  $G \not\cong C_5 \square K_2$  is a connected, cubic graph on  $n$  vertices that does not have a subgraph isomorphic to  $K_{2,3}$ , then  $i(G) \leq \frac{3}{8}n$ .*

A graph  $G$  is  $k$ -vertex connected, which we shall simply write as  $k$ -connected, if there does not exist a set of  $k - 1$  vertices whose removal disconnects the graph, i.e., the vertex connectivity of  $G$  is at least  $k$ . In particular, if a connected graph does not have a cut-vertex, then it is 2-connected. As a simple application of Theorem 3, we observe that Conjecture 2 is true for 2-connected, planar, cubic graphs.

**Theorem 4.** *If  $G \not\cong C_5 \square K_2$  is a 2-connected, planar, cubic graph on  $n$  vertices, then  $i(G) \leq \frac{3}{8}n$ .*

**Proof.** We show firstly that  $G$  has no subgraph isomorphic to  $K_{2,3}$ . Suppose, to the contrary, that  $G$  has a subgraph  $F$ , isomorphic to  $K_{2,3}$ , with partite sets  $\{a, f\}$  and  $\{b, c, d\}$ . Consider an embedding of  $G$  in the plane. For every embedding of  $K_{2,3}$  in the plane there is a cycle which has a vertex in its interior. Without loss of generality, suppose that  $c$  is a vertex in the interior of the cycle  $C$ , where  $C: abfda$ . Let  $x$  be the neighbor of  $c$  different from  $a$  and  $f$ . Either the vertex  $x$  is in the interior of the cycle  $C$  or the vertex  $x$  belongs to  $C$ , in which case  $x = b$  or  $x = d$ . If  $x = b$ , then the vertex  $d$  is a cut-vertex in  $G$ , contradicting the 2-connectivity of  $G$ . Hence,  $x \neq b$ . Analogously,  $x \neq d$ . Therefore, the vertex  $x$  is in the interior of  $C$ . Renaming vertices, if necessary, we may assume that  $x$  is in the interior of cycle  $abfca$ . Let  $X$  be the subgraph of  $G$  that lies in the interior of the cycle  $abfca$ . By assumption,  $x \in X$ . If the vertex  $b$  is adjacent to a vertex of  $X$ , then the vertex  $d$  is a cut-vertex of  $G$ , a contradiction. Therefore, the vertex  $b$  is not adjacent to a vertex of  $X$ . However, then, the vertex  $c$  is a cut-vertex of  $G$ , a contradiction. Hence,  $G$  has no subgraph isomorphic to  $K_{2,3}$ . Thus, by Theorem 3,  $i(G) \leq 3n/8$ . ■

We pose the following conjecture.

**Conjecture 5.** *If  $G \not\cong C_5 \square K_2$  is a connected, planar, cubic graph on  $n$  vertices, then  $i(G) \leq \frac{3}{8}n$ .*

The following conjecture was posed by Zhu and Wu [13].

**Conjecture 6** [13]. *If  $G$  is a 2-connected, planar, cubic graph of order  $n$ , then  $\gamma(G) \leq \frac{1}{3}n$ .*

We pose the following two conjectures.

**Conjecture 7.** *If  $G$  is a bipartite, planar, cubic graph of order  $n$ , then  $i(G) \leq \frac{1}{3}n$ .*

**Conjecture 8.** *If  $G$  is a bipartite, planar, cubic graph of order  $n$ , then  $\gamma(G) \leq \frac{1}{3}n$ .*

We remark that every bipartite, cubic graph has no cut-vertex, and therefore each of its components is a 2-connected, cubic (bipartite) graph. Hence, Conjecture 6 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 6. We also remark that Conjecture 7 implies Conjecture 8, and so Conjecture 8 is a weaker conjecture than Conjecture 7. A computer search confirms that Conjecture 7 is true when  $n \leq 24$ .

We have three immediate aims in this paper.

Our first aim is to provide an infinite family,  $\mathcal{G}_{\text{cubic}}$ , of 2-connected, planar, cubic graphs that achieve the upper bound of Theorem 4. The family  $\mathcal{G}_{\text{cubic}}$  is constructed in Section 2.

Our second aim is to provide an infinite family,  $\mathcal{F}_{\text{cubic}}$ , of connected, planar, cubic graphs that are not 2-connected that achieve the upper bound of Conjecture 5. The family  $\mathcal{F}_{\text{cubic}}$  is constructed in Section 3.

Our third aim is to provide an infinite family,  $\mathcal{H}_{\text{cubic}}$ , of bipartite, planar, cubic graphs that achieve the upper bound of Conjecture 7 and Conjecture 8. The family  $\mathcal{H}_{\text{cubic}}$  is constructed in Section 4.

For  $k \geq 1$ , we use the notation  $[k] = \{1, \dots, k\}$ .

## 2. THE GRAPH FAMILY $\mathcal{G}_{\text{cubic}}$

We denote the graph obtained from a 5-prism by deleting an edge that does not belong to a 5-cycle by  $(C_5 \square K_2)^-$ . The graph  $(C_5 \square K_2)^-$  is illustrated in Figure 2.

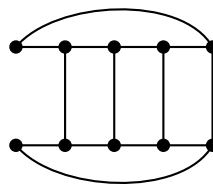


Figure 2. The graph  $(C_5 \square K_2)^-$ .

Let  $F \cong (C_5 \square K_2)^-$ , where  $V(F) = \{r_1, r_2, \dots, r_5, s_1, s_2, \dots, s_5\}$ , where  $r_1 r_2 \cdots r_5 r_1$  and  $s_1 s_2 \cdots s_5 s_1$  are the two 5-cycles in  $F$  and  $r_i s_i \in E(F)$  for  $i \in \{2, 3, 4, 5\}$ . Let  $H \cong (C_5 \square K_2)^-$ , where  $V(H) = \{p_1, p_2, \dots, p_5, q_1, q_2, \dots, q_5\}$ , where  $p_1 p_2 \cdots p_5 p_1$  and  $q_1 q_2 \cdots q_5 q_1$  are the two 5-cycles in  $H$  and  $p_i q_i \in E(H)$  for  $i \in \{2, 3, 4, 5\}$ . An infinite family,  $\mathcal{G}_{\text{cubic}}$ , of 2-connected, planar, cubic graphs can be constructed as follows. For  $k \geq 1$ , define the graph  $G_k$  as described below. Consider two copies of the path  $P_{4k+2}$  with respective vertex sequences

$c_0 d_0 a_1 b_1 c_1 d_1 \cdots a_k b_k c_k d_k$  and  $y_0 z_0 w_1 x_1 y_1 z_1 \cdots w_k x_k y_k z_k$ . Join  $c_0$  to  $z_0$ , and join  $d_0$  to  $y_0$ , and for each  $i \in [k]$ , join  $a_i$  to  $w_i$ ,  $b_i$  to  $x_i$ ,  $c_i$  to  $z_i$ , and  $d_i$  to  $y_i$ . To complete  $G_k$  add a disjoint copy of  $F$  and  $H$ , and join  $c_0$  to  $r_1$ ,  $y_0$  to  $s_1$ ,  $d_k$  to  $p_1$ , and  $z_k$  to  $q_1$ . We note that the graph  $G_k$  has order  $8k + 24$ . Let  $\mathcal{G}_{\text{cubic}} = \{G_k : k \geq 1\}$ . An embedding of the graph  $G_2 \in \mathcal{G}_{\text{cubic}}$  (of order 40) in the plane is illustrated in Figure 3.

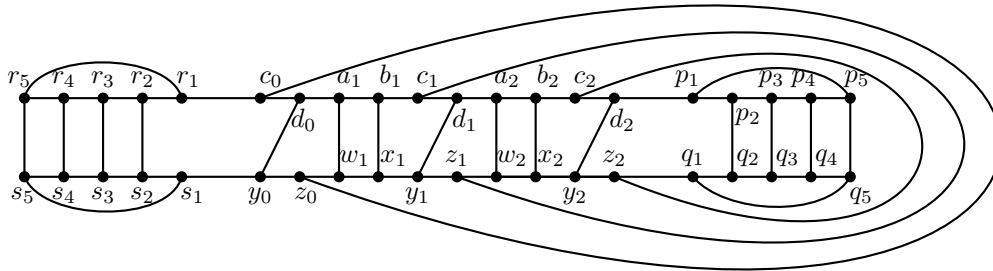


Figure 3. A planar drawing of the graph  $G_2$ .

For simplicity, the graph  $G_2$  is redrawn in Figure 4.

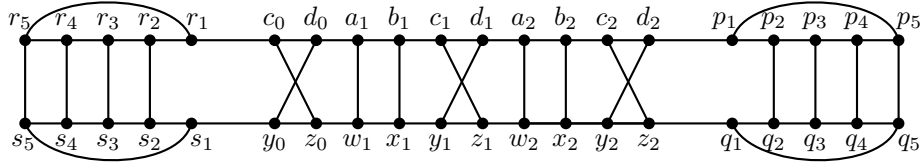


Figure 4. The graph  $G_2$ .

We are now in a position to prove the following result.

**Proposition 9.** *If  $G \in \mathcal{G}_{\text{cubic}}$  has order  $n$ , then  $i(G) = \frac{3}{8}n$ .*

**Proof.** Let  $G \in \mathcal{G}_{\text{cubic}}$  have order  $n$ . Then,  $G = G_k$  for some  $k \geq 1$ , and so  $G$  has order  $n = 8k + 24$ . We show that  $i(G) = 3k + 9$ . Let  $V_0 = \{c_0, d_0, y_0, z_0\}$ , and let  $V_i = \{a_i, b_i, c_i, d_i, w_i, x_i, y_i, z_i\}$  for  $i \in [k]$ . The set

$$\{r_2, r_4, s_1, s_3\} \cup \{p_2, p_4, q_1, q_3\} \cup \{z_0\} \cup \left( \bigcup_{i=1}^k \{a_i, c_i, y_i\} \right)$$

is an ID-set of  $G$  of cardinality  $3k + 9$ , implying that  $i(G) \leq 3k + 9$ . We show next that  $i(G) \geq 3k + 9$ . We adopt the following notation. If  $X$  is a subset of vertices of  $G$ , we let  $X_F = X \cap V(F)$  and let  $X_H = X \cap V(H)$ . Further, we let  $X_0 = V_0 \cap X$ , and for  $i \in [k]$ , we let  $X_i = V_i \cap X$ .

Let  $X$  be an  $i(G)$ -set. In order to dominate  $\{d_0, z_0\}$ , we note that  $|X_0| \geq 1$  since at most one of  $a_1$  and  $w_1$  belong to  $X$ . In order to dominate  $\{b_i, c_i, x_i, y_i\}$ , we note that  $|X_i| \geq 2$ . Let  $I_X = \{i \in [k] : |X_i| = 2\}$ . Among all  $i(G)$ -sets, let  $X$  be chosen so that

- (1)  $|X_F| + |X_H|$  is maximum.
- (2) Subject to (1),  $|X_0|$  is minimum.
- (3) Subject to (2),  $|I_X|$  is minimum.

We proceed further with the following series of claims. The statement and proof of our first claim is analogous to the statement and proof of a similar claim in [4]. For completeness, we include the proof of this claim.

**Claim A.** *If  $\{d_i, z_i\} \subseteq X_i$  for some  $i \in [k]$ , then  $|X_i| = 3$  or  $|X_i| = 4$ . Further, if  $|X_i| = 3$ , then either  $a_i$  or  $w_i$  is not dominated by  $X_i$ .*

**Proof.** If  $\{a_i, w_i\} \cap X_i \neq \emptyset$ , then either  $a_i \in X_i$ , in which case  $x_i \in X_i$  in order to dominate  $x_i$ , or  $w_i \in X_i$ , in which case  $b_i \in X_i$  in order to dominate  $b_i$ . In both cases,  $|X_i| = 4$ . On the other hand, if  $\{a_i, w_i\} \cap X_i = \emptyset$ , then either  $b_i \in X_i$ , in which case  $w_i$  is not dominated by  $X_i$ , or  $x_i \in X_i$ , in which case  $a_i$  is not dominated by  $X_i$ .  $\square$

**Claim B.**  *$3 \leq |X_H| \leq 4$ . Further, if  $|X_H| = 3$ , then neither  $p_1$  nor  $q_1$  belongs to  $X_H$ , and exactly one of  $p_1$  and  $q_1$  is not dominated by  $X_H$ .*

**Proof.** Suppose that  $\{p_1, q_1\} \subseteq X_H$ . In this case,  $p_3 \in X_H$  or  $q_3 \in X_H$ . We may assume, by symmetry, that  $p_3 \in X_H$ , which forces  $q_4$  to belong to  $X_H$ , and so  $|X_H| = 4$ . Suppose that exactly one of  $p_1$  and  $q_1$  belongs to  $X_H$ . We may assume, by symmetry, that  $p_1 \in X_H$ , and so  $q_1 \notin X_H$ . In order to dominate  $q_2$ , either  $q_2 \in X_H$  or  $q_3 \in X_H$ . If  $q_2 \in X_H$ , then in order to dominate  $p_3$  and  $q_5$ , we note that  $X_H$  contains two vertices in addition to  $p_1$  and  $q_2$ , and so  $|X_H| = 4$ . If  $q_3 \in X_H$ , then  $X_H = \{p_1, p_4, q_3, q_5\}$ , and once again  $|X_H| = 4$ . Suppose that neither  $p_1$  nor  $q_1$  belongs to  $X_H$ . In this case, either  $p_2 \in X_H$  or  $q_2 \in X_H$ . We may assume, by symmetry, that  $p_2 \in X_H$ . Now, either  $|X_H| = 4$  or  $X_H = \{p_2, p_5, q_3\}$  or  $X_H = \{p_2, p_5, q_4\}$ . In particular, if  $|X_H| = 3$ , then  $q_1$  is not dominated by  $X_H$ .  $\square$

By symmetry, the proof of Claim C is analogous to that of Claim B, and is therefore omitted.

**Claim C.**  *$3 \leq |X_F| \leq 4$ . Further, if  $|X_F| = 3$ , then neither  $r_1$  nor  $s_1$  belongs to  $X_F$ , and exactly one of  $r_1$  and  $s_1$  is not dominated by  $X_F$ .*

**Claim D.**  $|X_F| = |X_H| = 4$ .

**Proof.** Suppose, to the contrary, that  $|X_F| \neq 4$  or  $|X_H| \neq 4$ . By symmetry, we may assume that  $|X_H| \neq 4$ . Then, by Claim B,  $|X_H| = 3$ , neither  $p_1$  nor  $q_1$  belongs to  $X_H$ , and exactly one of  $p_1$  and  $q_1$  is not dominated by  $X_H$ . We may assume, by symmetry, that  $p_1$  is not dominated by  $X_H$ . In order to dominate the vertex  $p_1$ , we have that  $d_k \in X_k$ . But then  $z_k \in X_k$  in order to dominate  $z_k$ , noting that  $q_1 \notin X_H$ . Thus,  $\{d_k, z_k\} \subseteq X_k$ . By Claim A,  $|X_k| = 3$  or  $|X_k| = 4$ .

Suppose that  $|X_k| = 4$ . In this case, either  $X_k = \{a_k, d_k, x_k, z_k\}$  or  $X_k = \{b_k, d_k, w_k, z_k\}$ . We may assume, by symmetry, that  $X_k = \{a_k, d_k, x_k, z_k\}$ . But then removing the five vertices in  $X_H \cup \{d_k, z_k\}$  from  $X$ , and replacing them with the five vertices  $\{c_k, p_1, p_3, q_1, q_4\}$  produces a new  $i(G)$ -set  $X'$  satisfying  $|X'_F| = |X_F|$  and  $|X'_H| > |X_H|$ , which is contrary to our choice of the set  $X$ . Hence,  $|X_k| = 3$ .

Since  $|X_k| = 3$ , either  $X_k = \{b_k, d_k, z_k\}$  or  $X_k = \{x_k, d_k, z_k\}$ . We may assume, by symmetry, that  $X_k = \{b_k, d_k, z_k\}$ . But then removing the five vertices in  $X_H \cup \{d_k, z_k\}$  from  $X$ , and replacing them with the five vertices  $\{y_k, p_1, p_3, q_1, q_4\}$  produces a new  $i(G)$ -set  $X'$  satisfying  $|X'_F| = |X_F|$  and  $|X'_H| > |X_H|$ , which is contrary to our choice of the set  $X$ .  $\square$

**Claim E.**  $|X_0| = 1$ .

**Proof.** As observed earlier,  $|X_0| \geq 1$ . Suppose, to the contrary, that  $|X_0| \geq 2$ . Then, either  $X_0 = \{c_0, y_0\}$  or  $X_0 = \{d_0, z_0\}$ . If  $X_0 = \{c_0, y_0\}$ , then removing the four vertices in  $X_F$  from  $X$ , and replacing them with the three vertices  $\{r_2, r_5, s_3\}$  produces an ID-set of  $G$  of cardinality  $|X| - 1$ , contradicting the fact that  $X$  is an  $i(G)$ -set. Hence,  $X_0 = \{d_0, z_0\}$ . This implies that neither  $a_1$  nor  $w_1$  belongs to  $X$ , and at most one of  $b_1$  and  $x_1$  belongs to  $X$ . By symmetry, we may assume that  $b_1 \notin X$ . The set  $X' = (X \setminus \{d_0\}) \cup \{a_1\}$  produces a new  $i(G)$ -set satisfying  $|X'_F| + |X'_H| = |X_F| + |X_H|$  and  $|X'_0| < |X_0|$ , which is contrary to our choice of the set  $X$ .  $\square$

The proof of the following claim uses some of the arguments presented in [4].

**Claim F.**  $I_X = \emptyset$ .

**Proof.** Suppose, to the contrary, that  $|I_X| \geq 1$ . Let  $i$  be the largest integer such that  $|X_i| = 2$ . In order to dominate  $\{b_i, c_i, x_i, y_i\}$ , we may assume, by symmetry, that  $X_i = \{b_i, y_i\}$  or  $X_i = \{b_i, z_i\}$  or  $X_i = \{b_i, d_i\}$  or  $X_i = \{c_i, y_i\}$ . In all four cases, the vertex  $w_i$  is not dominated by  $X_i$ . If  $i = 1$ , then this would imply that in order to dominate the vertex  $w_i$ , we have that  $z_0 \in X_0$ . But then  $d_0 \in X_0$ , and so  $X_0 = \{d_0, z_0\}$ , contradicting Claim E.

Thus,  $i \geq 2$ . We now consider the set  $X_{i-1}$ . In order to dominate the vertex  $w_i$ , we have that  $z_{i-1} \in X_{i-1}$ . But then  $d_{i-1} \in X_{i-1}$  in order to dominate  $d_{i-1}$ . Thus,  $\{d_{i-1}, z_{i-1}\} \subseteq X_{i-1}$ . By Claim A, either  $|X_{i-1}| = 3$  or  $|X_{i-1}| = 4$ .

Suppose that  $|X_{i-1}| = 4$ . We may assume, by symmetry, that  $a_{i-1} \in X_{i-1}$ ; that is,  $X_{i-1} = \{a_{i-1}, d_{i-1}, x_{i-1}, z_{i-1}\}$ . But then the set  $X' = (X \setminus \{d_{i-1}, x_{i-1}, z_{i-1}\}) \cup \{c_{i-1}, y_{i-1}, w_i\}$  is an  $i(G)$ -set such that  $|X'_F| + |X'_H| = |X_F| + |X_H|$ ,  $|X'_0| = |X_0|$ , and  $|I_{X'}| < |I_X|$ , contradicting our choice of the set  $X$ . Hence,  $|X_{i-1}| = 3$ .

Since  $|X_{i-1}| = 3$ , either  $X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}$  or  $X_{i-1} = \{x_{i-1}, d_{i-1}, z_{i-1}\}$ . We may assume, by symmetry, that  $X_{i-1} = \{b_{i-1}, d_{i-1}, z_{i-1}\}$ . Thus,  $w_{i-1}$  is not dominated by  $X_{i-1}$ . If  $i = 2$ , then this would imply that in order to dominate the vertex  $w_{i-1}$ , we have that  $z_0 \in X_0$ . But then  $d_0 \in X_0$ , and so  $X_0 = \{d_0, z_0\}$ , contradicting Claim E. Thus,  $i \geq 3$ . We now consider the set  $X_{i-2}$ . In order to dominate the vertex  $w_{i-1}$ , we have that  $\{d_{i-2}, z_{i-2}\} \subseteq X_{i-2}$ .

Continuing this process, there is a smallest positive integer  $j < i$  such that  $\{d_{i-j}, z_{i-j}\} \subseteq X_{i-j}$  and  $|X_{i-j}| = 4$ . We may assume, by symmetry, that  $a_{i-j} \in X_{i-j}$ ; that is,  $X_{i-j} = \{a_{i-j}, d_{i-j}, x_{i-j}, z_{i-j}\}$ . We now define the set  $X'$  of vertices of  $G$  as follows. For  $\ell \in [k]$ , let  $X'_\ell = V_i \cap X'$  be the set defined as follows. Let  $X'_i = X_i \cup \{w_i\}$  and let  $X'_{i-j} = \{a_{i-j}, c_{i-j}, y_{i-j}\}$ . If  $j \geq 2$ , then for  $i - j + 1 \leq \ell \leq i - 1$ , let  $X'_\ell = \{a_\ell, c_\ell, y_\ell\}$ . If  $j \leq i - 1$ , then for  $0 \leq \ell \leq i - j - 1$ , let  $X'_\ell = X_\ell$ . If  $i < k$ , then for  $i + 1 \leq \ell \leq k$ , let  $X'_\ell = X_\ell$ . Then,  $|X'_i| = |X_i| + 1 = 3$ ,  $|X'_{i-j}| = |X_{i-j}| - 1 = 3$ , and  $|X'_\ell| = |X_\ell|$  for all  $\ell \notin \{i, i - j\}$ , where  $\ell \in [k] \cup \{0\}$ . Further, let  $X'_F = X_F$  and  $X'_H = X_H$ . Thus,

$$X' = X'_F \cup X'_H \cup \left( \bigcup_{i=1}^k X'_i \right),$$

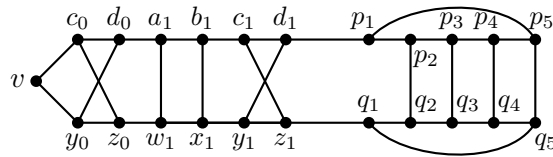
and  $|X'| = |X|$ . Since the set  $X$  is an ID-set, by construction so too is the set  $X'$ , implying that the set  $X'$  is an  $i(G)$ -set. However,  $|X'_F| = |X_F|$ ,  $|X'_H| = |X_H|$ ,  $|X'_0| = |X_0|$  and  $|I_{X'}| < |I_X|$ , contradicting our choice of the set  $X$ . Consequently,  $I_X = \emptyset$ .  $\square$

By Claim F,  $I_X = \emptyset$ , implying that  $|X_i| \geq 3$  for all  $i \in [k]$ . Thus, by Claim D and Claim E, we note that  $i(G) = |X| \geq 3k + 9$ . As observed earlier,  $i(G) \leq 3k + 9$ . Consequently,  $i(G) = 3k + 9 = 3n/8$ . This completes the proof of Proposition 9.  $\blacksquare$

### 3. THE GRAPH FAMILY $\mathcal{F}_{\text{cubic}}$

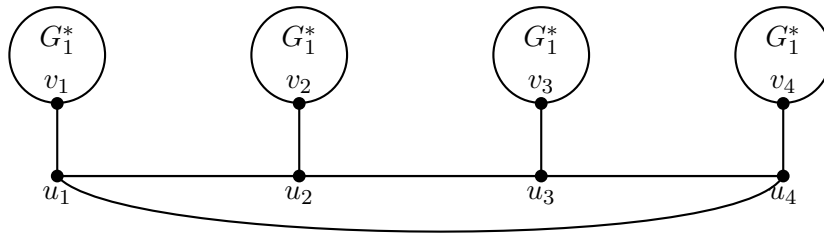
Following the notation introduced in Section 2, we construct an infinite family,  $\mathcal{F}_{\text{cubic}}$ , of connected, planar, cubic graphs that are not 2-connected as follows. Let  $G_1^*$  be the graph obtained from the graph  $G_1 \in \mathcal{G}_{\text{cubic}}$  by deleting the vertices in  $V(F)$ , and adding a new vertex  $v$  and adding the edges  $vc_0$  and  $vy_0$ . The resulting graph,  $G_1^*$ , is illustrated in Figure 5.



Figure 5. The graph  $G_1^*$ .

We note that  $G_1^*$  has order 23. An analogous, but simpler, proof than that of Proposition 9 (or simply use a computer) shows that  $i(G_1^*) = 9$ . The set  $\{p_1, p_3, q_1, q_4, c_1, y_1, d_0, z_0, v\}$  is an example of an  $i(G_1^*)$ -set.

For  $k \geq 3$ , let  $F_1, F_2, \dots, F_k$  be  $k$  vertex-disjoint copies of the graph  $G_1^*$ , and let  $v_i$  be the vertex of degree 2 in  $F_i$  for  $i \in [k]$ . Let  $C: u_1 u_2 \cdots u_k u_1$  be a  $k$ -cycle that has no vertex in common with these  $k$  copies of the graph  $G_1^*$ . Let  $F_k^*$  be the graph obtained from the disjoint union,  $F_1 \cup F_2 \cup \cdots \cup F_k \cup C$ , of these  $k+1$  graphs by adding the  $k$  edges  $u_i v_i$  for  $i \in [k]$ . Let  $\mathcal{F}_{\text{cubic}} = \{F_k^* : k \geq 3\}$ . The graph  $F_4^*$  (of order 96) in the family  $\mathcal{F}_{\text{cubic}}$  is illustrated in Figure 6.

Figure 6. The graph  $F_4^* \in \mathcal{F}_{\text{cubic}}$ .

For each  $k \geq 3$ , the graph  $F_k^*$  has order  $n = 24k$ . Further, since  $i(G_1^*) = 9$  and there exists an  $i(G_1^*)$ -set containing the vertex  $v$  of degree 2 in  $G_1^*$ , we observe that  $i(F_k^*) = 9k = 3n/8$ . We state this formally as follows.

**Proposition 10.** *If  $G \in \mathcal{F}_{\text{cubic}}$  has order  $n$ , then  $G$  is a connected, planar, cubic graph satisfying  $i(G) = \frac{3}{8}n$ .*

#### 4. THE GRAPH FAMILY $\mathcal{H}_{\text{cubic}}$

An infinite family,  $\mathcal{H}_{\text{cubic}}$ , of bipartite, planar, cubic graphs can be constructed as follows. For  $k \geq 2$ , define the graph  $H_k$  as described below. Consider two copies of the cycle  $C_{2k}$  with respective vertex sequences  $a_1 b_1 a_2 b_2 \cdots a_k b_k a_1$  and  $c_1 d_1 c_2 d_2 \cdots c_k d_k c_1$ . To complete  $H_k$ , add  $2k$  new vertices  $e_1, e_2, \dots, e_k$  and

$f_1, f_2, \dots, f_k$ , and for each  $i \in [k]$ , join  $e_i$  to  $a_i, c_i$  and  $f_i$ , and join  $f_i$  to  $b_i$  and  $d_i$ . We note that the graph  $H_k$  has order  $6k$ . Let  $\mathcal{H}_{\text{cubic}} = \{H_k : k \geq 2\}$ . The graph  $H_5$  (of order 30) in the family  $\mathcal{H}_{\text{cubic}}$  is illustrated in Figure 7.

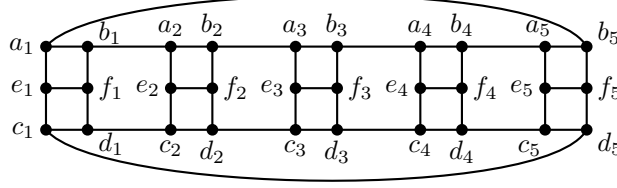


Figure 7. The bipartite, planar, cubic graph  $H_5$ .

Let  $S$  be a set of vertices in a graph  $G$  and let  $v \in S$ . The *open neighborhood* of  $v$  in  $G$  is  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$  and the *closed neighborhood* of  $v$  is  $N_G[v] = \{v\} \cup N_G(v)$ . The  *$S$ -private neighborhood* of  $v$  is defined by  $\text{pn}[v, S] = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$ . A classical result of Ore [11] states that if  $S$  is dominating set in a graph  $G$ , then  $S$  is a minimal dominating set of  $G$  if and only if for each  $v \in S$ ,  $\text{pn}[v, S] \neq \emptyset$ .

We are now in a position to prove the following result.

**Proposition 11.** *If  $G \in \mathcal{H}_{\text{cubic}}$  has order  $n$ , then  $\gamma(G) = i(G) = \frac{1}{3}n$ .*

**Proof.** Let  $G \in \mathcal{H}_{\text{cubic}}$  have order  $n$ . Then,  $G = H_k$  for some  $k \geq 2$ , and so  $G$  has order  $n = 6k$ . We show that  $\gamma(G) = i(G) = 2k$ . Let  $X_i = \{a_i, b_i, c_i, d_i, e_i, f_i\}$  for  $i \in [k]$ . The set  $D_k = \bigcup_{i=1}^k \{a_i, d_i\}$  is an ID-set of  $G$  of cardinality  $2k$ , implying that  $i(G) \leq |D_k| = 2k$ . We show next that  $\gamma(G) \geq 2k$ . Let  $S$  be a  $\gamma(G)$ -set. By the minimality of  $S$ , and by construction of the graph  $G$ , we note that  $1 \leq |S \cap X_i| \leq 4$  for all  $i \in [k]$ . For  $j \in [4]$ , let  $S_j = \{i \in [k] : |S \cap X_i| = j\}$ . Thus,  $(S_1, S_2, S_3, S_4)$  is a (weak) partition of the set  $[k]$ , where some of the sets may be empty. We note that  $|S| = \sum_{i=1}^4 i|S_i|$  and  $k = \sum_{i=1}^4 |S_i|$ .

In what follows, we take addition modulo  $k$ . Among all  $\gamma(G)$ -sets, we choose  $S$  so that  $|S_4|$  is a minimum. We proceed further with the following two claims.

**Claim I.**  $|S_4| = 0$ .

**Proof.** Suppose, to the contrary, that  $|S_4| \geq 1$ . Thus,  $|S \cap X_i| = 4$  for some  $i \in [k]$ . By the minimality of the set  $S$ , we note that  $S \cap X_i = \{a_i, b_i, c_i, d_i\}$ . By Ore's Theorem [11] and the structure of the graph  $G$ , we note that  $\text{pn}[a_i, S] = \{b_{i-1}\}$  and  $\text{pn}[c_i, S] = \{d_{i-1}\}$ . This implies that  $S \cap X_{i-1} = \{e_{i-1}\}$ . We now consider the set  $S' = (S \setminus \{a_i, c_i\}) \cup \{e_i, f_{i-1}\}$ . The resulting set  $S'$  is a dominating set of  $G$  satisfying  $|S'| = |S|$ , and is therefore a  $\gamma(G)$ -set. However,  $|S'_4| = |S_4| - 1$ , contradicting our choice of the set  $S$ . Therefore,  $|S_4| = 0$ .  $\square$

**Claim II.**  $|S_3| \geq |S_1|$ .

**Proof.** Suppose that  $i \in S_1$  for some  $i \in [k]$ , and so  $|S \cap X_i| = 1$ . In order to dominate  $e_i$  and  $f_i$ , we note that either  $e_i \in S$  or  $f_i \in S$ . Suppose that  $e_i \in S$ . In order to dominate  $b_i$ , the vertex  $a_{i+1} \in S$ , while in order to dominate  $d_i$ , the vertex  $c_{i+1} \in S$ . In order to dominate the vertex  $f_{i+1}$ , the set  $S$  contains a vertex of  $X_{i+1}$  different from  $a_{i+1}$  and  $c_{i+1}$ , implying that  $|S \cap X_{i+1}| \geq 3$ . By Claim I,  $|S \cap X_{i+1}| \leq 3$ . Consequently,  $|S \cap X_{i+1}| = 3$ , and so  $i+1 \in S_3$ . Hence, if  $e_i \in S$ , then  $i+1 \in S_3$ ,  $\{a_{i+1}, c_{i+1}\} \subset S$  and  $|S \cap \{b_{i+1}, d_{i+1}\}| \leq 1$ . Analogously, if  $f_i \in S$ , then  $i-1 \in S_3$ ,  $\{b_{i-1}, d_{i-1}\} \subset S$  and  $|S \cap \{a_{i-1}, c_{i-1}\}| \leq 1$ . This implies that if  $i \in S_1$ , then either  $e_i \in S$ , in which case we can uniquely associate  $i+1 \in S_3$  with  $i$ , or  $f_i \in S$ , in which case we can uniquely associate  $i-1 \in S_3$  with  $i$ . Therefore,  $|S_3| \geq |S_1|$ .  $\square$

We now return to the proof of Proposition 11. By Claim I,  $|S_4| = 0$ , and so  $|S| = \sum_{i=1}^3 i|S_i|$  and  $k = \sum_{i=1}^3 |S_i|$ .

By Claim II,  $|S_3| \geq |S_1|$ , and so  $k = |S_1| + |S_2| + |S_3| \geq 2|S_1| + |S_2|$ , or, equivalently,  $k - 2|S_1| - |S_2| \geq 0$ . Thus,

$$\begin{aligned} |S| &= |S_1| + 2|S_2| + 3|S_3| = |S_1| + 2|S_2| + 3(k - |S_1| - |S_2|) \\ &= 3k - 2|S_1| - |S_2| = 2k + (k - 2|S_1| - |S_2|) \geq 2k. \end{aligned}$$

Thus,  $2k \leq |S| = \gamma(G) \leq i(G) \leq 2k$ . Consequently, we must have equality throughout this inequality chain. In particular,  $\gamma(G) = i(G) = 2k = \frac{1}{3}n$ .  $\blacksquare$

## 5. SUMMARY OF RESULTS

In this paper, we consider five conjectures which we name as Conjectures 2, 5, 6, 7 and 8. We first consider Conjecture 2. We prove in Theorem 4 that Conjecture 2 is true for 2-connected graphs. Our first main result constructs an infinite family,  $\mathcal{G}_{\text{cubic}}$ , of 2-connected, planar, cubic graphs in Section 2 to show that in this case the bound is tight.

We next consider Conjecture 5. By of our previous result, it suffices to prove Conjecture 5 for connected, planar, cubic graphs that contain cut-vertices. Our second result constructs an infinite family,  $\mathcal{F}_{\text{cubic}}$ , of connected, planar, cubic graphs that contain cut-vertices in Section 3 to show that if Conjecture 5 is true for graphs with cut-vertices, then the bound is tight.

We finally consider Conjectures 6, 7 and 8. Our third result constructs an infinite family,  $\mathcal{H}_{\text{cubic}}$ , of bipartite, planar, cubic graphs in Section 4 to show that if Conjectures 7 and 8 are true, then the bounds are tight.

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