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# DOMINATION SUBDIVISION AND DOMINATION MULTISUBDIVISION NUMBERS OF GRAPHS

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## Abstract

The *domination subdivision number* $\mathrm{sd}(G)$ of a graph $G$ is the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the domination number of $G$. It has been shown [10] that $\mathrm{sd}(T) \leq 3$ for any tree $T$. We prove that the decision problem of the domination subdivision number is NP-complete even for bipartite graphs. For this reason we define the *domination multisubdivision number* of a nonempty graph $G$ as a minimum positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the domination number of $G$. We show that $\mathrm{msd}(G) \leq 3$ for any graph $G$. The domination subdivision number and the domination multisubdivision number of a graph are incomparable in general, but we show that for trees these two parameters are equal. We also determine the domination multisubdivision number for some classes of graphs.

**Keywords:** domination, domination subdivision number, domination multisubdivision number, trees, computational complexity.

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## 1. Introduction and motivation

For domination problems, multiple edges and loops are irrelevant, so we forbid them. Additionally, in this paper we consider connected graphs only. We use $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph $G$ and denote $|V(G)| = n$, $|E(G)| = m$.

A subset $D$ of $V(G)$ is *dominating* in $G$ if every vertex of $V(G) \setminus D$ has at least one neighbour in $D$. The *domination number* of $G$, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set in $G$. A minimum dominating set of a graph $G$ is called a $\gamma(G)$-*set*.

For a graph $G = (V, E)$, *subdivision* of the edge $e = uv \in E$ with vertex $x$ leads to a graph with vertex set $V \cup \{x\}$ and edge set $(E \setminus \{uv\}) \cup \{ux, xv\}$. Let $G_{e,t}$ denote graph obtained from $G$ by subdivision of the edge $e$ with $t$ vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For $t = 1$ we write $G_e$.

The *domination subdivision number*, $\mathrm{sd}(G)$, of a graph $G$ is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. Since the domination number of the graph $K_2$ does not increase when its only edge is subdivided, we consider the subdivision number for connected graphs of order at least 3. The domination subdivision number was defined by Velammal in 1997 (see [10]) and since then it has been studied widely in graph theory papers. This parameter was studied in trees by Aram, Sheikholeslami and Favaron [1] and also by Benecke and Mynhardt [3]. General bounds and properties have been studied for example by Haynes, Hedetniemi and Hedetniemi [8], by Bhattacharya and Vijayakumar [4], by Favaron, Haynes and Hedetniemi [5] and by Favaron, Karami and Sheikholeslami [6]. In this paper we continue the study of domination subdivision numbers of graphs by proving that the decision problem of domination subdivision number is NP-complete even for bipartite graphs. Additionally, we define $\mathrm{msd}(uv)$ to be the minimum number of subdivisions of the edge $uv$ such that $\gamma(G)$ increases. Moreover, let the *domination multisubdivision number* of a graph $G$, denoted by $\mathrm{msd}(G)$, be defined as

$$\mathrm{msd}(G) = \min\{\mathrm{msd}(uv) : \; uv \in E(G)\}.$$

The domination multisubdivision number is well defined for all graphs having at least one edge.

The concept of domination multisubdivision number has been studied for a total domination. A subset $D$ of $V(G)$ is *total dominating* in $G$ if every vertex of $G$ has at least one neighbour in $D$ and the *total domination number* of $G$, $\gamma_t(G)$, is the cardinality of a smallest total dominating set in $G$. The total domination multisubdivision number was defined by Avella-Alaminos *et al.* [2] as follows: $\mathrm{msd}_{\gamma_t}(uv)$ denotes the minimum number of subdivisions of the edge $uv$ such that $\gamma_t(G)$ increases and the *total domination multisubdivision number* of a graph $G$ of order at least two, denoted by $\mathrm{msd}_{\gamma_t}(G)$, is defined as $\mathrm{msd}_{\gamma_t}(G) = \min\{\mathrm{msd}_{\gamma_t}(uv) : \ uv \in E(G)\}$.

## 2. Notation

The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to $v$. The *degree* of a vertex $v$ is $d_G(v) = |N_G(v)|$.

We say that a vertex $v$ of a graph $G$ is a *leaf* if $v$ has exactly one neighbour in $G$. A vertex $v$ is called a *support vertex* if it is adjacent to a leaf. If $v$ is adjacent to more than one leaf, then we call $v$ a *strong support vertex*.

We call a path $(x, v_1, \ldots, v_l, y)$ connecting two vertices $x$ and $y$ in a graph $G$ an $(x-y)$-*path*. The vertices $v_1, \ldots, v_l$ are its *internal* vertices. The length of a shortest such path is called the *distance* between $x$ and $y$, and is denoted $d_G(x, y)$. The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the maximum distance between two vertices of $G$. For subsets $X$ and $Y$ of $V(G)$, an $(X-Y)$-*path* is a path which starts at a vertex of $X$, ends at a vertex of $Y$, and whose internal vertices belong to neither $X$ nor $Y$. If $X = \{x\}$, then we write $(x-Y)$-path.

The *private neighbourhood of a vertex $u$ with respect to a set $D \subseteq V(G)$*, where $u \in D$, is the set $\mathrm{PN}_G[u, D] = N_G[u] \setminus N_G[D \setminus \{u\}]$. If $v \in \mathrm{PN}_G[u, D]$, then we say that $v$ is a private neighbour of $u$ with respect to the set $D$.

For any unexplained terms and symbols, see [9].

## 3. NP-Completeness of Domination Subdivision Problem

The decision problem of domination subdivision problem is in this paper stated as follows.
DOMINATION SUBDIVISION NUMBER (DSN)
INSTANCE: Graph $G = (V, E)$ and the domination number $\gamma(G)$.
QUESTION: Is $\mathrm{sd}(G) > 1$?

**Theorem 1.** DOMINATION SUBDIVISION NUMBER *is* NP-*hard even for bipartite graphs.*

***Proof.*** The proof is by a transformation from 3-SAT, which was proven to be NP-complete in [7]. The problem 3-SAT is the problem of determining whether there exists an interpretation that satisfies a given Boolean formula. The formula in 3-SAT is given in conjunctive normal form, where each clause contains three literals. We assume that the formula contains the instance of any literal $u$ and its negation $\neg u$ (in the other case all clauses containing the literal $u$ are satisfied by the true assignment of $u$).

Given an instance, the set of literals $U = \{u_1, u_2, \ldots, u_n\}$ and the set of clauses $C = \{c_1, c_2, \ldots, c_m\}$ of 3-SAT, we construct the following graph $G$. For each literal $u_i$ construct a gadget $G_i$ on 6 vertices, where $u_i$ and $\neg u_i$ are the leaves (however $u_i$ and $\neg u_i$ are not necessarily leaves in $G$), see Figure 1.

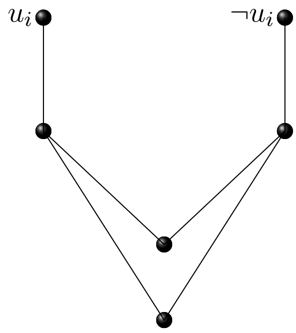


Figure 1. A gadget $G_i$.

For each clause $c_j$ we have a clause vertex $c_j$, where the vertex $c_j$ is adjacent to the literal vertices that correspond to the three literals it contains. For example, if $c_j = (u_1 \vee \neg u_2 \vee u_3)$, then the clause vertex $c_j$ is adjacent to the literal vertices $u_1$, $\neg u_2$ and $u_3$. Then add new vertices $x_0, x_1$ in such a way that $x_1$ is adjacent to every clause vertex $c_j$ and to $x_0$. Hence $x_0$ is of degree one and $x_1$ is of degree $m + 1$. Clearly we can see that $G$ is a bipartite graph and it can be built in polynomial time (see Figure 2).

First observe that at least two vertices from each gadget $G_i$ and either $x_1$ or $x_0$ must be contained in any minimum dominating set of $G$. Thus, $\gamma(G) \geq 2n+1$. On the other hand, it is possible to construct a dominating set of $G$ of cardinality $2n + 1$. Therefore, $\gamma(G) = 2n + 1$.

Denote by $G_1, G_2, \ldots, G_{m(G)}$ the graph obtained from $G$ by subdividing once edge $e_1, e_2, \ldots, e_{m(G)}$, respectively. For a given graph $G$ and its domination number $\gamma(G)$ it is possible to verify a certificate for the DSN problem, which are dominating sets of cardinality $\gamma(G)$ in $G_1, G_2, \ldots, G_{m(G)}$, in polynomial time.

Assume first $C$ has a satisfying truth assignment. If we subdivide any edge belonging to a gadget $G_i$, then we may construct a minimum dominating set of the resulting graph by adding to it two vertices from each gadget $G_i$ and additionally $x_1$. The situation is similar if we subdivide any edge incident with a clause vertex. Now let $x$ be the new vertex obtained by subdividing the edge $x_0x_1$ in $G$ and denote by $G_x$ the resulting graph. Since $C$ has a satisfying truth assignment, a minimum dominating set of $G_x$ is constructed by taking the vertices defined by the truth assignment together with one more vertex from each gadget $G_i$ together with $x$. Therefore we conclude that $\mathrm{sd}(G) > 1$.

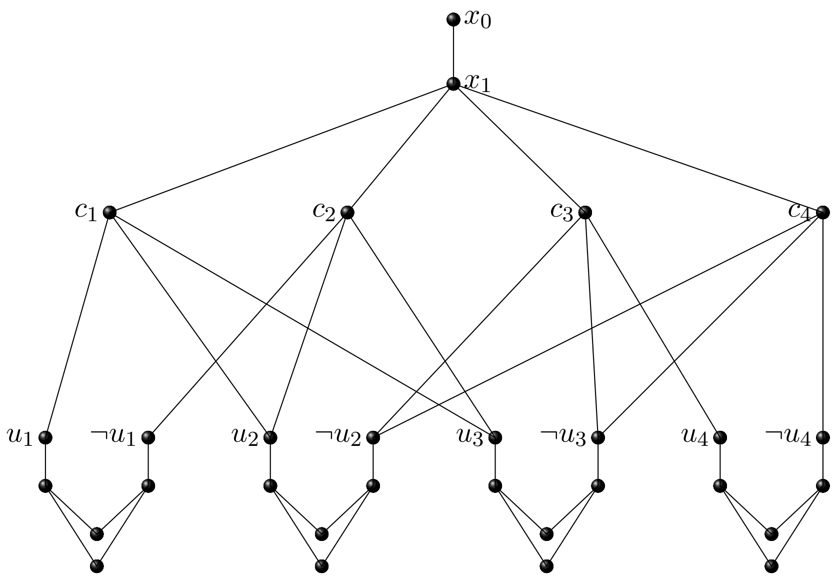


Figure 2. A construction of $G$ for $(u_1 \vee u_2 \vee u_3) \wedge (\neg u_1 \vee u_2 \vee u_3) \wedge (\neg u_2 \vee \neg u_3 \vee u_4) \wedge (\neg u_2 \vee \neg u_3 \vee \neg u_4)$.

Assume now $C$ does not have a satisfying truth assignment. Then subdivide the edge $x_0x_1$ to obtain the graph $G_x$. A minimum dominating set of $G_x$ must contain at least two vertices from each gadget $G_i$ and additionally $x$. However, since $C$ does not have a satisfying truth assignment, no subset of $2n$ vertices of $G_1 \cup G_2 \cup \cdots \cup G_n$ can dominate each gadget vertex and each clause vertex. Therefore, $\mathrm{sd}(G) = 1$. ■

The decision problem of domination multisubdivision problem may be stated similarly.
DOMINATION MULTISUBDIVISION NUMBER (DMN)
INSTANCE: Graph $G = (V, E)$ and the domination number $\gamma(G)$.
QUESTION: Is $\mathrm{msd}(G) > 1$?

**Observation 2.** *Let $G$ be a graph. Then*

$$\mathrm{sd}(G) = 1 \textit{ if and only if } \mathrm{msd}(G) = 1.$$

This observation implies that one may prove the following result in a similar manner as Theorem 1.

**Theorem 3.** DOMINATION MULTISUBDIVISION NUMBER *is* NP*-hard even for bipartite graphs.*

## 4. Results and Bounds for the Domination Multisubdivision Number

Determining the domination multisubdivision number is hard even for bipartite graphs, which is good motivation to study this parameter and give some general bounds and properties. Here we start with some basic properties of multisubdivision numbers.

The next two observations follow from Observation 2 and properties of graphs in which the subdivision number is one.

**Observation 4.** *If a graph $G$ has a strong support vertex, then*

$$\mathrm{sd}(G) = \mathrm{msd}(G) = 1.$$

**Observation 5.** *For a complete graph $K_n$ and a wheel $W_n$, $n \geq 3$, we have*

$$\mathrm{msd}(K_n) = \mathrm{sd}(K_n) = \mathrm{msd}(W_n) = \mathrm{sd}(W_n) = 1.$$

Since any cycle (any path) with an edge subdivided $k$ times is isomorphic to the cycle (the path) with $k$ edges subdivided once, we immediately obtain the following observation.

**Observation 6.** *For a cycle $C_n$ and a path $P_n$, $n \geq 3$, we have*

$$\mathrm{msd}(C_n) = \mathrm{sd}(C_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \pmod 3, \\ 2 & \textit{if } n \equiv 2 \pmod 3, \\ 3 & \textit{if } n \equiv 1 \pmod 3, \end{cases}$$

$$\mathrm{msd}(P_n) = \mathrm{sd}(P_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \pmod 3, \\ 2 & \textit{if } n \equiv 2 \pmod 3, \\ 3 & \textit{if } n \equiv 1 \pmod 3. \end{cases}$$

**Theorem 7.** *For a connected graph $G$,*

$$1 \leq \mathrm{msd}(G) \leq 3.$$

***Proof.*** Let $uv$ be an edge of a graph $G$. Since $\gamma(G_{uv}) \geq \gamma(G)$, we have $\mathrm{msd}(G) \geq 1$. Now, let us subdivide an edge $uv$ with three vertices $x$, $y$ and $z$ (we replace the edge $uv$ with the path $(u, x, y, z, v)$), and let $D$ be a $\gamma(G_{uv,3})$-set. Since $D$ is a minimum dominating set, it is easy to observe that $1 \leq |D \cap \{x, y, z\}| \leq 2$. It is again easy to observe that if $|D \cap \{x, y, z\}| = 2$, then we can exchange one vertex from $D \cap \{x, y, z\}$ with $u$ or $v$ to obtain minimum dominating set of $G_{uv,3}$ such that $|D \cap \{x, y, z\}| = 1$. Thus, if $x \in D$, then $v$ belongs to $D$ to dominate $z$ and $D \setminus \{x\}$ is a $\gamma(G)$-set. Similarly, if $z \in D$, then $u \in D$ and $D \setminus \{z\}$ a $\gamma(G)$-set. If $y \in D$, then obviously $D \setminus \{y\}$ is a $\gamma(G)$-set. In all the cases we can find a smaller dominating set in $G$ then in $G_{uv,3}$, it implies that $\mathrm{msd}(G) \leq 3$. ■

**Proposition 8.** *For a complete bipartite graph $K_{p,q}$, $p \leq q$, we have*

$$\mathrm{msd}(K_{p,q}) = \begin{cases} 1 & \text{if } p = 1 \text{ and } q > 1, \\ 2 & \text{if } p = q = 1, \\ 3 & \text{if } p \geq 2. \end{cases}$$

***Proof.*** The result is obvious for $p = 1$. Thus, we assume that $2 \leq p \leq q$.

Let $uv$ be any edge of $K_{p,q}$. Then $\{u, v\}$ is a minimum dominating set of the graph $K_{p,q}$ and of the graph $K_{p,q}$ with the edge $uv$ subdivided two times. This implies that $\mathrm{msd}(K_{p,q}) > 2$ and therefore, by Theorem 7, $\mathrm{msd}(G) = 3$. ■

Although the multisubdivision number of a graph is bounded from above by 3, it was proven by Favaron, Karami and Sheikholeslami [6] that the subdivision number can be arbitrary large: *For each pair of positive integers $r$ and $q$ such that $r+q \geq 4$, there exists a graph $G$ with $\delta(G) = r$ and* $\mathrm{sd}(G) \geq r+q$. Hence, the difference between $\mathrm{sd}(G)$ and $\mathrm{msd}(G)$ also cannot be bounded from above by any integer in general case. Although the multisubdivision number is always at most three and the subdivision number cannot be bounded from above by any integer, the inequality $\mathrm{msd}(G) \leq \mathrm{sd}(G)$ is not true, since $\mathrm{msd}(K_{p,q}) = 3$ and $\mathrm{sd}(K_{p,q}) = 2$ for $3 \leq p \leq q$. Thus, the subdivision number and the multisubdivision number are incomparable in general. In the next section we show that for trees these two domination parameters are the same.

## 5. Domination Multisubdivision Number of a Tree

Now we consider multisubdivision numbers for trees. The main result of this section is what follows.

**Theorem 9.** *Let $T$ be a tree with $n \geq 3$. Then*

$$\mathrm{sd}(T) = \mathrm{msd}(T).$$

Thus, in trees it does not matter if we subdivide a set of edges, each edge once, or if we multi-subdivide only one edge. In both cases the minimum number of subdivision vertices needed to increase the domination number is the same for a tree.

It has been shown by Velammal [10] that the domination subdivision number of a tree is either 1, 2 or 3. The classes of trees $T$ with $\mathrm{sd}(T) = 1$ and $\mathrm{sd}(T) = 3$ are characterized (see [1, 3]). Thus by Observation 2, in order to prove Theorem 9 it suffices to show that for a tree $T$ with at least 3 vertices we have

$$\mathrm{sd}(T) = 3 \text{ if and only if } \mathrm{msd}(T) = 3.$$

### 5.1. Trees with domination multisubdivision number equal to 3

The following constructive characterization of the family $\mathcal{F}$ of labeled trees $T$ with $\mathrm{sd}(T) = 3$ was given by Aram, Sheikholeslami and Favaron [1]. The label of a vertex $v$ is also called a status of $v$ and is denoted by $\mathrm{sta}(v)$. Let $\mathcal{F}$ be the family of labelled trees that

- contains $P_4$ where the two leaves have status $A$ and the two support vertices have status $B$; and
- is closed under the two operations $\mathcal{T}_1$ and $\mathcal{T}_2$, which extend the tree $T$ by attaching a path to a vertex $v \in V(T)$.

**Operation $\mathcal{T}_1$.** Assume $\mathrm{sta}(v) = A$. Then add a path $(x, y, z)$ and the edge $vx$. Let $\mathrm{sta}(x) = \mathrm{sta}(y) = B$ and $\mathrm{sta}(z) = A$.

**Operation $\mathcal{T}_2$.** Assume $\mathrm{sta}(v) = B$. Then add a path $(x, y)$ and the edge $vx$. Let $\mathrm{sta}(x) = B$ and $\mathrm{sta}(y) = A$.

If $T \in \mathcal{F}$, we let $A(T)$ and $B(T)$ be the set of vertices of statuses $A$ and $B$, respectively, in $T$.

**Theorem 10** [1]. *For a tree $T$ of order $n \geq 3$,*

$$\mathrm{sd}(T) = 3 \textit{ if and only if } T \in \mathcal{F}.$$

In order to prove Theorem 9, we will need the following Observation 11 and Lemma 12 made for trees belonging to the family $\mathcal{F}$.

**Observation 11** [1]. *Let $T \in \mathcal{F}$ and $v \in V(T)$.*

(1) *If $v$ is a leaf, then* $\mathrm{sta}(v) = A$.

(2) *If $v$ is a support vertex, then* $\mathrm{sta}(v) = B$.

(3) *If* $\mathrm{sta}(v) = A$, *then $N(v) \subseteq B(T)$.*

(4) *If* $\mathrm{sta}(v) = B$, *then $v$ is adjacent to exactly one vertex of $A(T)$ and at least one vertex of $B(T)$.*

(5) *The distance between any two vertices in $A(T)$ is at least* 3.

**Lemma 12** [1]. *If $T \in \mathcal{F}$, then $A(T)$ is a $\gamma(T)$-set.*

**Lemma 13.** *If $T$ is a tree with* $\mathrm{sd}(T) = 3$, *then* $\mathrm{msd}(T) = 3$.

***Proof.*** Let $T$ be a tree with $\mathrm{sd}(T) = 3$. Thus, by Theorem 10, $T \in \mathcal{F}$ and by Lemma 12, $A(T)$ is a $\gamma(T)$-set.

By Theorem 7, in order to prove the statement, it is enough to show that $\mathrm{msd}(T) > 2$.

Let $uv \in E(T)$ be any edge. Then by Observation 11, two cases are possible: either $\{\text{sta}(u), \text{sta}(v)\} = \{B\}$ or $\{\text{sta}(u), \text{sta}(v)\} = \{A, B\}$. We subdivide $uv$ with two vertices $x$ and $y$. Now we construct a minimum dominating set $D$ of $T_{uv,2}$ in a following way. First let $D := A(T)$ and then replace every vertex $a \in A(T)$ with a vertex $a' \in N(a)$ which belongs to the $(a - \{x, y\})$-path. If $\text{sta}(u) = \text{sta}(v) = B$, then $\{u, v\} \subset D$. If $\text{sta}(u) = A$ and $\text{sta}(v) = B$, then $\{x, v\} \subset D$. By Observation 11, it is easy to check that $D$ is a dominating set of $T_{uv,2}$ and that $|D| = |A(T)|$. Since subdivision of the edge cannot decrease the domination number of a graph, $D$ is a $\gamma(T_{uv,2})$-set. Hence, $\gamma(T) = \gamma(T_{uv,2})$, which implies $\text{msd}(T) = 3$. ■

**Lemma 14.** *If $T$ is a tree with* $\text{msd}(T) = 3$, *then* $\text{sd}(T) = 3$.

***Proof.*** Let $T$ be a tree with $\text{msd}(T) = 3$. By Theorem 10, it is enough to show that $T \in \mathcal{F}$. We consider trees with $\text{diam}(T) \geq 3$ (because for trees with $\text{diam}(T) \leq 2$ we have $\text{msd}(T) \leq 2$). Moreover, it is no problem to check that the result is true for all trees with at most 4 vertices: the only tree $T$ with $\text{msd}(T) = 3$ and with at most 4 vertices is $P_4$ which belongs to $\mathcal{F}$. We continue the proof by induction on $n$, the order of $T$. Assume that every tree $T'$ with $n' < n$ vertices such that $\text{msd}(T') = 3$ belongs to the family $\mathcal{F}$.

Now, let $T$ be a tree with $\text{msd}(T) = 3$, $\text{diam}(T) \geq 3$ and $n > 4$. Then $\gamma(T) = \gamma(T_{e,2})$ for every edge $e \in E(T)$. Let $P = (v_0, v_1, v_2, \ldots, v_k)$ be a longest path of $T$ such that the degree of a vertex $v_2$ is as big as possible. It follows by Observation 4 that $d(v_1) = 2$ (as otherwise $v_1$ is a strong support vertex and then $\text{msd}(T) = 1$). Now we consider three cases.

*Case* 1. $d(v_2) = 2$. Since $\text{msd}(T) = 3$, $v_3$ is neither a support vertex nor a neighbor of a support vertex (as otherwise $\gamma(T_{v_1v_2,2}) > \gamma(T)$). Thus, outside the path $P$, only $P_3$'s may be attached to $v_3$. We consider the tree $T' = T - \{v_0, v_1, v_2\}$. It is no problem to see that $\gamma(T) = \gamma(T') + 1$. Moreover, for every edge $e \in E(T')$ we have $\gamma(T'_{e,2}) = \gamma(T_{e,2}) - 1 = \gamma(T) - 1 = \gamma(T')$. Hence, $\text{msd}(T') = 3$ and from the induction hypothesis $T' \in \mathcal{F}$. From the construction of the family $\mathcal{F}$ we know $\text{sta}(v_3) = A$. Thus $T$ can be obtained from $T'$ by Operation $\mathcal{T}_1$, where $\text{sta}(v_2) = \text{sta}(v_1) = B$ and $\text{sta}(v_0) = A$.

*Case* 2. $d(v_2) > 2$ and $v_2$ is a support vertex, say $v_2'$ is the leaf adjacent to $v_2$. By Observation 4, $v_2$ is adjacent to only one leaf. We consider the tree $T' = T - \{v_0, v_1\}$. It is obvious that $\gamma(T) = \gamma(T') + 1$. Since $\text{msd}(T) = 3$ and $v_1, v_2$ are support vertices, we have $\gamma(T'_{e,2}) = \gamma(T_{e,2}) - 1 = \gamma(T) - 1 = \gamma(T')$ for every edge $e \in E(T') \setminus \{v_2v_2'\}$. This also implies that there exists a $\gamma(T')$-set $D'$ containing $v_2$ and $v_3$. We subdivide the edge $v_2'v_2$ with vertices $x$ and $y$. Then $(D' \setminus \{v_2\}) \cup \{x\}$ is a $\gamma$-set in $T'_{v_2v_2',2}$ and $\gamma(T') = \gamma(T'_{v_2v_2',2})$. Therefore $T' \in \mathcal{F}$ with

$\mathrm{sta}(v_2) = B$, and $T$ can be obtained from $T'$ by Operation $\mathcal{T}_2$, where $\mathrm{sta}(v_1) = B$ and $\mathrm{sta}(v_0) = A$.

*Case* 3. $d(v_2) > 2$ and $v_2$ is not a support vertex. Then $v_2$ is adjacent to at least two support vertices (otherwise $T$ has a longer path). Let $T' = T - \{v_0, v_1\}$. Again $\gamma(T) = \gamma(T') + 1$. Since $\mathrm{msd}(T) = 3$, there exists a minimum dominating set which contains $v_2$. Therefore for every edge $e \in E(T')$ we obtain $\gamma(T'_{e,2}) = \gamma(T_{e,2}) - 1 = \gamma(T) - 1 = \gamma(T')$. Hence, $T' \in \mathcal{F}$, $\mathrm{sta}(v_2) = B$ and $T$ can be obtained from $T'$ by Operation $\mathcal{T}_2$, where $\mathrm{sta}(v_1) = B$ and $\mathrm{sta}(v_0) = A$.

In all these cases $T \in \mathcal{F}$. ■

Now, Theorem 9 is an immediate consequence of Lemmas 13, 14 and Observation 2.

### 5.2. Trees with domination multisubdivision number equal to 1

In this subsection we briefly present a characterization of all trees $T$ with $\mathrm{msd}(T) = 1$. This characterization is an immediate consequence of Observation 2 and a result of Benecke and Mynhardt in [3], where they have characterized all trees with domination subdivision number equal to 1. Let $\mathcal{N}(G)$ consists of those vertices which are not contained in any $\gamma(G)$-set.

**Corollary 15.** *For a tree $T$ of order $n \geq 3$,* $\mathrm{msd}(T) = 1$ *if and only if $T$ has*

(i) *a leaf $u \in \mathcal{N}(T)$ or*

(ii) *an edge $xy$ with $x, y \in \mathcal{N}(T)$.*

## 6. Open Problems

We close with the following list of open problems that we have yet to settle.

**Problem 16.** Determine the class of graphs $G$ for which $\mathrm{sd}(G) = \mathrm{msd}(G) = 1$.

**Problem 17.** Do there exist domination multisubdivision critical graphs, i.e., if $\mathrm{msd}(G) = k$ for some $k \in \{1, 2, 3\}$, then subdivision of *any* edge exactly $k$ times leads to a graph with domination number greater than $\gamma(G)$?

**Problem 18.** Determine the computational complexity of subdivision and multisubdivision numbers for another classes of graphs (chordal graphs, planar graphs etc.).

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